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Bifurcation Analysis of A Nonlinear Pluse SIR Model with Media Coverage

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Abstract In this study, we propose a nonlinear pulse SIR model with media coverage to describe the vaccination and isolation measures determined by the size of susceptible individuals. The dynamic behaviours of the model without impulse were discussed and the basic reproduction number was defined. The existence and stability of disease-free periodic solutions(DFPS) is investigated when $R_0 < 1$. Even if $R_0 > 1$ the DFPS is still stable when $S_H < 1/R_0$, which indicates that the state-dependent pulse strategy is still effective in preventing the outbreak of infectious diseases by choosing suitable S_H. Further, by defining Poincaré map and using the bifurcation theorem, the transcritical and pitchfork bifurcations near the DFPS with respect to some key parameters were investigated. We find that complex dynamic behaviours and the rich biological significance of the model can be exhibited with the intervention of impulse control.

Keywords SIR Model · Media Coverage · Nonlinear Pulse · Bifurcations · Poincaré Map

1 Introduction

In recent dacades, numerous outbreaks of global infectious diseases have had a huge impact on public health systems, economic and social development . Effective prediction and control of infectious diseases is very helpful to preventing epidemics. Mathematical models play a crucial important role in the prevention and control of infectious diseases. It is not only provides threshold conditions for

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disease outbreak, but also offers theoretical support for the development of effective control strategies. Therefore, various models of infectious diseases have been proposed and studied [1–4]. The classic SIR model was first proposed by Kermack and McKendrick, who work on the epidemiological behaviour of the Black Death and plague in 1927 [5]. The total population is assumed to be a constant N, which is divided into three compartments: $S(t)$: the number of susceptible individuals, $I(t)$: the infected who are infectious, $R(t)$: the number of removed individuals, that is $N = S(t) + I(t) + R(t)$. The Kermack and McKendrick's model is as follows

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta SI - \mu S, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta SI - (\mu + \gamma)I, \\ \dfrac{\mathrm{d}R(t)}{\mathrm{d}t} = \gamma I - \mu R, \end{cases} \tag{1}$$

where Λ is the constant recruitment rate, μ denotes the death rate, β denotes the transmission coefficient, γ denotes the recovery rate of infected individuals. For simplicity, it is assumed that the removed individuals will not be infected again. That is, removed compartment will not enter the susceptible compartment. The dynamic behaviour of model (1) is determined by the following model

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta SI - \mu S, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta SI - (\mu + \gamma)I. \end{cases} \tag{2}$$

The results show that if the basic reproduction number $\mathcal{R}_0 \doteq \beta/(\gamma + \mu) > 1$ then the model (2) has a stable endemic equilibrium point, if $\mathcal{R}_0 < 1$ then the Infectious diseases is eventually eliminated. Thereafter, many academics extended the SIR model to understand the epidemiology of infectious diseases [6–13].

In recent years, with the progress of electronic communication technology, social media has become extremely significant in human social life. Media coverage has a strong impact not only on individual behaviour, but also on the implementation of control measures [14, 15]. Therefore, models describing the influence of media coverage on the spread of infectious diseases have attracted much attention in recent years [16–20]. On the one hand, if the susceptible individuals is alert enough, they will deliberately avoid unnecessary contact with infected individuals when the media reports confirmed cases. On the other hand, all infected people stay in the area where they are being cured. Therefore, the new transmission rates will be reduced by lower contact rates [21–23]. The effect of media coverage on the transmission coefficient could be described by a decreasing factor. The usual decreasing function is exponential. For simplicity, the transmission coefficient is supposed as $\beta \exp(-\alpha I)$, where α denotes the intensity of media coverage. Then, The classic SIR model with media coverage is as follows

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta \exp(-\alpha I) SI - \mu S, \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta \exp(-\alpha I) SI - (\mu + \gamma)I. \end{cases} \tag{3}$$

Vaccination and isolation are the principal methods of controlling the disease to prevent epidemics of infectious diseases [24]. Many experts have tried to find the ideal vaccination strategy by investigating the transmission dynamics of infectious diseases [25–27], some scholars suggested that continuous vaccination control should be implemented [28–31], others claimed that a fixed moment of vaccination should be implemented and impulse models have been proposed [32–36]. Therefore, it is more reasonable to assume that whether or not to vaccinate the susceptible population depends on the size of the infected population or the susceptible population [37]. Based on the assumptions, Zhang et al. proposed a state-dependent pulse model to describe whether or not vaccination is depends on the size of the susceptible population [37]. They found the threshold conditions for the existence and global stability of disease-free periodic solutions, and analyzed the bifurcations of the model. Li et al. studied the bifurcation phenomena with respect to key parameters near disease-free periodic solutions and observed that positive first-order periodic solutions can be bifurcated from disease-free periodic solutions through transcritical bifurcation [38]. Linear impulsive control is considered in the above models. However, the saturation phenomenon caused by limited resources should be nonlinear. Therefore, based on the model (3), we propose a state-dependent impulse model with media effects concerning susceptibles-guided impulse vaccination and isolation strategies. The model is expressed as follows

$$\begin{cases} \left.\begin{aligned} \frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \beta \exp(-\alpha I)SI - \mu S, \\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \beta \exp(-\alpha I)SI - (\mu+\gamma)I, \end{aligned}\right\} \quad S < S_H, \\ \left.\begin{aligned} S\left(t^+\right) &= \left(1 - \frac{qS}{S+h_1}\right)S, \\ I\left(t^+\right) &= \left(1 - \frac{pI}{I+h_2}\right)I, \end{aligned}\right\} \quad S = S_H, \end{cases} \tag{4}$$

where $q \in [0,1]$ denotes the maximal vaccination rate of susceptible individuals, $p \in [0,1]$ denotes the maximal treatment(or isolation) rate of infected individuals, $h_1 > 0$, $h_2 > 0$ denote the denote the half-saturation constants of susceptible and infected individuals, respectively. Denote $g(I) \doteq (1 - pI/(I+h_2))\, I$. And S_H represents the susceptible individuals threshold, i.e., when the number of susceptible individuals is less than the crucial value S_H, no control measures are implemented. Once susceptible individuals size surpasses the threshold value S_H, comprehensive control actions are taken.

In the next section, we first give some basic definitions of the planar impulsive dynamical systems and some very useful lemmas for bifurcations. In Section 3, the dynamics of the model without impulsive effects are introduced. The Poincare mapping is defined and its properties are discussed in Section 4 and then the existence and stability of DFPS are investigated. In Section 5, the transcritical and pitchfork bifurcations near the DFPS respect to q, p, Λ and S_H are studied. Finally, we numerically simulated our results, discussed the biological implications and conclusions.

2 Preliminaries

In this section, some definitions and theorems are stated, which are very helpful and useful in the following sections.

Definition 1 (Lambert W [39]) The Lambert W function is a multivalued function defined as the inverse of $x \to xe^x$, i.e., for any $x \in$ C, there is

$$Lambert\ W(x)\exp(Lambert\ W(x)) = x. \tag{5}$$

It's easy to see that

$$Lambert\ W'(x) = \frac{Lambert\ W(x)}{x(1 + Lambert\ W(x))}. \tag{6}$$

If $x > -1$, the inverse of the function xe^x on the interval $[-1, +\infty)$ is Lambert $W(0, x)$. Similarly, the inverse function of the function xe^x on the interval $(-\infty, -1]$ is Lambert $W(-1, x)$. In the practical problem, we only consider the functions Lambert $W(0, x)$ and Lambert $W(-1, x)$ defined on $x[-exp(-1), 0)$. For convenience, we note Lambert $W(x)$ as $W(x)$ through out the paper.

Lemma 1 *([40, 41]) The T-periodic solution$(x, y) = (\xi(t), \eta(t))$ of the system*

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = P(x, y), \quad \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = Q(x, y), \qquad if\ \psi(x, y) \neq 0, \\ \Delta x = \beta_1(x, y), \quad \Delta y = \beta_2(x, y), \qquad if \psi(x, y) = 0, \end{cases}$$

is orbitally asymptotically stable if the Floque multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \prod_{k=1}^{q} \Delta_k exp \left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)) \right) dt \right],$$

with

$$\Delta_k = \frac{P_+ \left(\frac{\partial \beta_2}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \beta_2}{\partial x}\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial x} \right) + Q_+ \left(\frac{\partial \beta_1}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial \beta_1}{\partial y}\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \right)}{P\frac{\partial \psi}{\partial x} + Q\frac{\partial \psi}{\partial y}},$$

and P, Q $\frac{\partial \beta_1}{\partial x}$, $\frac{\partial \beta_1}{\partial y}$, $\frac{\partial \beta_2}{\partial x}$, $\frac{\partial \beta_2}{\partial y}$, $\frac{\partial \psi}{\partial x}$, $\frac{\partial \psi}{\partial y}$ are calculated at the point $(\xi(t), \eta(t))$. $P_+ = P(\xi(t_k^+), \eta(t_k^+))$ and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$. Here $\psi(x, y)$ is a sufficiently smooth function such that grad $\psi(x, y) \neq 0$,and t_k is the time of the kth jump.

Lemma 2 *(Transcritical bifurcation [42]) Let $G : U \times I \to R$, G is C^r with $r \leqslant 2$, U and I are open intervals of the real line containing 0. If*

$$(i) G(0, \alpha) = 0\ for\ all\ \alpha; \qquad (ii) \frac{\partial G}{\partial x}(0, 0) = 1;$$

$$(iii) \frac{\partial^2 G}{\partial x \partial \alpha}(0, 0) > 0; \qquad (iv) \frac{\partial^2 G}{\partial x^2}(0, 0) > 0,$$

then, there are $\alpha_1 < 0 < \alpha_2$ and $\epsilon > 0$ such that
(i) if $\alpha_1 < \alpha < 0$, then $G_\alpha = G(\cdot, \alpha)$ has two fixed points, 0 and $x_{1\alpha} > 0$ in $(-\epsilon, \epsilon)$ with the origin being asymptotically stable and the other fixed point being unstable.
(ii) if $0 < \alpha < \alpha_2$, then G_α has two fixed points, 0 and $x_{1\alpha} < 0$ in $(-\epsilon, \epsilon)$ with the origin being unstable and the other fixed point being asymptotically stable.

It is revealed that case $\partial^2 G / \partial x \partial \alpha(0, 0) < 0$ can be analyzed by changing $\alpha \to -\alpha$.

Lemma 3 *(Pitchfork bifurcation [42]) Let $G : U \times I \to R$, G is C^r with $r \geqslant 3$, U and I are open intervals of the real line containing* 0*. If $\partial^2 G/\partial x \partial \alpha(0,0) > 0$, $\partial^2 G/\partial x^2(0,0) = 0$ and $\partial^3 G/\partial x^3(0,0) < 0$, then there exist with $\alpha_1 < 0 < \alpha_2$ and $\epsilon > 0$ such that*
(i) if $\alpha_1 < \alpha \leqslant 0$, then $G_\alpha = G(\cdot, \alpha)$ exists a unique fixed point in $(-\epsilon, \epsilon)$, which is asymptotically stable.
(ii) if $0 < \alpha < \alpha_2$, then G_α has three fixed points in $(-\epsilon, \epsilon)$ with the origin being unstable and the others $x_{1\alpha} < 0 < x_{2\alpha}$ being asymptotically stable.

The case $\partial^2 G/\partial x \partial \alpha(0,0) < 0$ can be discussed by changing $\alpha \to -\alpha$. And if $\partial^3 G/\partial x^3(0,0) > 0$, it is undergoes a subcritical pitchfork bifurcation.

3 Dynamics of the model without impulsive effects

In order to study the dynamics of model (4), we first need to investigate the dynamical behaviour of model (3). That is, the following model

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \beta \exp(-\alpha I)SI - \mu S = f_1(S, I), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta \exp(-\alpha I)SI - (\mu + r)I = f_2(S, I). \end{cases}$$

It's easy to see $\{(S, I) \in R_+^2 | S > 0, I > 0, 0 \leq S + I \leq \Lambda/\mu\} \doteq \Omega$ is an invariant domain of model (3). From model (3), we get

$$\left.\frac{\mathrm{d}S}{\mathrm{d}t}\right|_{S=0} = \Lambda > 0, \quad \left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{I=0} = 0, \quad \left.\frac{\mathrm{d}(S+I)}{\mathrm{d}t}\right|_{S+I=\Lambda/\mu} = \Lambda - \mu(S+I) - \gamma I < 0.$$

So all solution of model (3) will eventually enter into the region Ω. Thus, Ω is an invariant domain of model (3). Denote two isolines as follows

$$L_1 : \Lambda - \beta \exp(-\alpha I)SI - \mu S = 0, \quad L_2 : S = \frac{\mu + \gamma}{\beta} \exp(\alpha I).$$

Solve the following equation

$$\begin{cases} \Lambda - \beta \exp(-\alpha I)SI - \mu S = 0, \\ \beta \exp(-\alpha I)SI - (\mu + \gamma)I = 0. \end{cases} \tag{7}$$

Then, we get the disease-free equilibrium $E^0 = (K, 0)$ with $K = \Lambda/\mu$, and the possible endemic equilibrium $E^* = (S^*, I^*)$ provides

$$S^* = \frac{\mu + \gamma}{\beta} \exp(\alpha I^*), \quad \Lambda - (\mu + \gamma)I^* - \frac{\mu(\mu + \gamma)}{\beta} \exp(\alpha I^*) = 0.$$

The basic reproduction number $R_0 \doteq \beta K/(\mu + \gamma)$. Let

$$f(I) = \Lambda - (\mu + \gamma)I - \frac{\mu(\mu + \gamma)}{\beta} \exp(\alpha I).$$

The derivative of f with respect to I is given by

$$f'(I) = -(\mu + \gamma) - \frac{\mu\alpha(\mu + \gamma)}{\beta} \exp(\alpha I) < 0,$$

for $I \in \Omega$. Further, we have

$$f\left(\frac{\Lambda}{\mu+\gamma}\right) = -\frac{\mu(\mu+\gamma)}{\beta}\exp\left(\frac{\Lambda\alpha}{\mu+\gamma}\right) < 0, \tag{8}$$

and if $R_0 > 1$, then

$$\begin{aligned} f(0) &= \Lambda - \frac{\mu(\mu+\gamma)}{\beta} \\ &= \frac{\mu(\mu+\gamma)}{\beta}(R_0 - 1) > 0. \end{aligned} \tag{9}$$

It follws from the monotonicity of $f(I)$ and Equation (8) and Equation (9) that there exists a unique $I^* \in (0, \Lambda/(\mu+\gamma))$ such that $f(I^*) = 0$. That means there exists a endemic equilibrium E^* for $R_0 > 1$. Actually, according to Definition 1, using the Lambert W function it is possible to calculate I^*, and I^* satisfies the following equation

$$\Lambda - (\mu+\gamma)I^* = \frac{\mu(\mu+\gamma)}{\beta}\exp(\alpha I^*).$$

Both sides of the equation are multiplied by the factor $\alpha\exp(-\alpha\Lambda/(\mu+\gamma))$ at the same time as follows

$$\alpha\exp\left(\frac{-\alpha\Lambda}{\mu+\gamma}\right)(\Lambda - (\mu+\gamma)I^*) = \alpha\exp\left(\frac{-\alpha\Lambda}{\mu+\gamma}\right)\frac{\mu(\mu+\gamma)}{\beta}\exp(\alpha I^*).$$

Rewrite the above equation as follows

$$\left(\frac{\alpha\Lambda}{\mu+\gamma} - \alpha I^*\right)\exp\left(\frac{\alpha\Lambda}{\mu+\gamma} - \alpha I^*\right) = \frac{\alpha\mu}{\beta}\exp\left(\frac{\alpha\Lambda}{\mu+\gamma}\right).$$

By using the Lambert W function, we have

$$I^* = \frac{\alpha\Lambda}{\mu+\gamma} - \frac{1}{\alpha}W\left(\frac{\alpha\mu}{\beta}\exp\left(\frac{\alpha\Lambda}{\mu+\gamma}\right)\right).$$

Theorem 1 *If $R_0 < 1$, the disease-free equilibrium E^0 is globally asymptotically stable. If $R_0 > 1$, there exist a endemic equilibrium E^*, which is globally asymptotically stable.*

Proof If $R_0 < 1$, we choose Lyapunov function $V(t) = I(t) > 0$. Then calculate the total derivative of $V(t)$, we have

$$\dot{V}(t) = \frac{\mathrm{d}I(t)}{\mathrm{d}t} = \beta\exp(-\alpha I)SI - (\mu+\gamma)I < 0,$$

for all $(S, I) \in \Omega$, which indicates that the disease-free equilibrium $E^0 = (K, 0)$ is globally asymptotically stable(Fig. 1(a)).

If $R_0 > 1$, the jacobian matrix of model (3) as follows

$$J = \begin{bmatrix} -\beta\exp(-\alpha I)I - \mu & -\beta S\exp(-\alpha I)(1-\alpha I) \\ \beta\exp(-\alpha I)I & \beta S\exp(-\alpha I)(1-\alpha I) - (\mu+\gamma) \end{bmatrix}.$$

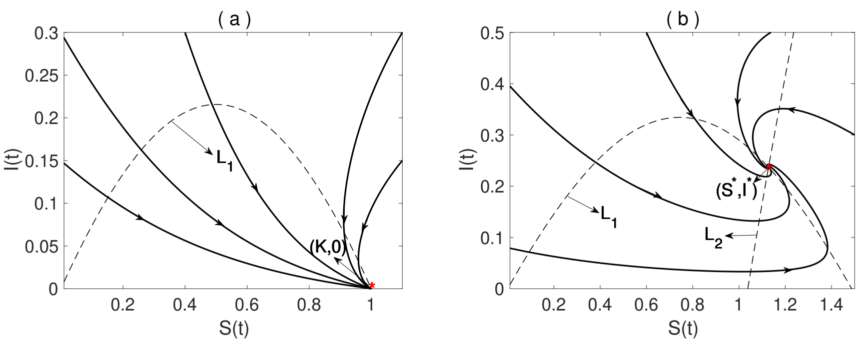

Fig. 1 Phase trajectories for model (3). (a)$R_0 < 1$; (b) $R_0 > 1$.

At the endemic equilibrium $E^* = (S^*, I^*)$, we have

$$J\left(E^*\right) = \begin{bmatrix} -\beta\exp(-\alpha I^*)I^* - \mu & -(\mu+\gamma)(1-\alpha I^*) \\ \beta\exp(-\alpha I^*)I^* & -(\mu+\gamma)\alpha I^* \end{bmatrix},$$

its eigenvalues are negative, which indicates that the endemic equilibrium E^* is locally asymptotically stable. Denote the Dulac function $B(S,I) = 1/(SI)$. By caculation, then

$$\begin{aligned}\frac{\partial(Bf_1)}{\partial S} + \frac{\partial(Bf_2)}{\partial I} &= \frac{-f_1(S,I)}{S^2 I} + \frac{1}{SI}(-\beta\exp(-\alpha I)I - \mu) + \frac{-f_2(S,I)}{SI^2} \\ &+ \frac{1}{SI}(\beta S\exp(-\alpha I)(1-\alpha I) - (\mu+\gamma)) \\ &= \frac{1}{SI}\left(-\frac{\Lambda}{S} - \beta\exp(-\alpha I)\alpha SI\right) < 0,\end{aligned}$$

for all $(S,I) \in \Omega$. According to the Bendixson-Dulac theorem, there is no limit cycle in model (3), which shows the endemic equilibrium E^* is globally asymptotically stable(Fig. 1(b)).

4 Poincaré map and Disease-free Periodic Solution

4.1 Poincaré map

We first define the Poincaré map in this subsection and study its properties. Since (S,I) is in the invariant domain, it is assumed that $S_H < K$ holds. Denote two straight lines as follows

$$L_3 : S = \left(1 - \frac{qS_H}{S_H + h_1}\right) S_H \doteq S_N, \quad L_4 : S = S_H.$$

Substituting $S = S_H$ into L_1 to obtain the intersection of L_4 and L_1, noted as $V_{S_H}(S_H, I_{S_H})$. By the definition of the Lambert W function in Definition 1, one has

$$I_{S_H} = -\frac{1}{\alpha} \cdot W\left(-\alpha \cdot \frac{\Lambda - \mu S_H}{\beta S_H}\right),$$

using the same method, denote the intersection of L_4 and L_1 as $V_{S_N} = (S_N, I_{S_N})$ with

$$I_{S_N} = -\frac{1}{\alpha} \cdot W\left(-\alpha \cdot \frac{\Lambda - \mu S_N}{\beta S_N}\right).$$

Define the open set as follows

$$\Omega_1 \doteq \{(S, I) \in R_+^2 | 0 \le S \le S_H, I > 0\}.$$

For $R_0 < 1$, the endemic equilibrium E^* does not exist, and the disease-free equilibrium $E^0(K, 0)$ is globally asymptotically stable. It implies that any solution of model (3) initiating from Ω_1 will reach at the line L_4 a finite time. Then, we could define the impulsive set $\mathcal{M}$ of the model (4)

$$\mathcal{M} = \{(S, I) | S = S_H, 0 \le I \le I_{S_H}\}.$$

Set the continuous function $H : (S_H, I) \in \mathcal{M} \to (S^+, I^+) = (S_N, g(I)) \in \Omega$ with $g(I) = (1 - pI/(I + h_2))\, I$. Therefore, the phase set $\mathcal{N}$ is defined as follows

$$\mathcal{N} = H(\mathcal{M}) = \{(S^+, I^+) \in \Omega | S^+ = S_N, 0 \le I^+ \le I_{S_N}\}.$$

Set the two sections as follows

$$S_{S_H} = \{(S, I) | S = S_H, I \ge 0\}, \quad S_{S_N} = \{(S, I) | S = S_N, I \ge 0\}.$$

Let the section S_{S_N} as a Poincaré section. Assume that point P_k^+ is on section S_{S_N}, denoted $P_k^+ = (S_N, I_k^+)$. According to the global properties of model (3), the trajectory from point P_k^+ will reach at section S_{S_H} in finite time, denoted the intersection point as $P_{k+1} = (S_H, I_{k+1})$. This indicates that I_{k+1} depends on I_k^+ ,i.e., we have, $I_{k+1} \doteq \mathcal{P}(I_k^+)$. After a single impulsive control is implemented at point P_{k+1}, then P_{k+1} jumps to point $P_{k+1}^+ = (S_N, I_{k+1}^+)$ with $I_{k+1}^+ = g(I_{k+1})$ on section S_{S_N}. Thus, we can define the Poincare map as follows

$$I_{k+1}^+ = g(I_{k+1}) = g(\mathcal{P}(I_k^+)) = I(S_N, I_k^+) \doteq \mathcal{P}_M(I_k^+). \tag{10}$$

To address the dynamic behavior of the model (4), we could describe the Poincare map in terms of the phase portrait, which is defined by the impulsive points in the phase set. In phase space, the scalar differential equation for model (4) is as follows

$$\begin{cases} \dfrac{\mathrm{d}I}{\mathrm{d}S} = \dfrac{\beta \exp(-\alpha I) S I - (\mu + \gamma) I}{\Lambda - \beta \exp(-\alpha I) S I - \mu S} \doteq R(S, I), \\ I(S_N) = I_0^+. \end{cases} \tag{11}$$

Denote

$$\Omega_2 = \left\{ (S, I) | S > 0, I > 0, I < -\frac{1}{\alpha} \cdot W\left(-\alpha \cdot \frac{\Lambda - \mu S}{\beta S}\right) \right\}. \tag{12}$$

It is easy to see that the function $R(S, I)$ is continuously differentiable on region Ω_2. Let $I_0^+ = I_0$, $S_0^+ = S_N$, $I_0 \in (N)$ with $I_0 < I_{S_N}$. That is, $(S_0^+, I_0^+) \in \Omega_2$. Then

$$I(S) = I(S : S_N, I_0) = I(S, I_0), \quad S_N \le S \le S_H,$$

from the model (11), we have

$$I(S, I_0) = I_0 + \int_{S_N}^{S} R(s, I(s, I_0))\mathrm{d}s. \tag{13}$$

Thus, the expression of the Poincaré map $\mathcal{P}_M$ in the Ω_2 as follows

$$\mathcal{P}_M(I_0) = g(I(S_H, I_0)).$$

According to the Cauchy and Lipschitz theorem with parameters of the ordinary differential equation [43], then we get the following results, which are quite useful in the following sections.

$$\begin{aligned}
\frac{\partial I(S, I_0)}{\partial I_0} &= \exp\left(\int_{S_N}^{S} \frac{\partial R(s, I(s, I_0))}{\partial I}\mathrm{d}s\right),\\
\frac{\partial^2 I(S, I_0)}{\partial I_0^2} &= \frac{\partial I(S, I_0)}{\partial I_0}\int_{S_N}^{S} \frac{\partial^2 R(s, I(s, I_0))}{\partial I^2}\frac{\partial I(s, I_0)}{\partial I_0}\mathrm{d}s,
\end{aligned}$$

with

$$\begin{aligned}
\left.\frac{\partial R(S, I)}{\partial I}\right|_{I=0} &= \frac{\beta S - (\mu + \gamma)}{\Lambda - \mu S},\\
\left.\frac{\partial^2 R(S, I)}{\partial I^2}\right|_{I=0} &= \frac{2\beta S(\beta S - (\mu + \gamma))}{(\Lambda - \mu S)^2}.
\end{aligned}$$

By simple calculations, one yields

$$\begin{aligned}
\frac{\partial \mathcal{P}_M}{\partial I_0} &= g'(I(S_H, I_0))\frac{\partial I(S, I_0)}{\partial I_0}\\
&= \left(1 - \frac{pI(S_H, I_0)(I(S_H, I_0) + 2h_2)}{(I(S_H, I_0) + h_2)^2}\right)\exp\left(\int_{S_N}^{S_H} \frac{\partial R(s, I(s, I_0))}{\partial I}\mathrm{d}s\right),
\end{aligned} \tag{14}$$

$$\begin{aligned}
\frac{\partial^2 \mathcal{P}_M}{\partial I_0^2} &= g''(I(S_H, I_0))\left(\frac{\partial I(S_H, I_0)}{\partial I_0}\right)^2 + g'(I(S_H, I_0))\frac{\partial^2 I(S_H, I_0)}{\partial I_0^2}\\
&= -\frac{2ph_2^2}{(I(S_H, I_0) + h_2)^3}\exp\left(2\int_{S_N}^{S_H} \frac{\partial R(s, I(s, I_0))}{\partial I}\mathrm{d}s\right)\\
&+ \left(1 - \frac{pI(S_H, I_0)(I(S_H, I_0) + 2h_2)}{(I(S_H, I_0) + h_2)^2}\right)\frac{\partial I(S_H, I_0)}{\partial I_0}\\
&\cdot \int_{S_N}^{S_H} \frac{\partial^2 R(s, I(s, I_0))}{\partial I^2}\frac{\partial I(s, I_0)}{\partial I_0}\mathrm{d}s
\end{aligned} \tag{15}$$

Theorem 2 *If $R_0 < 1$, then the Poincaré map $\mathcal{P}_M$ of model (4) satisfies the following properties:*

(i) The domain and range of $\mathcal{P}_M$ are $[0, +\infty)$ and $[0, \mathcal{P}_M] = [0, g(I(S_N, I_{S_N}))]$, respectively. It is increasing on $[0, I_{S_N}]$ and decreasing on $[I_{S_N}, +\infty)$.
(ii) $\mathcal{P}_M$ is continuously differentiable.
(iii) There exist a unique fixed point $I = 0$ for $\mathcal{P}_M$.
(iv) There is a horizontal asymptote $I = 0$ for $\mathcal{P}_M$ as $I_k^+ \to +\infty$.

Proof (i) Based on the vector field of model (3) that the definition domain of $\mathcal{P}_M$ can be expressed as $[0,+\infty)$. For any $I_{k_1}^+, I_{k_2}^+ \in [0, I_{S_N}]$ with $I_{k_1}^+ < I_{k_2}^+$. By the uniqueness of the solution of model (3), we have $I(S_N, I_{k_1}^+) < I(S_N, I_{k_2}^+)$. Note that the function $g(I) = (1 - pI/(I+h_2))\,I$ and the derivative of g with respect to I is $g'(I) = ((1-p)I^2 + 2Ih_2(1-p) + h_2^2)/(I+h_2)^2$, which shows that g is monotonically increasing for any $I > 0$. Therefore, $g(I(S_N, I_{k_1}^+)) < g(I(S_N, I_{k_2}^+))$,i.e., $\mathcal{P}_M(I_{k_1}^+) < \mathcal{P}_M(I_{k_2}^+)$.

For any $I_{k_1}^+, I_{k_2}^+ \in [I_{S_N}, +\infty)$ and $I_{k_1}^+ < I_{k_2}^+$, the trajectory from I_{k_i} will cross L_1 before it reaches L_3 according to the global property of model (3). The vertical coordinate of the intersection of the trajectory with L_3 are noted as $I_{\bar{k}_i}, i = 1, 2$. At this point, $I_{\bar{k}_1} > I_{\bar{k}_2}$. Similar to the previous case, we have

$$g(I(S_N, I_{k_1}^+)) = g(I(S_N, I_{\bar{k}_1})) > g(I(S_N, I_{\bar{k}_2})) = g(I(S_N, I_{k_2}^+)).$$

That is, $\mathcal{P}_M(I_{k_1}^+) > \mathcal{P}_M(I_{k_2}^+)$. Therefore, $\mathcal{P}_M$ is increasing on $[0, I_{S_N}]$ and decreasing on $[I_{S_N}, +\infty)$. At the same time, the rang of $\mathcal{P}_M$ is $[0, g(I(S_N, I_{S_N}))]$.

(ii) The function $R(S, I)$ is continuous and differentiable in Ω_2. Therefore, the continuity and differentiability of $I(S_H, S)$ can be confirmed by the continuity and differentiability theorems of solutions of ordinary differential equations with respect to their initial conditions, i.e., the Cauchy and Lipshitz theorem with parameters. Meanwhile, g is also continuously differentiable. Therefore, $\mathcal{P}_M = g(\cdot)$ is continuously differentiable.

(iii) It is esay to see that the Ω_2 is an invariant set of model (3), which indicates that $\mathrm{d}S/\mathrm{d}t > 0, \mathrm{d}I/\mathrm{d}t < 0$ for any $(S, I) \in \Omega_2$. We claim that $\mathcal{P}_M$ is increasing on $[0, I_{S_N}]$ and decreasing on $[I_{S_N}, +\infty)$. Therefore, it follows from $\mathrm{d}I/\mathrm{d}t < 0$ and $g(I) < I$ that $\mathcal{P}_M(I_k^+) < I_k^+$ for all $I_k^+ \in [0, I_{S_N}] \cup [I_{S_N}, +\infty)$. All of these results show that $\mathcal{P}_M$ has a unique fixed point $I = 0$.

(iv) Since Ω_2 is an invariant set of model (3), $I(S_H, +\infty) = 0$ with $(S_N, +\infty)$, i.e., $\mathcal{P}_M(+\infty) = 0$. Otherwise, there is a positive $\tilde{I}$ such that $\mathcal{P}_M(\tilde{I}) = g(I(S_N, \tilde{I})) = \tilde{I}$ with $\tilde{P} = (S_N, g(I(S_N, \tilde{I})))$. Take any poin $P_1 = (S_N, g(I(S_N, I_1)))$ with $0 < g(I(S_N, I_1)) < g(I(S_N, \tilde{I}))$. It follows from the monotonicity of the function g that we have $I(S_N, I_1) < I(S_N, \tilde{I}))$. Then, base on the uniqueness of the solution of model (3) and the backward orbit of the starting P_1 will reach a point $P_1^+ = (S_N, I_1^+)$ with $I_1^+ > +\infty$, which is a contradiction. Thus, there exist a horizontal asymptote $I = 0$ for $\mathcal{P}_M$ as $I_k^+ \to +\infty$.

4.2 Disease-Free Periodic Solution

Let $I(t) = 0$, model (4) reduces as follows

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda - \mu S & S < S_H, \\ S\left(t^+\right) = \left(1 - \dfrac{qS}{S + h_1}\right) S & S = S_H. \end{cases} \tag{16}$$

Integrate first equation with the initial condition $S(0^+) = (1 - (qS_H)/(S_H + h_1))S_H \doteq S_N$. Then

$$S(t) = \frac{\Lambda - (\Lambda - \mu S_N)\exp(-\mu t)}{\mu}.$$

$S(t)$ will reach L_4 in finite time due to the global attraction of the model (3) solution, assuming this period is T. So we get

$$S_H = K - (K - S_N)\exp(-\mu T).$$

Then we have the expression of T

$$T = -\frac{1}{\mu}\ln\left(\frac{K - S_H}{K - S_N}\right).$$

Therefore, the model (4) has a DFPS $(S^T(t), 0)$ with periodic T.

Next, we study the stability of periodic solution $(S^T(t), 0)$ by using the Analogue of Poincaré Criterion. It follws from Lemma 1, we have

$$P(S,I) = \Lambda - \beta\exp(-\alpha I)SI - \mu S, \quad Q(S,I) = \beta\exp(-\alpha I)SI - (\mu + r)I,$$

$$\beta_1(S,I) = -\frac{qS^2}{S + h_1}, \quad \beta_2(S,I) = -\frac{pI^2}{I + h_2}, \quad \psi(S,I) = S - S_H.$$

By simple calculation

$$\frac{\partial P}{\partial S} = -\beta\exp(-\alpha I)I - \mu, \quad \frac{\partial Q}{\partial I} = \beta S\exp(-\alpha I)(1 - \alpha I) - (\mu + \gamma),$$

$$\frac{\partial \beta_1}{\partial S} = -\frac{qS(S + 2h_1)}{(S + h_1)^2}, \quad \frac{\partial \beta_2}{\partial I} = -\frac{pI(I + 2h_2)}{(I + h_2)^2}, \quad \frac{\partial \psi}{\partial S} = 1,$$

$$\frac{\partial \beta_1}{\partial I} = \frac{\partial \beta_2}{\partial S} = \frac{\partial \psi}{\partial I} = 0,$$

and $(\xi(T), \eta(T)) = (S_H, 0), \quad (\xi(T^+), \eta(T^+)) = (S_N, 0),\ P_+ = \Lambda - \mu S_N, P = \Lambda - \mu S_H$ Thus, we have

$$\begin{aligned}
\Delta_1 &= \frac{P_+\left(\frac{\partial \beta_2}{\partial I}\frac{\partial \psi}{\partial S} - \frac{\partial \beta_2}{\partial S}\frac{\partial \psi}{\partial I} + \frac{\partial \psi}{\partial S}\right) + Q_+\left(\frac{\partial \beta_1}{\partial S}\frac{\partial \psi}{\partial I} - \frac{\partial \beta_1}{\partial I}\frac{\partial \psi}{\partial S} + \frac{\partial \psi}{\partial I}\right)}{P\frac{\partial \psi}{\partial S} + Q\frac{\partial \psi}{\partial I}} \\
&= \frac{P_+\left(1 - \frac{pI(I+2h_2)}{(I+h_2)^2}\right)}{P} \\
&= \left(1 - \frac{pI(I + 2h_2)}{(I + h_2)^2}\right)\frac{K - S_N}{K - S_H}
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\exp\left(\int_0^T \left(\frac{\partial P}{\partial E}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial T}(\xi(t), \eta(t))\right)\mathrm{d}t\right) \\
&= \exp\left(\int_0^T (-\mu + \beta S(t) - (\mu + \gamma))\mathrm{d}t\right) \\
&= \exp\left(\int_0^T \big[-\mu + \beta\,[K - (K - S_N)\exp(-\mu t)] - (\mu + \gamma)\big]\mathrm{d}t\right).
\end{aligned} \tag{18}$$

Denote

$$N_1 = \int_0^T -\mu\mathrm{d}t = -\mu T = \ln\left(\frac{K - S_H}{K - S_N}\right),$$

$$\begin{aligned} N_2 &= \int_0^T \beta\left[K - (K - S_N)\exp(-\mu t)\right]\mathrm{d}t \\ &= \beta KT - \frac{\beta}{\mu}(S_H - S_N) \\ &= -\frac{\beta K}{\mu}\ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}(S_H - S_N) \end{aligned} \tag{19}$$

and

$$N_3 = \int_0^T -(\mu+\gamma)\mathrm{d}t = \frac{\mu+\gamma}{\mu}\ln\left(\frac{K - S_H}{K - S_N}\right). \tag{20}$$

So that the Equation (18) can be rewritten as follows:

$$\exp\left(\int_0^T \left(\frac{\partial P}{\partial E} + \frac{\partial Q}{\partial T}\right)\mathrm{d}t\right) = \exp(N_1 + N_2 + N_3). \tag{21}$$

Therefore, we get

$$\mu_2 = \left(1 - \frac{pI(I + 2h_2)}{(I + h_2)^2}\right)\exp(N_2 + N_3).$$

Note that if $h_2 = 0$, then $\partial\beta_2/\partial I = -p$ with

$$\mu_2 = (1 - p)\exp(N_2 + N_3),$$

if $h_2 > 0$ and $I = 0$, then $\partial\beta_2/\partial I = 0$ with

$$\mu_2 = \exp(N_2 + N_3).$$

In summary, the expressionar of μ_2 is

$$\mu_2 = \begin{cases} (1 - p)\exp(N_2 + N_3), & \text{if } h_2 = 0, \\ \exp(N_2 + N_3), & \text{if } h_2 > 0. \end{cases} \tag{22}$$

Since $S_N < S_H < K$, we have

$$\begin{aligned} N_2 + N_3 &= -\frac{\beta K}{\mu}\ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}(S_H - S_N) \\ &\quad + \frac{\mu+\gamma}{\mu}\ln\left(\frac{K - S_H}{K - S_N}\right) \\ &= \frac{\mu+\gamma}{\mu}(1 - R_0)\ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}(S_H - S_N) \\ &< \frac{\mu+\gamma}{\mu}(1 - R_0)\ln\left(\frac{K - S_H}{K - S_N}\right). \end{aligned} \tag{23}$$

It follows from $R_0 < 1$ that $N_2 + N_3 < 0$, which implies that $\mu_2 < 1$ holds. Thus we have the following result.

Theorem 3 *If $R_0 < 1$, the DFPS $(S^T(t), 0)$ is orbitally asymptotically stable.*

It is worth noticing that when $R_0 > 1$, $N_2 > 0$ and $N_3 < 0$ holds. It follows from Equation (19) and Equation (20) that we can rewrite $N_2 + N_3$ as follows

$$N_2 + N_3 = \int_{S_N}^{S_H} \frac{\beta S - (\mu + \gamma)}{\Lambda - \mu S} \mathrm{d}S. \tag{24}$$

if $(\beta S - (\mu + \gamma))/(\Lambda - \mu S) < 0$, then $N_2 + N_3 < 0$ holds, which indicates that $\mu_2 < 1$. Therefore, we have following main result.

Theorem 4 *If $R_0 > 1$ and $S_H < 1/R_0$, the DFPS $(S^T(t), 0)$ is orbitally asymptotically stable.*

Proof According to Lemma 1, we only need to verify $\mu_2 < 1$, i.e., verify $N_2 + N_3 < 0$. From Equation (23) then we have

$$\begin{aligned} N_2 + N_3 &= -\frac{\beta K}{\mu} \ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}(S_H - S_N) + \frac{\mu + \gamma}{\mu} \ln\left(\frac{K - S_H}{K - S_N}\right) \\ &= \left(\frac{\mu + \gamma}{\mu} - \frac{\beta K}{\mu}\right) \ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}(S_H - S_N) \\ &= \frac{\beta K}{\mu}\left[\left(\frac{1}{R_0} - 1\right) \ln\left(\frac{1 - \frac{S_H}{K}}{1 - \frac{S_N}{K}}\right) - \left(\frac{S_H}{K} - \frac{S_N}{K}\right)\right]. \end{aligned}$$

It follow from the monotonicity of function $j(x) \doteq ln(1 - x) - x$ that

$$\begin{aligned} N_2 + N_3 &= \frac{\beta K}{\mu}\left[\left(\frac{1}{R_0} - 1\right) \ln\left(\frac{1 - \frac{S_H}{K}}{1 - \frac{S_N}{K}}\right) - \left(\frac{S_H}{K} - \frac{S_N}{K}\right)\right] \\ &= \frac{\beta K}{\mu}\left[\left(\frac{1}{R_0} - 1\right) \ln\left(1 - \frac{S_H}{K}\right) - \frac{S_H}{K}\right. \\ &\quad \left. - \left(\frac{1}{R_0} - 1\right) \ln\left(1 - \frac{S_H}{K}\right) + \frac{S_N}{K}\right] \end{aligned} \tag{25}$$

If $R_0 > 1$ and $S_H < 1/R_0$, then $N_2 + N_3 < 0$, i.e., $\mu_2 < 1$ holds. Therefore, the DFPS $(S^T(t), 0)$ is orbitally asymptotically stable. This completes the proof.

5 Bifurcations

In Section 4.2, we proved that there exists a DFPS $(S^T(t), 0)$ with period T for model (4). The DFPS $(S^T(t), 0)$ is orbitally asymptotically stable for $R_0 < 1$. Further, if $R_0 > 1$ and $S_H < 1/R_0$, the DFPS $(S^T(t), 0)$ is still orbitally asymptotically stable. But when $R_0 > 1$, $S_H > 1/R_0$, the stability of the DFPS of model 4 will changes. For $R_0 > 1$ and $S_H > 1/R_0$, the sign of $N_2 + N_3$ would transform from negative to positive due to the key parameters change. Therefore, the DFPS $(S^T(t), 0)$ will switch from stable to unstable, which indicates that the DFPS $(S^T(t), 0)$ may undergo transcritical and supercritical bifurcations. In the next subsections, the bifurcations with respect to the key parameters such as the maximal vaccination rate q, the maximal treatment(or isolation) rate p, the constant recruitment rate Λ and the threshold of susceptible individuals S_H are studied, which have very vital biological significance.

5.1 Bifurcations on q

Note that the expression for μ_2 depends on h_2. When $h_2 > 0$, we take μ_2 as a function of q, i.e.,

$$\mu_2(q) = \exp(N_{23}(q)),$$
$$N_{23}(q) = \left(\frac{\mu+\gamma}{\mu} - \frac{\beta K}{\mu}\right)\ln\left(\frac{K-S_H}{K-S_N(q)}\right) - \frac{\beta}{\mu}\left(S_H - S_N(q)\right).$$

where

$$S_N(q) = \left(1 - \frac{qS_H}{S_H+h_1}\right)S_H$$

The derivative of N_{23} with respect to q is as follows

$$\frac{\mathrm{d}N_{23}}{\mathrm{d}q} = \frac{\mu+\gamma-\beta S_N}{\mu(K-S_N)}\cdot\frac{\mathrm{d}S_N}{\mathrm{d}q}, \tag{26}$$

with

$$\frac{\mathrm{d}S_N}{\mathrm{d}q} = -\frac{S_H^2}{S_H+h_1} < 0. \tag{27}$$

Thus, we have the derivative of μ_2 with respect to q as

$$\frac{\mathrm{d}\mu_2}{\mathrm{d}q} = \exp(N_{23}(q))\cdot\frac{\mathrm{d}S_N}{\mathrm{d}q}\cdot\frac{\mu+\gamma-\beta S_N}{\mu(K-S_N)}, \tag{28}$$

Obviously, if $S_N < (\mu+\gamma)/\beta \doteq \bar{S}$, then μ_2 is monotonically decreasing. If $S_N > \bar{S}$, then μ_2 is monotonically increasing. Since S_N is monotonically decreasing for any $q \in [0,1]$, there exists a unique $\bar{q}$ such that $S_N = \bar{S}$. Sovling $S_N = \bar{S}$, we have

$$\bar{q} = \frac{S_H-\bar{S}}{S_H}\cdot\frac{S_H+h_1}{S_H},$$

and $0 \le \bar{q} \le 1$ provided that $S_H h_1/(S_H+h_1) < \bar{S} < S_H$. Thus, if $q < \bar{q}$, then $S_N > \bar{S}$ and then $\mathrm{d}\mu_2/\mathrm{d}q > 0$ holds, if $q > \bar{q}$, then $S_N < \bar{S}$ and then $\mathrm{d}\mu_2/\mathrm{d}q < 0$ holds.

From Equation (24) that $\mu_2(0) = 1$ and $\mu_2(\bar{q}) > 1$, which indicates that the DFPS $(S^T(t), 0)$ is unstable for all $q \in [0,\bar{q})$. Further, it follows from the monotonicity of $\mu_2(q)$ on the interval $(\bar{q}, 1]$ that if $\mu_2(1) < 1$, there exists a unique q^* such that $\mu_2(q^*) = 1$. In summary, if $\bar{q} < q < q^*$, the DFPS $(S^T(t), 0)$ is unstable, and if $q^* < q < 1$, the $(S^T(t), 0)$ is orbitally asymptotically stable. This implies that the bifurcation may occur at point q^*.

Note that, if $q = 1$, then $S_N = S_H h_1/(S_H+h_1)$. Since $(K-S_N)/(K-S_H) > 1$, then

$$\ln\left(\frac{K-S_N}{K-S_H}\right) < \frac{S_H-S_N}{\sqrt{(K-S_H)(K-S_N)}}.$$

Thus, from Equation (23) we have

$$\begin{aligned} N_{23}(1) &= \frac{\beta K}{\mu}\ln\left(\frac{K-S_H}{K-S_N}\right) - \frac{\beta}{\mu}\left(S_H-S_N\right) + \frac{\mu+\gamma}{\mu}\ln\left(\frac{K-S_H}{K-S_N}\right) \\ &= -\frac{\beta}{\mu}\left[(K-\bar{S})\ln\left(\frac{K-S_N}{K-S_H}\right) - (S_H-S_N)\right] \\ &< -\frac{\beta}{\mu}\left(S_H-S_N\right)\left(\frac{K-\bar{S}}{\sqrt{(K-S_H)(K-S_N)}} - 1\right), \end{aligned} \tag{29}$$

which indicates that if $S_N < \bar{S}$, then we have $N_{23}(1) < 0$, i.e., $\mu_2(1) < 1$. From the above analysis, we have the following result.

Theorem 5 *If $R_0 > 1$, $S_H > 1/R_0$ and $\mu_2(1) < 1$, $J \neq 2p/h_2$, then the $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $q = q^*$.*

Proof Choosing q as the bifurcation parameter, therefore the Poincaré map $\mathcal{P}_M$ can be rewritten as follows

$$\mathcal{P}_M(I_0, q) = g(I(S_H, I_0)).$$

In order to prove Theorem 5, only need to verify the four conditions of Lemma 2.

(i) Let $I(S; S_N, I_0) = I(S, I_0)$, then $\mathcal{P}_M(0, q) = g(I(S_H, 0)) = 0$.

(ii) It follows from Equation (14) and Equation (24) that

$$\frac{\partial \mathcal{P}_M(0,q)}{\partial I_0} = \exp\left(\int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S\right) = \exp(N_2 + N_3) = \mu_2(q). \tag{30}$$

Thus

$$\frac{\partial \mathcal{P}_M(0,q^*)}{\partial I_0} = \mu_2(q^*) = 1.$$

(iii) From Equation (30), then

$$\begin{aligned}
\frac{\partial^2 \mathcal{P}_M(0,q)}{\partial I_0 \partial q} &= \frac{\partial}{\partial q}\left(\frac{\partial \mathcal{P}_M(0,q)}{\partial I_0}\right) \\
&= \exp\left(\int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S\right) \cdot \frac{(\mu+\gamma) - \beta S_N}{\Lambda - \mu S_N} \cdot \frac{\partial S_N}{\partial q} \\
&= \exp(N_{23}(q)) \cdot \frac{(\mu+\gamma) - \beta S_N}{\Lambda - \mu S_N} \cdot \frac{\partial S_N}{\partial q} = \frac{\mathrm{d}\mu_2(q)}{\mathrm{d}q},
\end{aligned}$$

which means that

$$\frac{\partial^2 \mathcal{P}_M(0,q^*)}{\partial I_0 \partial q} = \frac{\mathrm{d}\mu_2(q^*)}{\mathrm{d}q} < 0.$$

(iv) Further, according to Equation (15) then

$$\begin{aligned}
\frac{\partial^2 \mathcal{P}_M(0,q)}{\partial I_0^2} &= -\frac{2p}{h_2} \exp\left(2\int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S\right) \\
&\quad + \exp\left(\int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S\right) \int_{S_N}^{S_H} \frac{\partial^2 R(s, I(s,0))}{\partial I^2} \frac{\partial I(s,0)}{\partial I_0} \mathrm{d}s \\
&= -\frac{2p}{h_2} (\mu_2(q))^2 + \mu_2(q) \int_{S_N}^{S_H} \frac{\partial^2 R(s, I(s,0))}{\partial I^2} \frac{\partial I(s,0)}{\partial I_0} \mathrm{d}s.
\end{aligned}$$

Let

$$b_1(s) = \int_{S_N}^{s} \frac{\beta s - (\mu+\gamma)}{\Lambda - \mu s} \mathrm{d}s, b_2(s) = \frac{\partial I(s,0)}{\partial I_0} = \exp\left(\int_{S_N}^{s} \frac{\beta s - (\mu+\gamma)}{\Lambda - \mu s} \mathrm{d}s\right).$$

It is easy to caculation that $b_2(S_N) = 1$, $b_2(S_H) = \mu_2(q)$ and

$$\frac{\mathrm{d}b_1(s)}{\mathrm{d}s} = \frac{\beta s - (\mu+\gamma)}{\Lambda - \mu s}. \tag{31}$$

Denote

$$h_1(s) = \frac{\partial^2 R(s, I(s,0))}{\partial I^2} = \frac{2\beta S(\beta S - (\mu+\gamma))}{(\Lambda - \mu S)^2}, \quad h_2(s) = \frac{h_1(s)}{b_1'(s)} = \frac{2\beta s}{\Lambda - \mu s}.$$

Then

$$\frac{\mathrm{d}h_2(s)}{\mathrm{d}s} = \frac{2\Lambda\beta}{(\Lambda - \mu s)^2} > 0.$$

Therefore

$$\begin{aligned}\frac{\partial^2 \mathcal{P}_M(0,q^*)}{\partial I_0^2} &= -\frac{2p}{h_2} + \int_{S_{N_{q^*}}}^{S_H} \frac{\partial^2 R(s, I(s,0))}{\partial I^2}\frac{\partial I(s,0)}{\partial I_0}\mathrm{d}s \\ &= -\frac{2p}{h_2} + J,\end{aligned}$$

where

$$\begin{aligned}J\Big|_{q=q^*} &\doteq \int_{S_{N_{q^*}}}^{S_H} \frac{\partial^2 R(s, I(s,0))}{\partial I^2}\frac{\partial I(s,0)}{\partial I_0}\mathrm{d}s \\ &= \int_{S_{N_{q^*}}}^{S_H} h_1(s) b_2(s)\mathrm{d}s \\ &= \int_{S_{N_{q^*}}}^{S_H} h_2(s)\mathrm{d}b_2(s) \\ &= h_2(s)b_2(s)\Big|_{S_{N_{q^*}}}^{S_H} - \int_{S_{N_{q^*}}}^{S_H} b_2(s)\mathrm{d}h_2(s) \\ &= h_2(S_H) - h_2(S_N) - \int_{S_{N_{q^*}}}^{S_H} b_2(s)h_2'(s)\mathrm{d}s \\ &= \int_{S_{N_{q^*}}}^{S_H} (1 - b_2(s))\mathrm{d}h_2(s).\end{aligned}$$

It follows from Equation (31) that we have $b_2(s)$ is monotonically decreasing on interval $[S_N, \bar{S}]$ and monotonically increasing on interval $[\bar{S}, S_H]$. Further, $b_2(S_{N_{q^*}}) = \mu_2(q^*) = 1$. Thus, $b_2(\bar{S}) \le b_2(s) \le 1$ for any $s \in [S_{N_{q^*}}, S_H]$. Since $\mathrm{d}h_2(s)/\mathrm{d}s > 0$, then $h_2(s)$ is monotonically increasing on $[S_{N_{q^*}}, S_H]$. Therefore, $0 < J < (1 - b_2(\bar{S}))(h_2(S_{N_{q^*}}) - h_2(S_H))$.

Summarize the above, if $J \neq 2p/h_2$, then $\partial^2\mathcal{P}_M(0,q^*)/\partial I_0^2 \neq 0$. That is, all four conditions of Lemma (2) are verified, which implies that the $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $q = q^*$(Fig. 2(a)). This completes the proof.

Furthermore, if $J = 2p/h_2$, then $\partial^2\mathcal{P}_M(0,q^*)/\partial I_0^2 = 0$. In order to consider the pitchfork bifurcation, only need to verify the conditions of Lemma (3) as follows

$$\begin{aligned}\frac{\partial^3 \mathcal{P}_M(I_0,q)}{\partial I_0^3} &= g'''(I(S_H, I_0))\left(\frac{\partial I(S_H, I_0)}{\partial I_0}\right)^3 \\ &\quad + 3g''(I(S_H, I_0))\frac{\partial I(S_H, I_0)}{\partial I_0}\frac{\partial^2 I(S_H, I_0)}{\partial I_0^2} \\ &\quad + g'(I(S_H, I_0))\frac{\partial^3 I(S_H, I_0)}{\partial I_0^3}.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial^3 \mathcal{P}_M(0,q^*)}{\partial I_0^3} &= \frac{6p}{h_2^2} - \frac{6p}{h_2}J + \left.\frac{\partial^3 I(S_H,0)}{\partial I_0^3}\right|_{q=q^*} \\ &= \frac{6p}{h_2^2} - \frac{12p^2}{h_2^2} + \left.\frac{\partial^3 I(S_H,0)}{\partial I_0^3}\right|_{q=q^*},\end{aligned}$$

where

$$\begin{aligned}\left.\frac{\partial^3 I(S_H,0)}{\partial I_0^3}\right|_{q=q^*} &= \left(\int_{S_{N_{q^*}}}^{S_H} \frac{\partial^2 R(s,I(s,0))}{\partial I^2}\frac{\partial I(s,0)}{\partial I_0}\mathrm{d}s\right)^2 \\ &+ \frac{\partial}{\partial I_0}\left(\int_{S_{N_{q^*}}}^{S_H} \frac{\partial^2 R(s,I(s,0))}{\partial I^2}\frac{\partial I(s,0)}{\partial I_0}\mathrm{d}s\right) \\ &= J^2 + \int_{S_{N_{q^*}}}^{S_H}\left(\frac{3}{2}h_2(s)h_1(s)b_2^2(s) + h_1(s)\frac{\partial^2 I(s,0)}{\partial I_0^2}\right)\mathrm{d}s \\ &\doteq \frac{4p^2}{h_2^2} + Q.\end{aligned}$$

Thus

$$\frac{\partial^3 \mathcal{P}_M(0,q^*)}{\partial I_0^3} = \frac{6p - 8p^2}{h_2^2} + Q.$$

That is, if $Q \neq (8p^2 - 6p)/h_2^2$ then $\partial^3\mathcal{P}_M(0,q^*)/\partial I_0^3 \neq 0$ which means that conditions of Lemma (2) holds. Therefore, we have the following result.

Corollary 1 *If $R_0 > 1$, $S_H > 1/R_0$, $\mu_2(1) < 1$, $J = 2p/h_2$ and $Q \neq (8p^2 - 6p)/h_2^2$, then the $\mathcal{P}_M(I_0,q)$ undergoes a pitchfork bifurcation at $q = q^*$.*

For $h_2 = 0$, the expression of μ_2 is as follows

$$\mu_2(q) = (1-p)\exp(N_{23}(q)).$$

Then,

$$\frac{\mathrm{d}\mu_2}{\mathrm{d}q} = (1-p)\exp(N_{23}(q))\cdot\frac{\mathrm{d}N_{23}(q)}{\mathrm{d}q}.$$

According to Equation (26) and Equation (27), there is a q such that $\mathrm{d}N_{23}/\mathrm{d}q = 0$. Thus it remains that $\mu_2(q)$ is increasing on the interval $[0,\overline{q})$, and decresing on the interval $(\overline{q},1]$. From Equation (24) that $\mu_2(0) < 1$, and $\mu_2(1) < 1$ due to Equation (29). Therefore, if $\mu_2(\overline{q}) > 1$ that there exist q^* such that $\mu_2(q^*) = 1$ and q^{**} such that $\mu_2(q^{**}) = 1$. That is, the possible bifurcation could occur at q^* and q^{**}(Fig. 2(b)). By calculation we have

$$\begin{aligned}&\frac{\partial\mathcal{P}_M(0,q^*)}{\partial I_0} = \mu_2(q^*) = 1, &&\frac{\partial\mathcal{P}_M(0,q^{**})}{\partial I_0} = \mu_2(q^{**}) = 1,\\ &\frac{\partial^2\mathcal{P}_M(0,q^*)}{\partial I_0\partial q} = \frac{\mathrm{d}\mu_2(q^*)}{\mathrm{d}q} > 0, &&\frac{\partial^2\mathcal{P}_M(0,q^{**})}{\partial I_0\partial q} = \frac{\mathrm{d}\mu_2(q^{**})}{\mathrm{d}q} < 0,\\ &\frac{\partial^2\mathcal{P}_M(0,q^*)}{\partial I_0^2} = J > 0, &&\frac{\partial^2\mathcal{P}_M(0,q^{**})}{\partial I_0^2} = J > 0.\end{aligned}$$

Then, we have the following result:

Theorem 6 *if $h_2 = 0$, $R_0 > 1$, $S_H > 1/R_0$ and $\mu_2(\overline{q}) > 1$, then $\mathcal{P}_M(I_0,q)$ transcritical bifurcation occurs at q^* and q^{**}.*

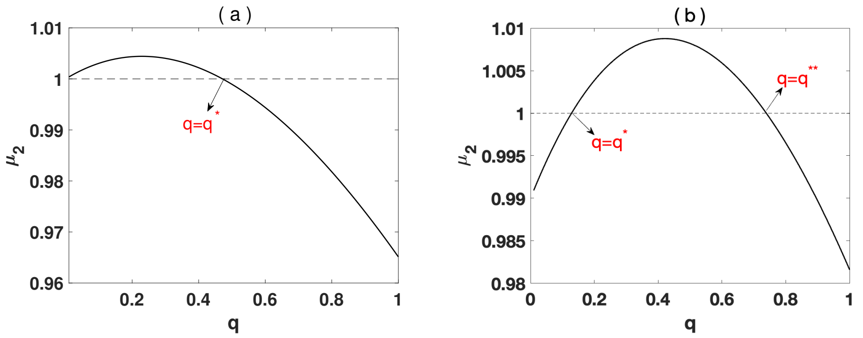

Fig. 2 The functions of μ_2 with respect to q. (a)$S_H = 1.05$; $p = 0.5$; $h_2 = 2$ (b)$S_H = 1.1$; $p = 0.01$; $h_2 = 0$. The other parameter values were fixed as follows: $\Lambda = 0.8$; $\mu = 0.5$; $\beta = 1$; $\alpha = 0.5$; $\gamma = 0.5$; $h_1 = 4$.

5.2 Bifurcations on p

Particularly, when $h = 0$, we can choose p as the bifurcation parameter. So we have

$$\mu_2(p) = (1 - p)\exp(N_2 + N_3).$$

Taking the derivative of $\mu_2(p)$ with respect to p as follow

$$\frac{\mathrm{d}\mu_2(p)}{\mathrm{d}p} = -\exp(N_2 + N_3) < 0,$$

which means that $\mu_2(p)$ is monotonically decreasing on interval $p \in (0, 1)$. It follows from $\mu_2(0) > 1$ and $\mu_2(1) = 0$ that there exists a p^* such that $\mu_2(p^*) = 1$. Actually, from Equation (25), if $S_N > 1/R_0$ then $N_2+N_3 > 0$. that is $\mu_2(0) = \exp(N_2+N_3) > 1$. By employing the same methods as Theorem (5), we get

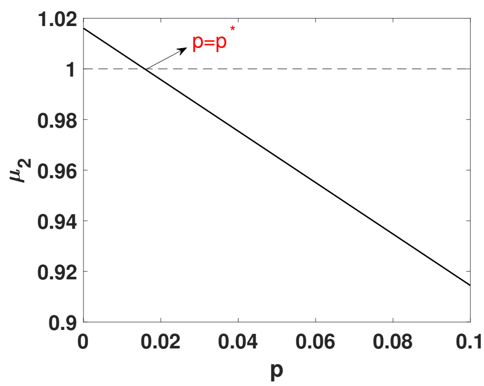

Fig. 3 The functions of μ_2 with respect to p. The other parameter values were fixed as follows: $\Lambda = 0.8; \mu = 0.5; \beta = 1; \alpha = 0.5; \gamma = 0.5; q = 0.6; h1 = 4; h2 = 2; S_H = 1.1$.

$$\frac{\partial \mathcal{P}_M(0,p^*)}{\partial I_0} = \mu_2(p^*) = 1, \frac{\partial^2 \mathcal{P}_M(0,p^*)}{\partial I_0 \partial p} = \frac{\mathrm{d}\mu_2(p^*)}{\mathrm{d}p} > 0, \frac{\partial^2 \mathcal{P}_M(0,p^*)}{\partial I_0^2} = J > 0.$$

Then, we have the following result.

Theorem 7 *If $R_0 > 1$, $h_2 = 0$ and $S_N > 1/R_0$, then the $\mathcal{P}_M(I_0, p)$ undergoes a transcritical bifurcation at $p = p^*$ (Fig. 3).*

5.3 Bifurcations on Λ

When $h_2 > 0$, we choose Λ as a bifurcation parameter, i.e., let $\mu_2(\Lambda) = \exp(N_{23}(\Lambda))$ and

$$N_{23}(\Lambda) \doteq \left(\frac{\mu+\gamma}{\mu} - \frac{\beta\Lambda}{\mu^2}\right) \ln\left(\frac{\Lambda - \mu S_H}{\Lambda - \mu S_N}\right) - \frac{\beta}{\mu}(S_H - S_N)$$

with

$$\lim_{\Lambda \to +\infty} N_{23}(\Lambda) = 0, \qquad \lim_{\Lambda \to \mu S_H} N_{23}(\Lambda) = +\infty. \tag{32}$$

Taking the derivative of $\mu_2(\Lambda)$ with respect to Λ as

$$\frac{\mathrm{d}\mu_2(\Lambda)}{\mathrm{d}\Lambda} = \exp(N_{23}(\Lambda)) \cdot \frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda},$$

where

$$\frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda} = -\frac{\beta}{\mu^2} \ln\left(\frac{\Lambda - \mu S_H}{\Lambda - \mu S_N}\right) + \left(\mu + \gamma - \frac{\beta\Lambda}{\mu}\right) \frac{S_H - S_N}{(\Lambda - \mu S_H)(\Lambda - \mu S_N)}$$

and

$$\lim_{\Lambda \to +\infty} \frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda} = 0.$$

Further, we have

$$\frac{\mathrm{d}^2 N_{23}(\Lambda)}{\mathrm{d}\Lambda^2} = \frac{\beta(S_H - S_N)\left[\Lambda(S_H + S_N - 2\overline{S}) - \mu(2S_H S_N - \overline{S}(S_H + S_N))\right]}{(\Lambda - \mu S_H)^2(\Lambda - \mu S_N)^2}.$$

If $\overline{S} = (S_H + S_N)/2$, then $\overline{S} > 2S_H S_N/(S_H + S_N)$ and

$$\frac{\mathrm{d}^2 N_{23}(\Lambda)}{\mathrm{d}\Lambda^2} = \frac{\beta(S_H - S_N)\left[-\frac{\mu}{S_H + S_N}\left(\frac{2S_H S_N}{S_H + S_N} - \overline{S}\right)\right]}{(\Lambda - \mu S_H)^2(\Lambda - \mu S_N)^2} > 0,$$

which indicates that $\frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda}$ is monotonically increasing on the interval $(\mu S_H, +\infty)$. Thus, $\frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda} < 0$ on the interval $(\mu S_H, +\infty)$, i.e., $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$. Moreover, from Equation (32) that $N_{23}(\Lambda) > 0$ for all $\Lambda \in (\mu S_H, +\infty)$, which indicates that the DFPS is unstable and no bifurcation occurs.

If $\overline{S} \neq (S_H + S_N)/2$, then solving

$$\frac{\mathrm{d}^2 N_{23}(\Lambda)}{\mathrm{d}\Lambda^2} = 0,$$

one has

$$\bar{\Lambda} = \frac{\mu\left(\frac{2S_H S_N}{S_H+S_N} - \overline{S}\right)}{\left(\frac{S_H+S_N}{2} - \overline{S}\right)\frac{2}{S_H+S_N}}.$$

(i) If $\overline{S} < 2S_H S_N/(S_H+S_N) < (S_H+S_N)/2$, then $\bar{\Lambda} > 0$. Moreover, if $\bar{\Lambda} > \mu S_H$, then we have $\mathrm{d}^2 N_{23}(\Lambda)/\mathrm{d}\Lambda^2 < 0$ on the interval $(\mu S_H, \bar{\Lambda})$ and $\mathrm{d}^2 N_{23}(\Lambda)/\mathrm{d}\Lambda^2 > 0$ on the interval $(\bar{\Lambda}, +\infty)$, i.e., $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda$ monotonically decreasing on the interval $(\mu S_H, \bar{\Lambda})$ and monotonically increasing on the interval $(\bar{\Lambda}, +\infty)$. It follows from $\lim_{\Lambda\to+\infty} \frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda} = 0$ that $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda < 0$ for all $\Lambda \in (\bar{\Lambda}, +\infty)$. In fact, $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda < 0$ for all $\Lambda \in (\mu S_H, \bar{\Lambda})$. Otherwise, there exists a $\hat{\Lambda} \in (\mu S_H, \bar{\Lambda})$ such that $\mathrm{d}N_{23}(\hat{\Lambda})/\mathrm{d}\Lambda = 0$, then $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda > 0$ for all $\Lambda \in (\mu S_H, \hat{\Lambda})$ and $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda < 0$ for all $\Lambda \in (\hat{\Lambda}, +\infty)$. That is, $N_{23}(\Lambda)$ is monotonically increasing on the interval $(\mu S_H, \hat{\Lambda})$ and monotonically decreasing on the interval $(\hat{\Lambda}, +\infty)$, which contradicts $\lim_{\Lambda\to+\infty} N_{23}(\Lambda) = 0$. Thus, $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$ and $N_{23}(\Lambda) > 0$ for all $\Lambda \in (\mu S_H, +\infty)$, which means no bifurcation occurs with respect to Λ. When $\bar{\Lambda} < \mu S_H$, then $\mathrm{d}^2 N_{23}(\Lambda)/\mathrm{d}\Lambda^2 > 0$ for all $\Lambda \in (\mu S_H, +\infty)$ and then $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$. In summary, $\mu_2(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$(Fig. 4(a)). According to Equation (32), we have $N_{23}(\Lambda) > 0$ for all $\Lambda \in (\mu S_H, +\infty)$. All these results confirm that the bifurcation does not occur for this case.

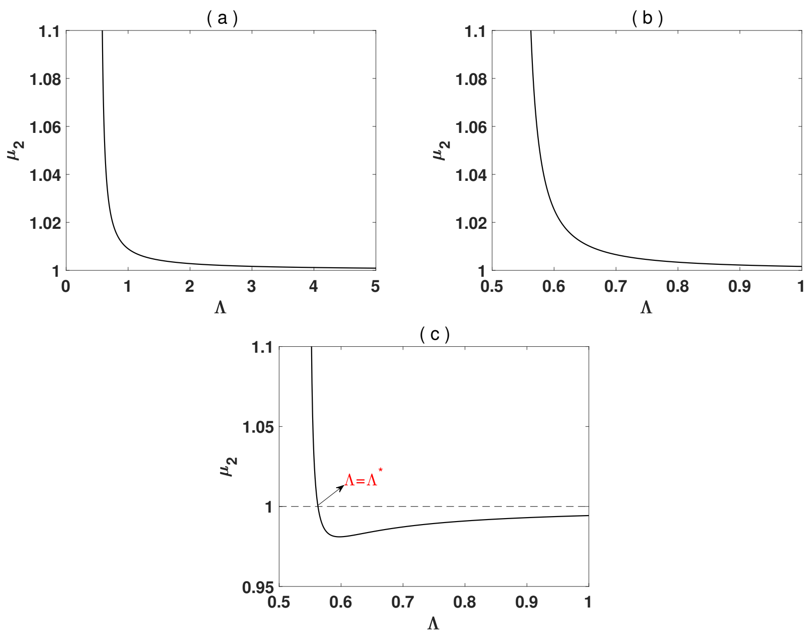

Fig. 4 The functions of μ_2 with respect to Λ. (a) $\gamma = 0.5$; (b) $\gamma = 0.525$; (c) $\gamma = 0.55$. The other parameter values were fixed as follows: $\Lambda = 0.8; \mu = 0.5; \beta = 1; \alpha = 0.5; q = 0.6; h1 = 4; h2 = 2; S_H = 1.1$.

(ii) If $2S_HS_N/(S_H+S_N) < \overline{S} < (S_H+S_N)/2$, then $\bar{\Lambda} < 0$ and $\mathrm{d}^2N_{23}(\Lambda)/\mathrm{d}\Lambda^2 > 0$ for all $\Lambda \in (\mu S_H, +\infty)$. Using the same methods, we have $N_{23}(\Lambda) > 0$ for all $\Lambda \in (\mu S_H, +\infty)$, which indicates that the bifurcation does not occur with respect to Λ(Fig. 4(b)).

(iii) If $2S_HS_N/(S_H + S_N) < (S_H + S_N)/2 < \overline{S}$, then $\bar{\Lambda} > 0$. Actually, $\bar{\Lambda} > \mu S_H$. Otherwise, if $\bar{\Lambda} < \mu S_H$ then $\mathrm{d}^2N_{23}(\Lambda)/\mathrm{d}\Lambda^2 < 0$ for all $\Lambda \in (\mu S_H, +\infty)$ and $\mathrm{d}N_{23}(\hat{\Lambda})/\mathrm{d}\Lambda$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$. And $\lim_{\Lambda\to+\infty} \frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda} = 0$, thus $\mathrm{d}N_{23}(\hat{\Lambda})/\mathrm{d}\Lambda > 0$ for all $\Lambda \in (\mu S_H, +\infty)$ which means that $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$. This contradicts $\lim_{\Lambda\to+\infty} N_{23}(\Lambda) = 0$, and $\lim_{\Lambda\to\mu S_H} N_{23}(\Lambda) = +\infty$. Therefore, $\mathrm{d}^2N_{23}(\Lambda)/\mathrm{d}\Lambda^2 > 0$ on the interval $(\mu S_H, \bar{\Lambda})$ and $\mathrm{d}^2N_{23}(\Lambda)/\mathrm{d}\Lambda^2 < 0$ on the interval $(\bar{\Lambda}, +\infty)$. It is easy to see that there is a $\hat{\Lambda} \in (\mu S_H, \bar{\Lambda})$ such that $\mathrm{d}N_{23}(\hat{\Lambda})/\mathrm{d}\Lambda = 0$. Then, $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda < 0$ for all $\Lambda \in (\mu S_H, \hat{\Lambda})$ and $\mathrm{d}N_{23}(\Lambda)/\mathrm{d}\Lambda > 0$ for all $\Lambda \in (\hat{\Lambda}, +\infty)$. From Equation (32) that there is a unique $\Lambda^* \in (\mu S_H, \hat{\Lambda})$ such that $N_{23}(\Lambda^*) = 0$, i.e., $\mu_2(\Lambda^*) = 1$. That is, the possible bifurcation will occur at $\Lambda = \Lambda^*$(Fig. 4(c)). By employing the similar methods as Theorem (5), we have

$$\frac{\partial \mathcal{P}_M(0,\Lambda^*)}{\partial I_0} = \mu_2(\Lambda^*) = 1, \quad \frac{\partial^2 \mathcal{P}_M(0,\Lambda^*)}{\partial I_0 \partial \Lambda} = \frac{\mathrm{d}\mu_2(\Lambda^*)}{\mathrm{d}\Lambda} < 0,$$
$$\frac{\partial^2 \mathcal{P}_M(0,\Lambda^*)}{\partial I_0^2} = -\frac{2p}{h_2} + J, \quad \frac{\partial^3 \mathcal{P}_M(0,\Lambda^*)}{\partial I_0^3} = \frac{6p - 8p^2}{h_2^2} + Q.$$

Theorem 8 *If $R_0 > 1$, $2S_HS_N/(S_H + S_N) < (S_H + S_N)/2 < \overline{S}$ and $J \neq 2p/h_2$, then the $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $\Lambda = \Lambda^*$.*

Corollary 2 *If $R_0 > 1$, $2S_HS_N/(S_H + S_N) < (S_H + S_N)/2 < \overline{S}$, $J = 2p/h_2$ and $Q \neq (8p^2 - 6p)/h_2^2$, then the $\mathcal{P}_M(I_0, q)$ undergoes a pitchfork bifurcation at $\Lambda = \Lambda^*$.*

For the special case $h_2 = 0$, we have $\mu_2(\Lambda) = (1 - p)\exp(N_{23}(\Lambda))$. Then

$$\frac{\mathrm{d}\mu_2(\Lambda)}{\mathrm{d}\Lambda} = (1 - p)\exp(N_{23}(\Lambda)) \cdot \frac{\mathrm{d}N_{23}(\Lambda)}{\mathrm{d}\Lambda}.$$

In the cases (i) and (ii), $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, +\infty)$, in the case (iii), $N_{23}(\Lambda)$ is monotonically decreasing on the interval $(\mu S_H, \hat{\Lambda})$ and monotonically increasing on the interval $(\hat{\Lambda}, +\infty)$. All of these confirm that there exists a unique $\Lambda^* \in (\mu S_H, +\infty)$ such that $N_{23}(\Lambda^*) = \ln(1/(1 - p))$ and $\mathrm{d}N_{23}(\Lambda^*)/\mathrm{d}\Lambda < 0$, i.e., $\mu_2(\Lambda^*) = 1$ and $\mathrm{d}\mu_2(\Lambda^*)/\mathrm{d}\Lambda < 0$. Furthermore, we have

$$\frac{\partial \mathcal{P}_M(0,\Lambda^*)}{\partial I_0} = \mu_2(\Lambda^*) = 1, \frac{\partial^2 \mathcal{P}_M(0,\Lambda^*)}{\partial I_0 \partial \Lambda} = \frac{\mathrm{d}\mu_2(\Lambda^*)}{\mathrm{d}\Lambda} < 0, \frac{\partial^2 \mathcal{P}_M(0,\Lambda^*)}{\partial I_0^2} = J > 0.$$

Then we have the follow result.

Theorem 9 *If $R_0 > 1$, $h_2 = 0$ and $S_N > 1/R_0$, then the $\mathcal{P}_M(I_0, \Lambda)$ undergoes a transcritical bifurcation at $\Lambda = \Lambda^*$.*

5.4 Bifurcations on S_H

In this subdection, we choose S_H as a bifurcation parameter when $h_2 > 0$. Thus $\mu_2(S_H) = \exp(N_{23}(S_H))$ and

$$N_{23}(S_H) \doteq \left(\frac{\mu+\gamma}{\mu} - \frac{\beta K}{\mu}\right) \ln\left(\frac{K - S_H}{K - S_N}\right) - \frac{\beta}{\mu}\left(S_H - S_N\right),$$

where

$$S_N = (1 - \frac{qS_H}{S_H + h_1})S_H,$$

with

$$\begin{aligned}\frac{\mathrm{d}S_N}{\mathrm{d}S_H} &= 1 - \frac{2qS_H(S_H + h_1) - qS_H^2}{(S_H + h_1)^2} \\ &= \frac{(1-q)S_H^2 + 2h_1 S_H(1-q) + h_1^2}{(S_H + h_1)^2} > 0.\end{aligned}$$

Calculating the derivative of $\mu_2(S_H)$ with respect to S_H gives

$$\frac{\mathrm{d}\mu_2(S_H)}{\mathrm{d}S_H} = \exp(N_{23}(S_H)) \cdot \frac{\mathrm{d}N_{23}(S_H)}{\mathrm{d}S_H},$$

and

$$\begin{aligned}\frac{\mathrm{d}N_{23}(S_H)}{\mathrm{d}S_H} &= \frac{-\left(\mu+\gamma-\beta K\right)}{\mu(K - S_H)} + \frac{-\left(\mu+\gamma-\beta K\right)}{\mu(K - S_N)} \cdot \frac{\mathrm{d}S_N}{\mathrm{d}S_H} - \frac{\beta}{\mu} + \frac{\beta}{\mu} \cdot \frac{\mathrm{d}S_N}{\mathrm{d}S_H} \\ &= \frac{\beta(S_H - \overline{S})}{\mu(K - S_H)} - \frac{\beta(S_N - \overline{S})}{\mu(K - S_N)} \cdot \frac{\mathrm{d}S_N}{\mathrm{d}S_H}.\end{aligned}$$

Let $f(x) \doteq \beta(x - \overline{S})/\mu(K - x)$, then $f'(x) = \beta(K - \overline{S})/\mu(K - x)^2 > 0$. Since $S_N < S_H$ then

$$\begin{aligned}\frac{\mathrm{d}N_{23}(S_H)}{\mathrm{d}S_H} &= f(S_H) - f(S_N) \cdot \frac{\mathrm{d}S_N}{\mathrm{d}S_H} \\ &> f(S_N)\left(1 - \frac{\mathrm{d}S_N}{\mathrm{d}S_H}\right) = f(S_N)\frac{2qS_H h_1 + qS_H^2}{(S_H + h_1)^2},\end{aligned}$$

if $S_N > \overline{S}$, then $\mathrm{d}N_{23}(S_H)/\mathrm{d}S_H > 0$, if $S_N < \overline{S} < S_H$, then $\mathrm{d}N_{23}(S_H)/\mathrm{d}S_H > 0$, which indicates that as long as $\overline{S} < S_H$ has $N_{23}(S_H)$ is monotonically increasing on the interval $(\overline{S}, K)$. Moreover, it follows from Equation (24) that

$$\begin{aligned}\lim_{S_H \to \overline{S}} N_{23}(S_H) &= \lim_{S_H \to \overline{S}} \int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S < 0, \\ \lim_{S_H \to +\infty} N_{23}(S_H) &= \lim_{S_H \to +\infty} \int_{S_N}^{S_H} \frac{\beta S - (\mu+\gamma)}{\Lambda - \mu S} \mathrm{d}S = +\infty.\end{aligned}$$

All these results show that $N_{23}(S_H)$ is monotonically increasing on the interval $(\overline{S}, K)$, which means there is a unque $S_H^* \in (\overline{S}, K)$ such that $N_{23}(S_H^*) = 0$ and $\mathrm{d}N_{23}(S_H^*)/\mathrm{d}S_H > 0$. Furthermore, we have

$$\begin{aligned}\frac{\partial \mathcal{P}_M(0, S_H^*)}{\partial I_0} &= \mu_2(S_H^*) = 1, \quad & \frac{\partial^2 \mathcal{P}_M(0, S_H^*)}{\partial I_0 \partial S_H} &= \frac{\mathrm{d}\mu_2(S_H^*)}{\mathrm{d}S_H} > 0, \\ \frac{\partial^2 \mathcal{P}_M(0, S_H^*)}{\partial I_0^2} &= -\frac{2p}{h_2} + J, \quad & \frac{\partial^3 \mathcal{P}_M(0, S_H^*)}{\partial I_0^3} &= \frac{6p - 8p^2}{h_2^2} + Q.\end{aligned}$$

Then we have following results

Theorem 10 *If $R_0 > 1$, $S_H > 1/R_0$, $S_H > \overline{S}$ and $J \neq 2p/h_2$, then the $\mathcal{P}_M(I_0, S_H)$ undergoes a transcritical bifurcation at $S_H = S_H^*$(Fig. 5).*

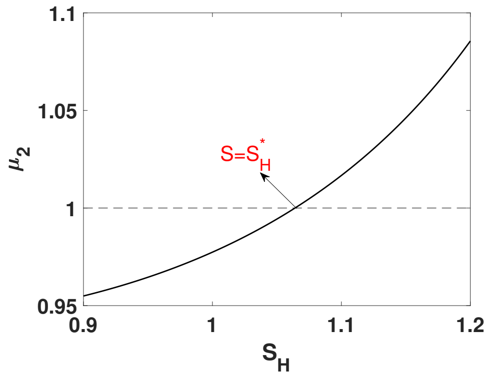

Fig. 5 The functions of μ_2 with respect to Λ. The other parameter values were fixed as follows: $\Lambda = 0.8; \mu = 0.5; \beta = 1; \alpha = 0.5; \gamma = 0.5; q = 0.6; h_1 = 4; h_2 = 2;$.

Corollary 3 *If $R_0 > 1$, $S_H > 1/R_0$, $S_H > \overline{S}$, $J = 2p/h_2$ and $Q \neq (8p^2 - 6p)/h_2^2$, then the $\mathcal{P}_M(I_0, S_H)$ undergoes a pitchfork bifurcation at $S_H = S_H^*$.*

When $h_2 = 0$, then $\mu_2(S_H) = (1-p)\exp(N_{23}(S_H))$. It follows from the monotonicity of fuction $N_{23}(S_H)$ that $\mathrm{d}\mu_2(S_H)/\mathrm{d}S_H < 0$ for all $S_H \in (1/R_0, \overline{S})$ and $\mathrm{d}\mu_2(S_H)/\mathrm{d}S_H < 0$ for all $S_H \in (\overline{S}, K)$, which shows that there exists a unque S_H^* such that $N_{23}(S_H^*) = \ln(1/(1-p))$, i.e., $\mu_2(S_H^*) = 1$. By caculation, we get

$$\frac{\partial \mathcal{P}_M(0, S_H^*)}{\partial I_0} = \mu_2(S_H^*) = 1,$$

$$\frac{\partial^2 \mathcal{P}_M(0, S_H^*)}{\partial I_0 \partial S_H} = \frac{\mathrm{d}\mu_2(S_H^*)}{\mathrm{d}\Lambda} > 0, \frac{\partial^2 \mathcal{P}_M(0, S_H^*)}{\partial I_0^2} = J > 0.$$

Theorem 11 *If $R_0 > 1$, $h_2 = 0$ and $S_N > 1/R_0$, then the $\mathcal{P}_M(I_0, S_H)$ undergoes a transcritical bifurcation at $S_H = S_H^*$.*

6 Discussion

With the progress of information technology, the changes in individual behaviors are greatly affected by media coverage, which has a strong impact on the spread of infectious diseases. Therefore, the subject of the impact of media coverage on infectious diseases has attracted a great deal of academic's attention [16–23]. Meanwhile, vaccination and isolation of patients are important means to prevent disease outbreak. In fact, the continuous and fixed-time vaccination control models are widely studied [29–33]. A more reasonable assumption is to consider the size of susceptible individuals or patients as the threshold to decide whether to vaccinate susceptible individuals and isolate patients. Therefore, we propose a nonlinear pulse SIR model with media coverage to describe the vaccination and isolation measures determined by the size of susceptible individuals. The complex dynamic behaviors of the model are studied and its biological significance is discussed.

Firstly, the dynamical behaviours of the model (3) without impulse control are investigated in Section 3. The basic reproduction number is defined as $R_0 = \beta K/(\mu+\gamma)$ and when $R_0 < 1$, there exists a disease-free equilibrium $E^0 = (K, 0)$, which is globally asymptotically stable(Fig. 1(a)). If $R_0 > 1$, there exist a endemic equilibrium E^* by using the *Lambert W* function, which is globally asymptotically stable(Fig. 1(b)). Then, we define the Poincaré map $\mathcal{P}_M$ of model (4) and investigate its properties. The result indicates that the Poincaré map $\mathcal{P}_M$ has no positive fixed point when $R_0 < 1$. Moreover, the existence and stability of DFPS of model (4) are studied in Section 4.2. The results show that the DFPS are always orbitally asymptotically stable whether $R_0 < 1$ or $R_0 > 1$ and $S_H < 1/R_0$. However, if $R_0 > 1$ and $S_H > 1/R_0$, by utilizing the bifurcation theory and Poincaré map we analyze the transcritical and pitchfork bifurcations near the DFPS with respect to some key parameters such as the maximal vaccination rate q, the maximal treatment(or isolation) rate p, the constant recruitment rate Λ and the threshold of susceptible individuals S_H. When fixing other parameters and changing one of the key parameters, the DFPS of the model (4) may undergo transcritical bifurcation or pitchfork bifurcation as follows:

- If $\mu_2(1) < 1$, $J \neq 2p/h_2$, then the $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $q = q^*$; for $h_2 = 0$, if $\mu_2(\overline{q}) > 1$, then $\mathcal{P}_M(I_0, q)$ transcritical bifurcation occurs at q^* and q^{**}. Further, if $J = 2p/h_2$ and $Q \neq (8p^2 - 6p)/h_2^2$, then the $\mathcal{P}_M(I_0, q)$ undergoes a pitchfork bifurcation at $q = q^*$.
- If $h_2 = 0$, we are able to chose the maximal vaccination rate q as parameter, then the $\mathcal{P}_M(I_0, p)$ undergoes a transcritical bifurcation at $p = p^*$.
- If $2S_HS_N/(S_H + S_N) < (S_H + S_N)/2 < \overline{S}$, $J \neq 2p/h_2$ or $h_2 = 0$, then the $\mathcal{P}_M(I_0, q)$ undergoes a transcritical bifurcation at $\Lambda = \Lambda^*$.
- If $S_H > \overline{S}$, $J \neq 2p/h_2$ or $h_2 = 0$, then the $\mathcal{P}_M(I_0, S_H)$ undergoes a transcritical bifurcation at $S_H = S_H^*$. Further, if $J = 2p/h_2$ and $Q \neq (8p^2 - 6p)/h_2^2$, then the $\mathcal{P}_M(I_0, S_H)$ undergoes a pitchfork bifurcation at $S_H = S_H^*$.

It is worth noting that the DFPS is orbital asymptotically stable when $\mu_2 < 1$ and unstable when $\mu_2 > 1$. Therefore, μ_2 can actually be regarded as a control reproduction number for model (4). And q represents the maximum vaccination rate and reflects the extent to which the susceptible population is protected. Thus, μ_2 is considered as a function of q. Then, plotting the changing curve of the μ_2 with respect to q on the interval $[0, 1]$ in Fig. 2. The other parameters were fixed as $\Lambda = 0.8$; $\mu = 0.5$; $\beta = 1$; $\alpha = 0.5$; $\gamma = 0.5$; $h_1 = 4$. Since the form of μ_2 will be different in cases of $h_2 > 0$ and $h_2 = 0$, we fix the parameters $S_H = 1.05$, $h_2 = 2 > 0$ in Fig. 2(a) and $S_H = 1.05$, $p = 0.01$, $h_2 = 0$ in Fig. 2(b). It is easy to calculate that $R_0 = \beta K/(\mu + \gamma) = 1.6 > 1$ and $S_H > 1/R_0$. Further there is that $\mu_2(0) > 1$ and $\mu_2(1) < 1$. As we analyzed in Section 5.1, there is a unique $\bar{q}$ such that $\mathrm{d}\mu_2/\mathrm{d}q = 0$, which shows that $\mu_2(q)$ is monotonically increasing on interval $(0, \bar{q})$ and monotonically decreasing on interval $(\bar{q}, 1)$(Fig. 2(a)). Therefore, there exists a unique q^* such that $\mu_2(q^*) = 1$, implying that the value of μ_2 is constantly greater than 1 on the interval $(0, q^*)$ and less than 1 on the interval $(q^*, 1)$ as

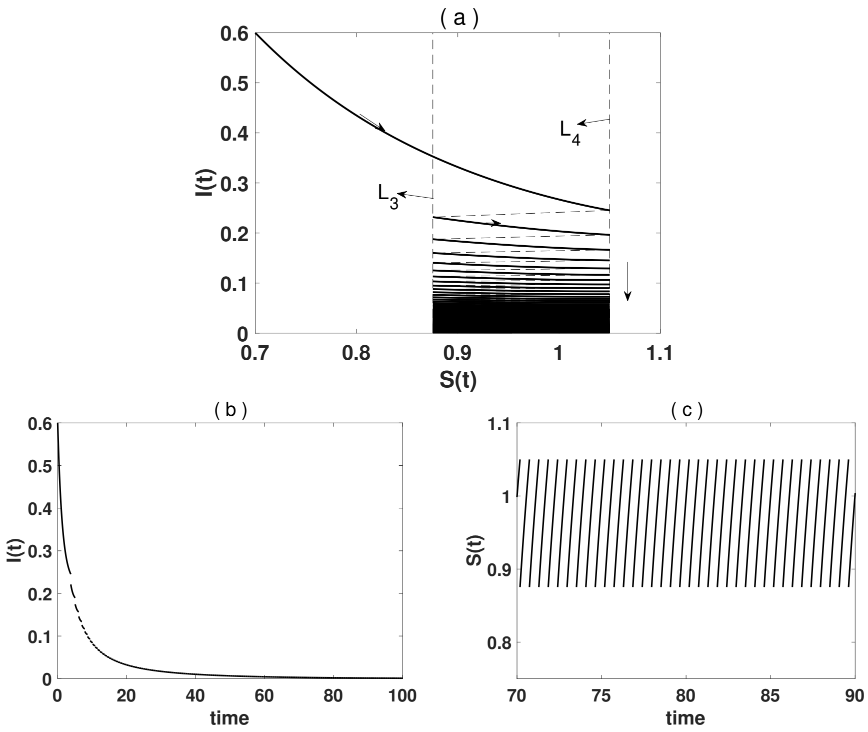

Fig. 6 (a)Phase portrait for stable DFPS; (b) and (c) Time series for stable DFPS. The other parameter values are as follows: $\Lambda = 0.8$; $\mu = 0.5$; $\beta = 1$; $\alpha = 0.5$; $\gamma = 0.5$; $q = 0.8$; $h_1 = 4$; $h_2 = 2$; $p = 0.5$ $S_H = 1.05$.

shown in Fig. 2(a). That is, the DFPS of model (4) is stable for $q \in (0, q^*)$ and unstable for $q \in (q^*, 1)$, which indicates that the transcritical bifurcation will occur at $q = q^*$ as shown in Theorem 5. In Fig. 2(b), we plot the function μ_2 with respect to q for the special case $h_2 = 0$. Compared with the case of $h_2 > 0$, the form of μ_2 has changed but the monotonicity remains the same. There exist two points of bifurcation q^* and q^{**}, which shows the DFPS are stable when the value of q is less than q^* or q is greater than q^{**} unstable between q^* and q^{**}. Moreover, we fix $q = 0.8 > q^*$, while the other parameters are the same as in Fig. 2(a). So in Fig. 6(a), we plot the phase portrait of the stable DFPS and the time series diagrams of $S(t)$ and $I(t)$. This is consistent with the conclusion of Theorem 5. We then reduce the value of q making it less than q^* and fix it to $q = 0.3$, holding the other parameters constant. Then, there generated an unstable internal periodic solution(Fig. 7).

In particularly, choosing the maximal treatment(or isolation) rate p as the bifurcation parameter in the case of $h_2 = 0$. As shown in Fig. 3, μ_2 is a linear function with respect to p and the DFPS undergoes a transcritical bifurcattion at $p = p^*$. Three cases exist when the constant recruitment rate Λ is chosen as the bifurcation parameter. In cases (i) and (ii), $\mu_2(\Lambda)$ is monotonically descending and eventually does not reach 1(Fig. 4(a) and Fig. 4(b)). In case (iii), $\mu_2(\Lambda)$ is

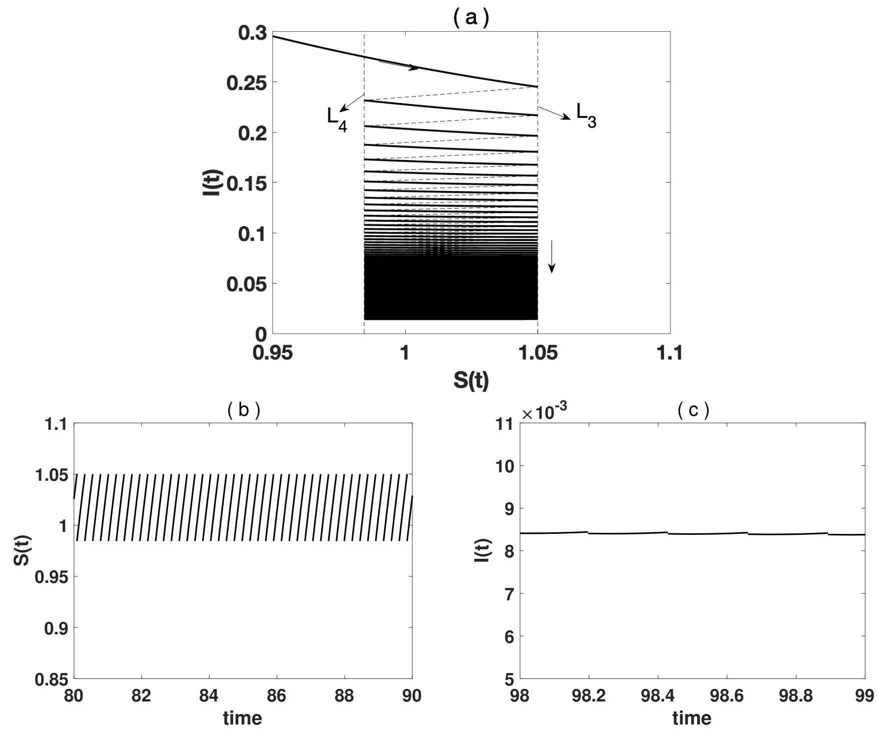

Fig. 7 (a)Phase portrait for unstable DFPS; (b) and (c) Time series for unstable DFPS. The other parameter values are as follows: $\Lambda = 0.8$; $\mu = 0.5$; $\beta = 1$; $\alpha = 0.5$; $\gamma = 0.5$; $q = 0.3$; $h_1 = 4$; $h_2 = 2$; $p = 0.5$ $S_H = 1.05$.

monotonically decreasing on the interval $(\mu S_H, \Lambda^*)$ and monotonically increasing on the interval $(\Lambda^*, +\infty)$(Fig. 4(c)). This means the transcritical bifurcation only occurs in case (iii).

Further, even if $R_0 > 1$, the DFPS is still stable when $S_H < 1/R_0$ as shown in Theorem 4. This indicates that by choosing the suitable threshold of susceptible individuals S_H, even with $R_0 > 1$ in Model 4, the state-dependent pulse strategy is still effective in preventing the epidemic of infectious diseases and eventually eradicate infectious diseases. In addition, choosing the threshold of susceptible individuals S_H as the bifurcation parameter. By changing S_H in the interval $(\overline{S}, K)$ and the value of other parameters is fixed. Plotting the curve of μ_2 with respect to S_H in Fig. 5. It follows from Fig. 5 that there is a critical value S_H^* such that $\mu_2(S_H^*) = 1$, implying that the DFPS of model (4) changes from orbitally asymptotically stable to unstable as S_H crosses S_H^* from left to right.

Biologically, the bifurcation parameters of the maximal vaccination rate q, the maximal treatment(or isolation) rate p, the constant recruitment rate Λ and the threshold of susceptible individuals S_H play essential roles for the control reproduction number μ_2. The results suggest that q should be identified as high as possible to eradicate the disease(Fig. 2(a)). When $h_2 = 0$, in which there is no saturation, such as in the pre-infectious period, the two feasible ways to prevent disease epidemics are to maintain high q and low p or low q and high p(Fig. 2(b) and Fig. 3). The recovery rate γ play a significant effect on the functions of μ_2 with respect to Λ when plotting the bifurcation diagrams with Λ. Fig. 4 shows that the DFPS is stable at high recovery rates. In other words, a high recovery rate benefits the disease's eradication. The threshold of susceptible individuals S_H is very important for implementing a state-dependent pulse strategy and should be lower than S_H^* when setting the threshold for the model.

In this paper, we propose a nonlinear pulse state-dependent SIR model with media coverage to describe implement control measures determined by the size of susceptible individuals. The innovation of this paper are as follows: (1) Compared to [15], we investigated the dynamics of the SIR model with media coverage using state-dependent pulses instead of the filippov system. (2) Considering nonlinear pulse to describe the saturation phenomenon due to resource constraints which is different from [37]. (3) Our main focus is to discuss the transcritical bifurcation and pitchfork bifurcation of the proposed model.

All these results enrich and improve the study of infectious diseases models. In this paper, the threshold condition S_H is defined as a straight line. In the future, we will further study the dynamic behaviors of the model, which considering the threshold condition includes both the susceptible population and the growth rate of susceptible individuals.

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Conflict of interest

The authors declare that they have no conflict of interest.

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