Modelling optimal control of air pollution to reduce respiratory diseases

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Abstract

Respiratory diseases caused by inhalation of air pollutants are affected by seasonal changes and mitigated by air pollution control, resulting in complex dynamics. In order to investigate the effects of various factors such as random noise and air pollution control on respiratory diseases, we developed deterministic and stochastic two-dimensional coupled SIS models with multiple control measures. The proposed models and parameter estimation methods, including determinations of unknown parameter values, were used to fit the Air Quality Index (AQI) data for Xi'an city in recent 10 years. The existences of the optimal solutions for the deterministic and stochastic models were analyzed theoretically and provided to compare the parameter fitting solutions with the optimal solutions, and give theoretical support for seeking a more reasonable air pollution optimization prevention and control scheme. To show this, we conducted numerical simulations of the optimal control solution and state evolution trajectories under different weight coefficient ratios and control objectives. The results show that the stochastic optimal control problem is more consistent with the practical scenario. We also formulate the optimal control problem assuming that the control variable depends on the concentration of air pollutants. The optimal control solution reflected the periodic variation of the air pollution con-

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trol strategy well. A comparison of cost values for different combinations of the three control measures illustrated that air pollution reduction is the most effective control measure.

Keywords: Air pollution; Respiratory disease; Optimal control; Stochastic model; Data validation

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1. Introduction

Severe air pollution issues often accompany rapid economic growth in urban conurbations [1, 2, 3]. Exposure to high concentrations of pollutants in the air such as sulphur dioxides (SO2), nitrogen oxides (NOx), and fine particulates can diminish lung function and cause negative respiratory impacts [4, 5, 6]. The health hazards of air pollution have led to increased medical expenditures and the loss of labour productivity, resulting in substantial social costs [7, 8]. Effects on human health associated with air pollution constitute a major proportion of such social costs [9]. Many efforts have been undertaken to control air pollution in China [10] where source of energy and climate policies are leading to significant changes. The air pollution prevention and control strategies include monitoring various data indicators related to air quality, energy-saving, and reduction policies [11, 12, 13, 14]. The most direct and effective air pollution control measure is to reduce the emission of pollutants. This is reflected in the pollution-related industry closures, restricted use of motor vehicles, or the promotion of new cleaner energy techniques [15, 16]. However, a primary goal of air pollution control is to protect human health and promote sustainable development. The management of air pollution requires careful consideration of various measures to develop the most cost-effective actions.

Epidemic models have been widely used to explored transmission dynamics and the effects of control interventions, and have provided feasible control intervention strategies. Similarly, optimal control models have been widely used to identify effective strategies for minimizing the economic impacts of infectious diseases [17, 18, 19, 20, 21, 22, 23, 24]. We can apply optimal control theory to evaluate the total number of infected individuals and the costs of medical treatments [25, 26, 27]. There are usually two approaches in these studies. One is to reduce the disease's infection rate by taking preventive measures, whereby researches focus on investigating behaviour changes, vaccines, isolation, and other measures [28, 29, 30]. The other is to reduce the number of infections through treatment and many studies have been sought to devise the most effective strategies in relation to the available resources [31, 32, 33]. In the study [34], Wang et al. determined the optimal vaccination strategy for a time-varying SEIR (Susceptible, Exposed, Infected, and Recovered) epidemic model with seasonally varying coefficients. Their analytical and numerical results imply that taking time-varying factors into account is more reasonable and reliable for a control strategy. Based on a simple SIR model, Bolzoni et al. investigated a multi-objective optimal control problem by minimizing an epidemic's size and duration through isolation or vaccination. They also considered limitations on the total resources available for controlling the epidemic. They discussed optimal solutions under different conditions [35]. There have also been some studies of the optimal control problem using stochastic differential equation models [36, 37, 38, 39, 40, 41, 42]. Gani et al. [43] conducted optimal control analysis by considering the effects of media awareness programmes and the treatment of infectives. They obtained numerical results from both deterministic and s-tochastic differential equation models. The air pollution problem has also been studied using optimal control theory of deterministic and stochastic models. For example, Zhu and Zeng have established a partial differential forecast system and discussed the issue of optimal air pollution control based on the weather forecasts [44]. The optimal control solution obtained by models is usually for idealized situations, lacking comparison with actual data. The challenges presented by optimal control problems is how to combine the results of theoretical analysis with complex actual situations to give a more practical and reliable control strategy.

Respiratory disease infections depend on air conditions and environmental changes, showing seasonal characteristics. The main purpose of the present paper is to develop mathematical models with interactions between multiple control measures, and to explore the optimal control strategy according to the actual data. To investigate the control strategies for the respiratory disease caused by air pollution, we will consider a variety of measures, including indirect control measures for air pollution and direct control measures for disease treatment, based on a coupled epidemic model studied in [45]. The two-dimensional deterministic model is

$$\begin{cases} \frac{dI(t)}{dt} = \beta F(t)\frac{(N-I(t))I(t)}{N} - \gamma I(t), \\ \frac{dF(t)}{dt} = c - \mu(t)F(t). \end{cases} \tag{1}$$

The variable $I(t)$ in the model represents the number of infected cases and the variable $F(t)$ measures the air quality. The dynamics of disease transmission are described by the simple SIS model with the assumption that the total population is constant N. γ is the recovery rate for infected individuals. The disease transmission rate depends on the level of air pollution. A simple and reasonable assumption is that the infection rate is linearly related to the concentration of air pollutants. The clearance rate of air pollutants is assumed to be a seasonally time-varying function, i.e. $\mu(t) = \mu_0 + \mu_1 \sin(\omega t + \phi_0)$. Our work focuses on analyzing the practical control measures and then mathematically incorporating them into the model. The problem is stated as determining the optimal strategy for the set of control measures. The multiple control measures discussed in this study include reducing the inflow during air pollution control, decreasing the infection rate, etc. The parameters in the model will be estimated based on the air quality index (AQI) of the city of Xi'an and associated data on the numbers of cases of influenza-like illnesses (ILI) for the period 15 November 2010 to 14 November 2016 described by He et al. [45]. Theoretically, we will develop an analytical method for studying both the deterministic and stochastic models and compare the optimal control solutions of the two models. The remaining part of the paper is organized as follows: the formulation and analysis of the coupled deterministic model with multiple control measures are given in Section 2. The corresponding stochastic model is addressed in Section 3. Section 4 presents the data fitting results and the numerical simulations under different scenarios including several special cases and different control measure combinations. The discussion and conclusions follow in Section 5.

2. The deterministic coupled system with multiple control measures

2.1. Model formulation

To investigate the optimal prevention and treatment strategies for the control of air pollution and related respiratory diseases, we propose an optimal control problem for the deterministic model (1). The model includes relevant control variables governed by the following equations

$$\begin{cases} \frac{dI(t)}{dt} = (1-u_1(t))\beta F(t)\frac{(N-I(t))I(t)}{N} - \gamma I(t) - \frac{ku_2(t)I(t)}{1+\alpha I(t)}, \\ \frac{dF(t)}{dt} = (1-u_3(t))c - \mu(t)F(t). \end{cases} \tag{2}$$

Our goal is to reduce the number of infected individuals and decrease the value of the air pollution index. There are three intervention methods, called controls, that are included in the model (2). Control efficacy is represented as a time-varying function and assigned upper bound one and lower bound zero. The meaning of the control variables we considered in the model are as follows: $u_1(t)$ is the percentage of susceptible individuals taking protective measures such as wearing masks, and $u_2(t)$ represents treatment of the infected population. We use a saturated treatment rate function to depict the limited medical resources (medical diagnosis, medical beds, treatment, health care, etc), where k is positive and α is non-negative. The parameter α measures the reverse effect of the infected being delayed for treatment. When $\alpha = 0$, the saturated treatment function returns to the linear one. $u_3(t)$ represents the reduction in the influx of pollutants per unit time caused by the measures taken by the government. The control functions are assumed to be $L^1(0,T)$ functions, belonging to a set of admissible controls $\mathcal{U}$ defined by

$$\mathcal{U} = \{u_1, u_2, u_3 | u_i \ measurable, 0 \le u_i(t) \le 1, t \in [0,T]\}, i = 1,2,3.$$

For simplicity, we re-write system (2) as follows

$$\frac{dX(t)}{dt} = Z(X(t), u(t)), \tag{3}$$

where $X(t) = [I(t), F(t)]'$,$u(t) = [u_1(t), u_2(t), u_3(t)]'$ and $Z(t) = [z_1(t), z_2(t)]'$.

Controls $u_1(t)$, $u_2(t)$ and $u_3(t)$ are used to minimize the number of infectives, improve air quality, economize manpower and material resources over a certain period $[0, T]$. Here, the costs are assumed to be proportional to the square of the corresponding control function. The cost functional is given by

$$J(u_1(t), u_2(t), u_3(t)) = \int_0^T \left[AI(t) + BF(t) + \frac{1}{2}C_1u_1^2(t) + \frac{1}{2}C_2u_2^2(t) + \frac{1}{2}C_3u_3^2(t) \right] dt. \tag{4}$$

For convenience, we define the following functions

$$l(I(t), F(t), u(t)) = AI(t) + BF(t) + \frac{1}{2}C_1u_1^2(t) + \frac{1}{2}C_2u_2^2(t) + \frac{1}{2}C_3u_3^2(t).$$

The initial time is 0, and the termination time of the investigation is T. The control cost function is a nonlinear quadratic function. The positive constants A and B are weight factors in the cost of air pollution control and the number of people infected with respiratory diseases, respectively. The positive constants C_1, C_2, and C_3 are weight coefficients to reflect the cost of three different control measures. The quadratic expressions of the controls indicate nonlinear costs potentially arising at strong control efforts.

The optimal control problem is to minimize the objective functional over $[0, T]$, i.e. to find $u^* \in \mathcal{U}$ satisfying

$$J(u^*) = \inf_{u \in \mathcal{U}} J(t, X, u).$$

2.2. Necessary condition of optimal control

We derive the first-order necessary condition for the optimal control solution by constructing the Hamiltonian $\mathcal{H}$ and then applying Pontryagin's minimum principle. This method introduced the idea of adjoint functions to the objective function, which appends constraints to minimize or maximize the Hamiltonian in terms of controls rather than the objective functional. The Hamiltonian of the problem is made up of the integrand of the objective functional, adjoint function and the right-hand side of (2). Thus, the Hamiltonian for the control problem is given by

$$\begin{aligned}\mathcal{H}(I(t), u_1(t), u_2(t), u_3(t)) &= l(I(t), F(t), u(t)) + \lambda_1 \tfrac{dI(t)}{dt} + + \lambda_2 \tfrac{dF(t)}{dt} \\ &= AI(t) + BF(t) + \tfrac{1}{2}C_1 u_1^2(t) + \tfrac{1}{2}C_2 u_2^2(t) + \tfrac{1}{2}C_3 u_3^2(t) \\ &\quad + \textstyle\sum_{i=1}^{2} \lambda_i z_i(t).\end{aligned} \tag{5}$$

Here $\lambda_i(t)$, for $i \in \{1, 2\}$ are the adjoint variables, which were evaluated at the optimal controls and corresponding states. By applying Pontryagin's minimum principle, we obtain the following theorem

Theorem 2.1. *Let $X^*(t)$ be the optimal state solution with associated optimal control variables $u_1^*(t)$, $u_2^*(t)$ and $u_3^*(t)$ for the optimal control problem (2). There exist adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ satisfying the following system of differential equations*

$$\begin{aligned}\tfrac{\partial \lambda_1(t)}{\partial t} &= -A - \lambda_1 \left[(1-u_1)\tfrac{\beta F(N-2I)}{N} - \gamma - \tfrac{k u_2}{(1+\alpha I)^2}\right], \\ \tfrac{\partial \lambda_2(t)}{\partial t} &= -B + \lambda_2 \mu(t) - \lambda_1 (1-u_1)\tfrac{\beta (N-I) I}{N},\end{aligned} \tag{6}$$

with transversality (or boundary) conditions

$$\lambda_1(T) = 0, \lambda_2(T) = 0.$$

Moreover, the optimal control is given by

$$\begin{aligned}u_1^* &= min\left\{1, max\left\{0, \tfrac{\lambda_1 \beta F(N-I)I}{C_1 N}\right\}\right\}, \\ u_2^* &= min\left\{1, max\left\{0, \tfrac{\lambda_1 k I}{C_2(1+\alpha I)}\right\}\right\}, \\ u_3^* &= min\left\{1, max\left\{0, \tfrac{\lambda_2 c}{C_3}\right\}\right\}.\end{aligned} \tag{7}$$

Proof. We use the Hamiltonian (5) to determine the adjoint equations and the transversality conditions. By differentiating the Hamiltonian with respect to all the state variables to obtain time derivatives, the adjoint system can be written as

$$\begin{aligned}\frac{\partial \lambda_1(t)}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial I} = -A - \lambda_1 \left[(1-u_1)\frac{\beta F(N-2I)}{N} - \gamma - \frac{ku_2}{(1+\alpha I)^2}\right], \\ \frac{\partial \lambda_2(t)}{\partial t} &= -\frac{\partial \mathcal{H}}{\partial F} = -B + \lambda_2\mu(t) - \lambda_1(1-u_1)\frac{\beta(N-I)I}{N}.\end{aligned} \tag{8}$$

To obtain the characterization of the optimality control, we differentiate the Hamiltonian with respect to (u_1, u_2, u_3) and set it to zero,

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial u_1} &= -\lambda_1 \frac{\beta F(N-I)I}{N} + C_1 u_1 = 0, \\ \frac{\partial \mathcal{H}}{\partial u_2} &= -\lambda_1 \frac{kI}{1+\alpha I} + C_2 u_2 = 0, \\ \frac{\partial \mathcal{H}}{\partial u_3} &= C_3 u_3 - \lambda_2 c = 0.\end{aligned} \tag{9}$$

In the interior of U, the optimal control can be expressed as follows by solving (9)

$$u_1^* = \frac{\lambda_1 \beta F(N-I)I}{C_1 N},\ u_2^* = \frac{\lambda_1 kI}{C_2(1+\alpha I)},\ u_3^* = \frac{\lambda_2 c}{C_3}.$$

Since the control is bounded by 0 and 1, then $u_i = 0$ if $u_i < 0$ and $u_i = 1$ if $u_i > 1$ otherwise $u_i = u_i$. Using the property of the control space, we can obtain the desired characterization (7). This completes the proof. □

2.3. Existence and uniqueness of the optimality system

In this section, we will prove the existence and uniqueness of the optimal control for system (2).

Theorem 2.2. *There exists an optimal control $u^*(t)$ such that*

$$J(u^*(t)) = \min_u J(u(t))$$

subject to the control system (2) with the initial conditions.

Proof. By Theorem 4.1 in reference [27], the following conditions in this minimization problem can be easily verified:

(D1) $(X(0), u_1^*, u_2^*, u_3^*) : (u_1(t), u_2(t), u(t)) \in \mathcal{U} \neq \emptyset$;

(D2) The admissible control set $\mathcal{U}$ is closed and convex;

(D3) The right hand side of the state system $Z(X(t), u(t))$ is continuous and

$$Z(X(t), u(t)) = \phi(t, X(t)) + \psi(t, X(t))u(t),$$

where $\phi(t, X(t))$ and $\psi(t, X(t))$ are two dimensional matrix function of t and X;

(D4) The integrand of the objective functional J with $l(t)$ is convex on $\mathcal{U}$;

(D5) There exist constants $K_1 > 0$, $K_2 > 0$ and $\alpha > 1$ such that the integrand of the objective function is bounded below by $K_1(|u_1|^2 + |u_2|^2 + |u_3|^2)^\alpha - K_2$.

The conclusion is that there exists an optimal control. □

3. The corresponding stochastic system

In this section, we consider that the transmitted rate of disease and the clearance of air pollutants are disturbed by random noise, i.e. $\beta(F(t))$ is replaced by $\beta(F(t)) + \sigma_1\eta_1(t)$ and $\mu(t)$ is replaced by $\mu(t) + \sigma_2\eta_2(t)$ in the deterministic model, where $\eta_1(t)$ and $\eta_1(t)$ are white noises. We obtain the corresponding stochastic differential equation model and formulate the stochastic version of the optimization problem. The stochastic model is

$$\begin{cases} dI(t) &= \left[(1-u_1(t))\beta F(t)\frac{(N-I(t))I(t)}{N} - \gamma I(t) - \frac{ku_2(t)I(t)}{1+\alpha I(t)}\right] dt \\ & \quad +\sigma_1(1-u_1(t))\frac{(N-I(t))I(t)}{N} dW_1(t), \\ dF(t) &= [(1-u_3(t))c - \mu(t)F(t)]\, dt - \sigma_2 F(t) dW_2(t). \end{cases} \tag{10}$$

where $W_1(t)$ and $W_2(t)$ are independent standard Brownian motions. We write system (10) as

$$dX(t) = Z(X(t), u(t))dt + G(X(t))dW(t), \tag{11}$$

where $G(t) = [g_1(t), g_2(t)]'$ with

$$g_1(t) = \sigma_1(1-u_1(t))\frac{(N-I(t))I(t)}{N}, \;\; g_2(t) = -\sigma_2 F(t),$$

and $W(t) = [W_1(t), W_2(t)]'$.

Our objective is to find an optimal control solution u^* that minimizes the objective functional for an initial state X_0. The objective functional is

$$J_S(t, X, u) = E\left\{\int_0^T l(I(s), F(s), u(s))dt\right\}, \quad (12)$$

The expectation is conditional on the state of the system being a fixed value X at time t. Based on the above deterministic problem, the set of admissible control $\mathcal{A}$ is given by

$$\mathcal{A} = \{u_1, u_2, u_3 | u_i \ measurable, 0 \le u_i(t) \le 1, t \in [0, T]\}, i = 1, 2, 3.$$

We define the value function $V(t, x) \in C^{1,2}(R \times R^2)$ as follows

$$V(t, X) = \inf_{u \in \mathcal{A}} J_S(t, X, u) = J_S(t, X, u^*). \quad (13)$$

The stochastic optimal control problem can be stated as follows: Given the system (10) and $\mathcal{A}$ with J as in (12), find the value of the function $V(t, X)$ and an optimal control function

$$u_i^*(t) = arg \inf_{u_i(t) \in \mathcal{U}} J_S(x; u_i(t)) \in \mathcal{U}, i = 1, 2, 3. \quad (14)$$

We introduce the Hamiltonian $H(X, u, p, q)$ defined by

$$H(x, u, p, q) = \langle z(x, u), p\rangle + l(x, u) + \langle g(x), q\rangle, \quad (15)$$

where $\langle \cdot, \cdot\rangle$ denotes a Euclidean inner product, $p(t) = [p_1(t), p_2(t)]'$ and $q(t) = [q_1(t), q_2(t)]$ are a pair of adjoint vectors satisfying the following adjoint backward stochastic differential equations

$$\begin{cases} dp(t) = -\{b_x(t, x, u)p(t) + \sum_{j=1}^m \sigma_x^j(t, x(t), u(t))q_j(t) + l_x(t, x, u)\}dt + q(t)dw(t), \\ p(T) = -h_x(x(T)). \end{cases} \quad (16)$$

It follows from the stochastic minimum principle that the following relations hold:

$$dx^*(t) = \frac{\partial H(x^*, u^*, p, q)}{\partial p}dt + g(x^*(t))dw(t), \quad (17)$$

$$H(x^*, u^*, p, q) = \min_{u \in \mathcal{U}} H(x^*, u, p, q), \tag{18}$$

where $x^*(t)$ is an optimal trajectory of $x(t)$. The adjoint variables $p_1(t)$ and $p_2(t)$ satisfy the following stochastic differential equations:

$$\begin{aligned} dp_1(t) &= (-A - p_1\left((1-u_1)\tfrac{\beta F(N-2I)}{N} - \gamma - \tfrac{ku_2}{(1+\alpha I)^2}\right) \\ &\quad +\sigma_1(1-u_1)\tfrac{(N-2I)}{N}q_1)dt + q(t)dW_1(t), \\ dp_2(t) &= \left(-B + p_2\mu(t) - p_1(1-u_1)\tfrac{\beta(N-I)I}{N} - \sigma_2 q_2\right)dt + q(t)dW_2(t). \end{aligned} \tag{19}$$

The initial and terminal conditions of (17) and (16) are given by

$$x^*(0) = [I_0, F_0],$$

$$p(T) = [0, 0, 0]'.$$

Since (18) implies that the optimal control $u^*(t)$ is a function of $p(t)$,$q(t)$ and $x^*(t)$, we have

$$H(x^*, u^*, p, q) = H(x^*, p, q). \tag{20}$$

By the following theorem, we can obtain the optimal control $u_1^*(t)$, $u_2^*(t)$ and $u_3^*(t)$. Let us define by L the differential operator associated with the function displayed in (11),

$$LV = \frac{\partial V}{\partial t} + Z^T \frac{\partial V}{\partial x} + \frac{1}{2}(G^T \frac{\partial^2 V}{\partial t^2} G)^T. \tag{21}$$

Theorem 3.1. *The optimal solution for the problem (18) exists and has the following form*

$$u_1^* = min\left\{1, max\left\{0, \frac{\beta F(N-I)IV_I}{C_1 N}\right\}\right\},$$

$$u_2^* = min\left\{1, max\left\{0, \frac{1}{C_2}\left(\frac{kI}{1+\alpha I}V_I\right)\right\}\right\},$$

$$u_3^* = min\left\{1, max\left\{0, \frac{cV_F}{C_3}\right\}\right\}.$$

Proof: We calculate

$$\begin{aligned}\mathcal{LV}(t) &= z_1(t)V_I(t) + z_2(t)V_F(t) + \frac{1}{2}g_1^2V_{II}(t) + \frac{1}{2}g_2^2V_{FF}(t) + g_1g_2V_{IF}(t)\\ &= \left((1-u_1(t))\frac{\beta F(t)(N-I)I}{N} - \gamma I(t) - \frac{ku_2(t)I(t)}{1+\alpha I(t)}\right)V_I(t) +\\ &\quad ((1-u_3(t))c - \mu(t)F(t))\, V_F(t)\frac{1}{2}\left(\sigma_1\frac{(N-I)I}{N}\right)^2 V_{II}(t) +\\ &\quad \frac{1}{2}\left(\sigma_2 F(t)\right)^2 V_{FF}(t) - \left(\sigma_1\sigma_2\frac{(N-I)I}{N}F(t)\right)V_{IF}(t).\end{aligned} \tag{22}$$

where V_X is the partial derivative and V_{XY} are second-order partial derivatives of $V(t)$ (X and Y can be taken as I and F).

Applying Hamiltonian Jacobi Bellman theory, the infimum of (14) is

$$\inf_{u\in\mathcal{U}}\left[AI(t) + BF(t) + \frac{1}{2}C_1u_1^2 + \frac{1}{2}C_2u_2^2 + \frac{1}{2}C_3u_3^2 + \mathcal{LV}\right]. \tag{23}$$

To obtain the optimal control $u_1(t)$ and $u_2(t)$, it follows from (23) that

$$\begin{cases} C_1u_1 - \frac{\beta F(t)(N-I)I}{N}V_I = 0,\\ -\frac{kI(t)}{1+\alpha I}V_I + C_2u_2 = 0,\\ C_3u_3 - cV_F = 0.\end{cases} \tag{24}$$

We consider the bounds of u_1, u_2 and u_3, and can obtain the asserted expressions for u_1^*, u_2^* and u_3^*, respectively.

We use the results of the deterministic control problem to find an approximate numerical solution for the stochastic control problem. In particular, we use p_1, p_2 as a proxy for V_I, V_F in the calculation of u^* in this case. We note that the presence of $I(t)$ makes V become a stochastic variable even with the proxy (in the stochastic case)[36, 43].

4. Model with pollution-dependent interventions

The implementation of intervention measures for controlling air pollution is dynamically adjusted, which usually depends on the air quality. For instance, when the value of AQI reaches the level of severity, the government will carry out strict policies, such as issuing orders restricting the number of vehicles and bans of factory operations to reduce emissions of air pollutants. Thus, the control variable $u_3(t)$ in the deterministic model (2) and stochastic model (10) changes with time t and also $F(t)$. Replacing the control variable $u_3(t)$ with $u_3(t)\frac{F(t)}{F(t)+\eta}$, the deterministic model (2) and stochastic model (10) become

$$\begin{cases} \frac{dI(t)}{dt} = (1-u_1(t))\beta F(t)\frac{(N-I(t))I(t)}{N} - \gamma I(t) - \frac{ku_2(t)I(t)}{1+\alpha I(t)}, \\ \frac{dF(t)}{dt} = (1-u_3(t)\frac{F}{F+\eta})c - \mu(t)F(t), \end{cases} \tag{25}$$

and

$$\begin{cases} dI(t) &= \left[(1-u_1(t))\beta F(t)\frac{(N-I(t))I(t)}{N} - \gamma I(t) - \frac{ku_2(t)I(t)}{1+\alpha I(t)}\right] dt \\ & \quad +\sigma_1(1-u_1(t))\frac{(N-I(t))I(t)}{N} dW_1(t), \\ dF(t) &= \left[(1-u_3(t)\frac{F}{F+\eta})c - \mu(t)F(t)\right] dt - \sigma_2 F(t) dW_2(t), \end{cases} \tag{26}$$

where the Hill function $\frac{F}{F+\eta}$ describes how the level of air pollution influences the intervention measures. It is an increasing function of $F(t)$ and η is the AQI value at which the intensity of the control measure is half its maximum.

Following the same method in previous section, we can calculate the solution of the optimal control problem for the deterministic system (25). It is given by

$$\begin{aligned} u_1^* &= min\left\{1, max\left\{0, \frac{\lambda_1\beta F(N-I)I}{C_1 N}\right\}\right\}, \\ u_2^* &= min\left\{1, max\left\{0, \frac{\lambda_1 kI}{C_2(1+\alpha I)}\right\}\right\}, \\ u_3^* &= min\left\{1, max\left\{0, \frac{\lambda_2 cF}{C_3(F+\eta)}\right\}\right\}, \end{aligned} \tag{27}$$

where the adjoint variables $\lambda_1(t)$ and $\lambda_2(t)$ satisfy the following system of differential equations

$$\begin{aligned} \frac{\partial\lambda_1(t)}{\partial t} &= -A - \lambda_1\left[(1-u_1)\frac{\beta F(N-2I)}{N} - \gamma - \frac{ku_2}{(1+\alpha I)^2}\right], \\ \frac{\partial\lambda_2(t)}{\partial t} &= -B + \lambda_2\left[\frac{u_3 c\eta\mu(t)}{(F+\eta)^2} - \mu(t)\right] - \lambda_1(1-u_1)\frac{\beta(N-I)I}{N}. \end{aligned} \tag{28}$$

The solution of the optimal control problem for the stochastic model (26) takes the form

$$u_1^* = min\left\{1, max\left\{0, \frac{\beta F(N-I)IV_I}{C_1 N}\right\}\right\},$$

$$u_2^* = min\left\{1, max\left\{0, \frac{1}{C_2}\left(\frac{kI}{1+\alpha I}V_I\right)\right\}\right\},$$

$$u_3^* = min\left\{1, max\left\{0, \frac{cFV_F}{C_3(F+\eta)}\right\}\right\}.$$

The adjoint vectors p_1 and p_2 are used to replace V_I and V_F, satisfying the following equations:

$$\begin{aligned} dp_1(t) &= (A - p_1\left((1-u_1)\frac{\beta F(N-2I)}{N} - \gamma - \frac{ku_2}{(1+\alpha I)^2}\right) \\ &\quad +\sigma_1(1-u_1)\frac{(N-2I)}{N}q_1)dt + q(t)dW_1(t), \\ dp_2(t) &= \left(B + p_2(\frac{cu_3\eta}{(F+\eta)^2} + \mu(t)) - p_1(1-u_1)\frac{\beta(N-I)I}{N} - \sigma_2 q_2\right)dt + q(t)dW_2(t). \end{aligned} \tag{29}$$

5. Numerical results

In this section we will present the results of fitting the optimal control model to the actual data. Further, the results of numerical simulations generated by the implementation of the intervention strategies under various scenarios to investigate the effect of optimal control strategies on the transmission dynamics of respiratory disease with air pollution are described.

5.1. Data fitting

To investigate the utility of the deterministic and stochastic models discussed in Section 3 with respect to real data on influenza-like illness (ILI) cases and the air quality index (AQI), we collected data related to air pollution for the period of 15 November 2012 to 14 November 2021 from Xi'an, Shaanxi Province, China, which are shown in Figure 1 (a). It can be seen from the time series data that the air pollution is serious in autumn and winter every year, and the AQI value is between 150 and 500. The air quality is good in spring and summer every year, when the AQI value is less than 150 most of the time. Since the winter of 2017, the local government has taken vehicle control measures in Xi'an to improve the air quality. The normalized traffic restriction measure is that 20% of cars (according to the last two digits of their licence plate numbers) are banned from travelling every day. Thus, we assumed that the control variables $u_3(t)$ in the deterministic model (2) and the stochastic model (10) are switched in the light of the policy implementation time. Based on the AQI data , we estimated parameters associated with the $F(t)$ variable with control measures implemented. The estimation results related to the concentration of air pollutants are given in Table 1. The estimates of control variables u_3 in the policy implementation phase is 0.3. The fitting results in Figure 1 (b) show that government responses did indeed have achieved some success.

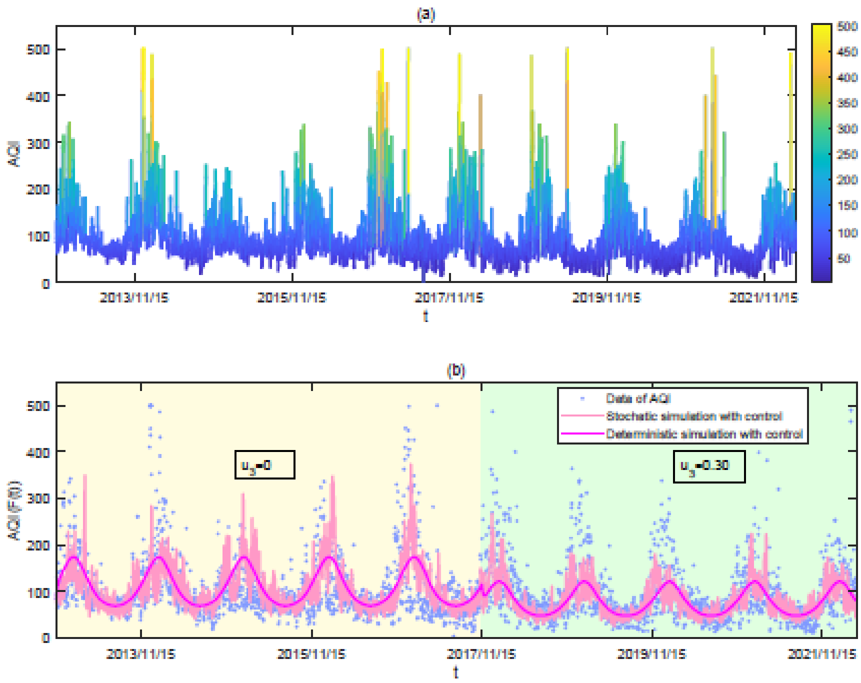

Figure 1: Time series and results of fitting the AQI data from 15 Nov. 2012 to 14 Nov. 2021. (a) The AQI for Xi'an, Shaanxi province; (b): The blue points are actual data, the dark pink curves and light pink curves are correspond to the deterministic and stochastic simulation results, respectively.

The above result is the estimation of the control variables based on the actual data, and the optimal control is numerically solved by the backward-forward sweep method for the deterministic model. We use the fourth-order Runge-Kutta algorithm to solve the state system and the backward fourth-order Runge-Kutta to solve the adjoint system. The whole process is repeated until convergence. The process is the same for the stochastic model except for the numerical method for solving the stochastic differential equations. In the following simulation, the initial conditions of the infected population and parameters associated with respiratory disease infections refer to the parameter estimation results in [45], which are shown in Table 1. The period we choose for simulation is 2000 days.

5.2. Special case $B = 0$

We consider a case in which the goal is to reduce the number of infected individuals. The weight B is set to zero in the cost functional. To investigate how the optimal control depends upon various input costs and the intensity of control measures in the model, we plot the control $u(t)$ and the evolution of state variables for different values of weight coefficients. Figure 2 presents the numerical solution of the optimal control and the simulated path of the number of infected cases and the concentration of air pollutants for the deterministic model (2). We also provide the corresponding simulated paths without control for comparison. The values of weights are chosen to balance the weights of variables in the objective function. The baseline for the cost weight coefficients of the three different control measures are assumed to be the same ($C1 = 1$, $C2 = 1$, $C3 = 1$). The optimal trajectories and the state evolution are shown in Figure 2 (a1-c1). The optimal control solution in Figure 2 (a1) shows some decrease with the decline in the infected population. It is observed from Figure 2 (b1-c1) that the infected population will be well controlled when the strength of the three control measures reaches the maximum from the beginning. The concentration of air pollutants can also be well controlled if the strictest measures are taken. It will return to its original state when the number of infectives is zero. It is clear that there are no constraints on the concentration of air pollutants. Notice that the optimal trajectories of the first control variable $u_1(t)$ have a sudden increase. This is caused by the sudden increase in the concentration of air pollutants. The optimal results are similar when increasing the cost weight coefficient of treatment for the infected population ($C2$ changes from 1 to 1000). We obtain similar optimal results when increasing the cost weight coefficient of air pollution control ($C2$ changes from 1 to 100, and $C2$ changes from 1 to 10000). Regardless of the variation in the weight cost, it is optimal to implement the three control measures to control the respiratory disease in the end.

Figure 3 presents the numerical solution of the optimal control, the simulated path of the number of infected cases, and the concentration of air pollutants for the stochastic model (10). The cost weight coefficients of the three different control measures are the same as those in the deterministic model. However, the optimal solution of the third control variable $u_3(t)$ for the stochastic model is different from the corresponding deterministic solution. As shown in Figure 3 (a1), the strength of the optimal strategy of air pollution control fluctuates dynamically between 0 and 1 when the combination weight factors are $C1 = 1$, $C2 = 1$, $C3 = 1$. The optimal strategy for the respiratory control is mainly based on the first two control strategies, while the third control strategy does not play a major role when the combination weight factors are $C1 = 1000$, $C2 = 100$, $C3 = 1$ or $C1 = 1$, $C2 = 100$, $C3 = 10000$, which are shown in Figure 3 (a2) and (a3), respectively. If the weight of air pollution control cost is much higher than the other two control cost, then the proportion of air pollution control in the optimal control strategy is very small. In addition, the simulated results for optimal trajectories show that the stochastic optimization control is more practical and reasonable.

5.3. General case $B \neq 0$

In this section, some numerical simulations for the deterministic model (2) and corresponding stochastic model (10) are performed to evaluate the effect of the control strategy in a general senario, where the weight factor of the cost of air pollution control is not zero, i.e., $B \neq 0$. Figure 4 presents the numerical solution of the optimal control, simulation path of the number of infected cases, and the concentration of air pollutants for the deterministic model (2). When we consider that all the costs are equal (the weight factors are $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$), the optimal control strategy is that the three control measures are all in full force. In this case, the number of infected cases and the concentration of air pollutants will quickly drop to zero. As shown

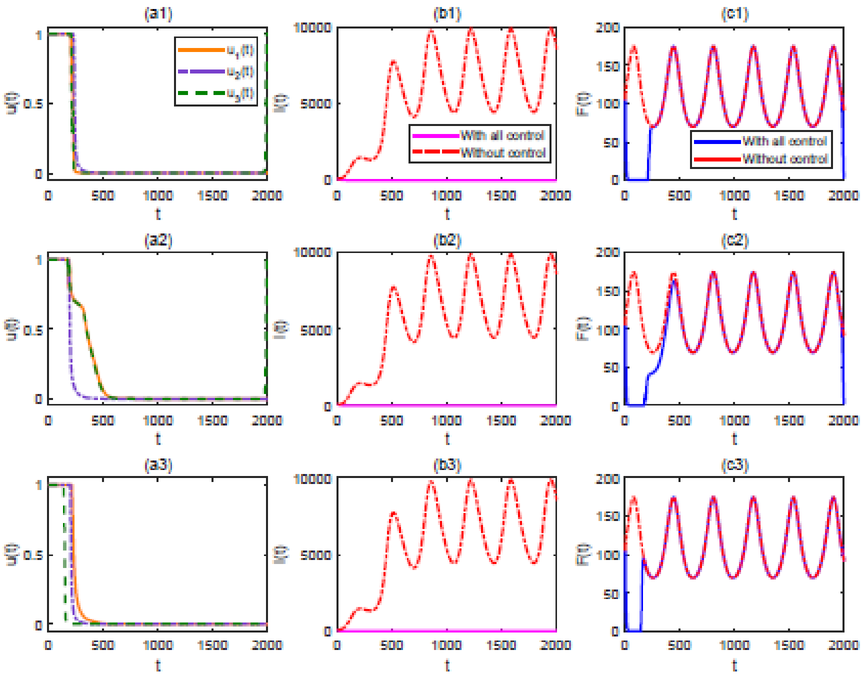

Figure 2: Optimal trajectories and comparison of the state evolution with and without optimal control for the deterministic model (2). (a1-c1): $A = 1$, $B = 0$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 0$, $C1 = 1$, $C2 = 1000$, $C3 = 1$; (a3-c3): $A = 1$, $B = 0$, $C1 = 1$, $C2 = 100$, $C3 = 10000$.

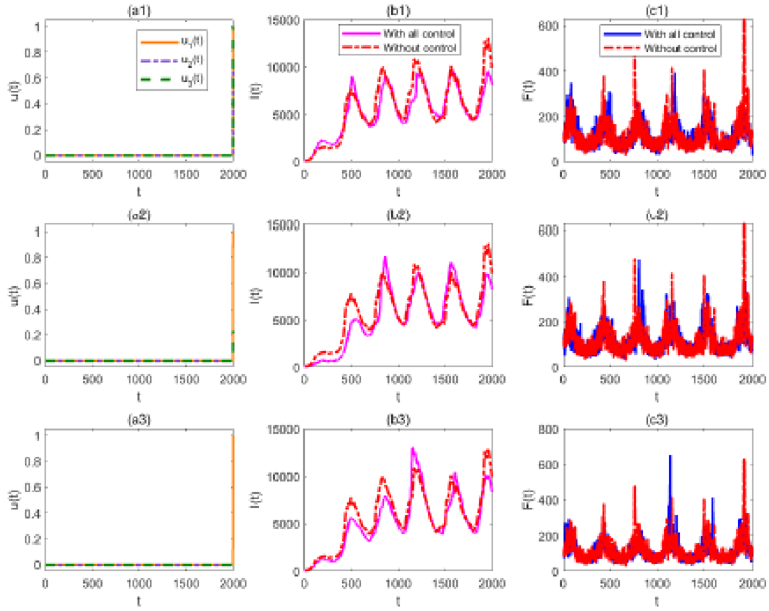

Figure 3: Optimal trajectories and comparison of the state evolution with and without optimal control for the stochastic model (10). (a1-c1): $A = 1$, $B = 0$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 0$, $C1 = 1000$, $C2 = 100$, $C3 = 1$; (a3-c3): $A = 1$, $B = 0$, $C1 = 1$, $C2 = 100$, $C3 = 10000$.

in Figure 4 (a1-c1), the first two control measures can stop when the number of infected cases reaches zero and the third measure for air pollution control needs to be implemented all the time to ensure that the concentration of air pollutants does not rise. However, this is an extreme case. It is impractical to completely eliminate the emission of pollutants into the air. As the weight factor of the cost of control measures increases (set as $C1 = 1000$, $C2 = 100$, $C3 = 100$), the optimal trajectory of $u_3(t)$ follows a periodic change (see Figure 4 (a2-c2)). When the concentration of air pollutants is high, the intensity of control measures is large, and vice versa. Suppose the proportion of air pollutant control is reduced. In that case, the value of B is reduced ($A = 1$, $B = 0.001$), and the intensity of the third measure for air pollution control decreases once the number of infected cases reaches zero (see in Figure 4 (a3-c3)). Figure 5 shows the simulated optimal control results for the corresponding stochastic model (10). When the weight factors are the same as those in Figure 4 (a3-c3), the optimal stochastic solution of the third control variable $u_3(t)$ for the stochastic model is different from the corresponding deterministic solution. As shown in Figure 3 (a3), the intensity of the optimal strategy of air pollution control fluctuates dynamically between zero and one. It is stronger than that of the corresponding deterministic model. For the other two sets of weight factors, the simulated results of the stochastic model, shown in Figure 3 (a1-c1) and Figure 3 (a2-c2), are similar to the corresponding deterministic model.

5.4. Numerical results of the model with pollution-dependent interventions

In this section, we study the optimal control solutions of the deterministic model (25) and the corresponding stochastic model (26) in which the control variable $u_3(t)$ depends on the air pollution $F(t)$. The optimal trajectories of control variables $u_1(t)$, $u_2(t)$, $u_3(t)$ and the combination term $\frac{u_3(t)F(t)}{F+\eta}$ for the deterministic model with three different weight factor sets are shown in Figure 6 (a1-a3), where the value of the parameter η is 200. Regardless of the ratio of the weight factors, the number of infected cases can be well controlled, as shown in Figure 6 (b1-b3). For the control of air pollution, the concentration of

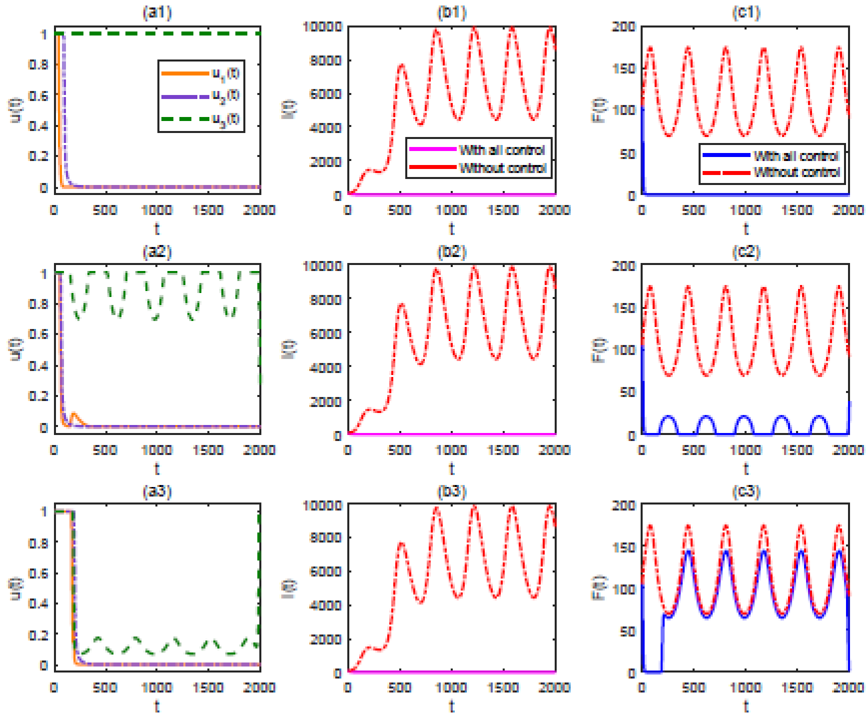

Figure 4: Optimal trajectories and comparison of the state evolution with and without optimal control for the deterministic model (2). (a1-c1): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 1$, $C1 = 1000$, $C2 = 100$, $C3 = 100$; (a3-c3): $A = 1$, $B = 0.001$, $C1 = 1$, $C2 = 1$, $C3 = 1$.

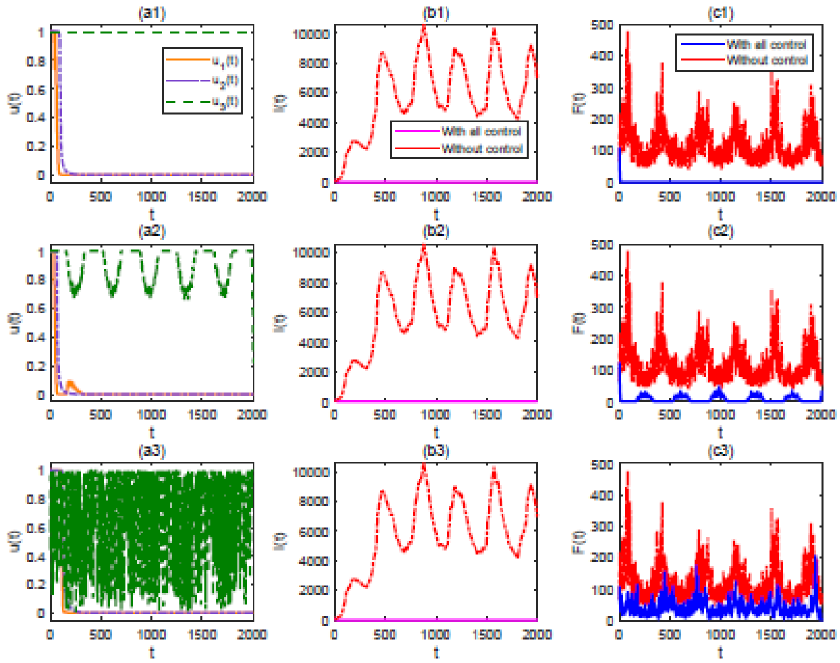

Figure 5: Optimal trajectories and comparison of the state evolution with and without optimal control for the deterministic model (10). (a1-c1): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1000$, $C3 = 100$; (a3-c3): $A = 1$, $B = 0.001$, $C1 = 1$, $C2 = 1$, $C3 = 1$.

air pollutants can be reduced during the severe period of a cycle. The trend of air pollutant concentration changes remains unchanged, which is different from the result in Figure 4 (c2)). It illustrates that the optimal control solution of the new model (25) is more in line with reality. By comparing the results under three sets of different weight factors shown in Figure 6 (c1-c3), we find that the concentration of air pollutants will be better controlled when the cost of air pollution control is lower than the cost of disease treatment. If the proportion of air pollutant control is reduced, that is, the value of B is reduced ($A = 1$, $B = 0.001$), then the air pollution control would not be improved significantly. Figure 7 presents the simulated results of the corresponding stochastic model (26). We have conclusions similar to the two cases in previous sections.

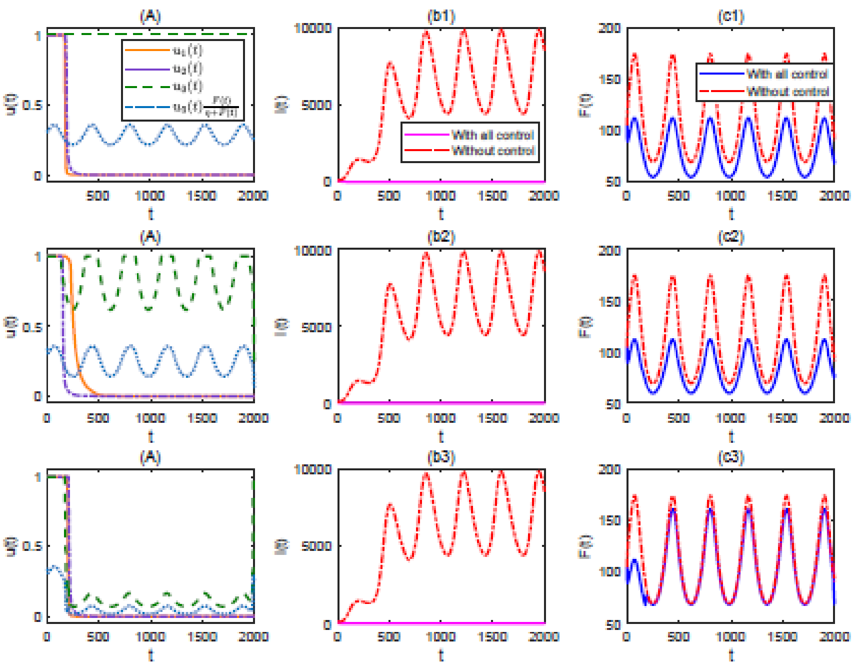

Figure 6: Optimal trajectories and comparison of the state evolution with and without optimal control for the stochastic model (25). (a1-c1): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1000$, $C3 = 100$; (a3-c3): $A = 1$, $B = 0.001$, $C1 = 1$, $C2 = 1$, $C3 = 1$.

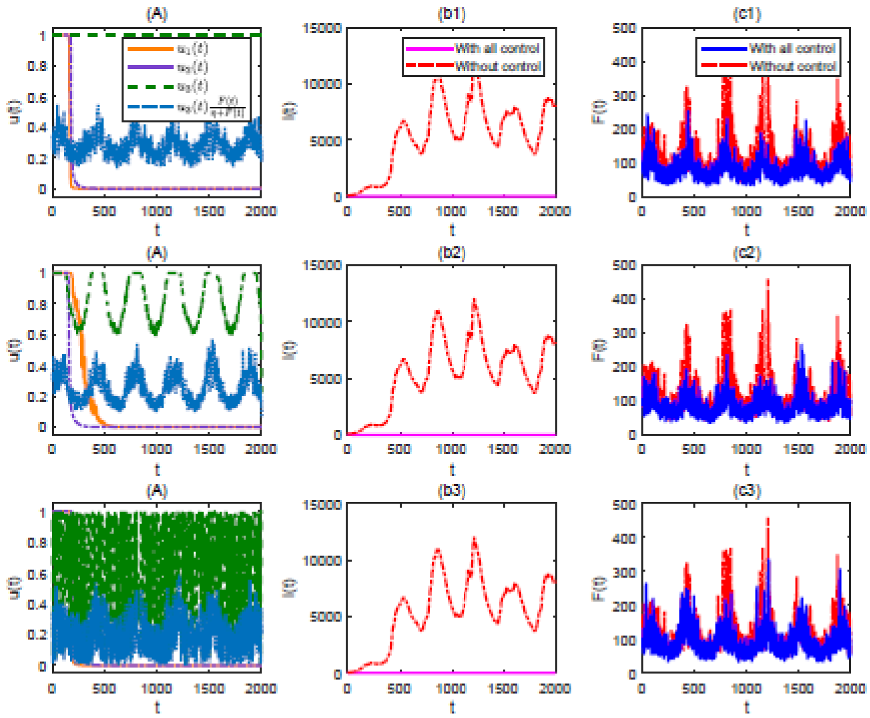

Figure 7: Optimal trajectories and comparison of the state evolution with and without optimal control for the stochastic model (26). (a1-c1): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$; (a2-c2): $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1000$, $C3 = 100$; (a3-c3): $A = 1$, $B = 0.001$, $C1 = 1$, $C2 = 100$, $C3 = 10000$.

5.5. Comparison of stochastic and deterministic model

The range of control variables is between 0 and 1 in the above numerical simulations, which show that the solution of optimal trajectories reaches the maximum value of 1 at the initial stage, and the number of infected people decreases rapidly and finally tends to zero with control measures. However, the estimate of control variables u_3 is 0.3 when the policy is implemented. This shows that it is not easy to achieve the maximum control level. Thus, we will explore the difference in the optimal control solution between the deterministic model (25) and stochastic model (26) by narrowing the range of control variables. In the simulations, we fixed the cost weight factor of control measures (set as $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$), increased the parameter value of the disease infection rate to $2.9769 * 10^{-4}$, and set three ranges for control variables as $[0, 0.2]$, $[0, 0.28]$, $[0, 0.35]$. It can be seen from the results in Figure 8 (a1-c1) that when the maximum intensity of the control variable is 0.4, the disease will go extinct with control measures both in stochastic and deterministic models. The results in Figure 8 (a3-c3) show that the disease will persist when the maximum intensity of the control variable is 0.2. However, when the maximum intensity of the control variable is set to 0.285, the optimal control trajectory of the number of infected people $I(t)$ in the deterministic model still persists. In contrast, the optimal control trajectory of the number of infected people $I(t)$ in the corresponding stochastic model is finally goes to extinction, as shown in Figure 8 (a2-c2). This illustrates that the disease control based on the stochastic optimal control model is more effective under the same conditions than the deterministic model. These findings have also been verified in the actual data fitting results shown in Figure 1.

5.6. The value of the objective function

When the government decides what control measures to be implemented or when individuals decide whether to take protective measures, they usually consider the cost of different measures. Thus, we will calculate the value of the objective function under different combinations of three control measures to

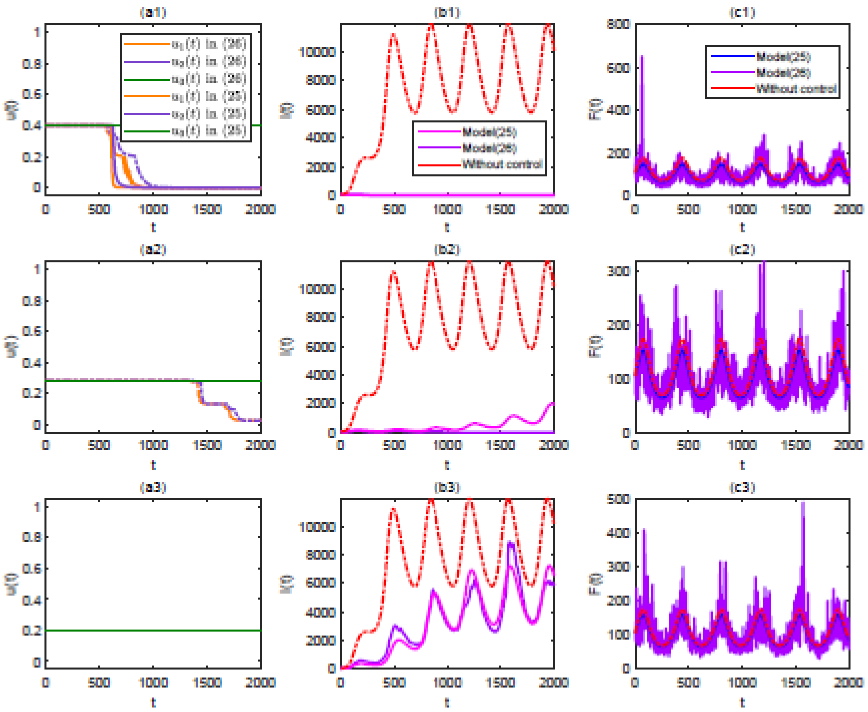

Figure 8: Optimal trajectories of the deterministic model (25) and stochastic model (26), and comparison of the state evolution with and without optimal control. Here $A = 1$, $B = 1$, $C1 = 1$, $C2 = 1$, $C3 = 1$, $\beta = 2.9769 * 10^{-4}$, and the other parameter values are the same with those in Figure 7. (a1-c1): the value range of control variables is $[0, 0.4]$; (a2-c2): the value range of control variables is $[0, 0.285]$; (a3-c3): the value range of control variables is $[0, 0.2]$.

Table 1: Parameters and initial conditions of the system.

Parameters	Definition	Values
c	Inflow rate of pollutants	24.6706
μ_0	Parameter in the rate of pollutant clearance	0.2477
μ_1	Parameter in the rate of pollutant clearance	0.1071
ϕ_0	Parameter in the rate of pollutant clearance	3.5444
δ_2	Noise intensity of pollutant clearance	0.1793
β	Baseline transmission coefficient	$2.3838 * 10^{-4}$
γ	Recovery rate for infectives	0.017
δ_1	Noise intensity of transmission coefficient	0.0025
k	Cure rate for infectives	0.017
α	The magnitude of the effect of the infected being delayed for treatment	0.0025
$I(0)$	Initial value of infected population	86
$F(0)$	Initial value of concentration of air pollutants	105

analyze control strategies. Eight combinations of the three control measures are listed in Table 2. For comparison, the weight coefficients A, B, $C1$, $C2$ and $C3$ are set to 1 in numerical simulations. By comparing the results of J in Table 2, we find that it is the highest when no measure is taken. Taking any of the three measures is conducive to respiratory disease control. The control measures can reduce the number of infected cases of respiratory diseases and the concentration of air pollutants, and the total cost of measures is low. As long as $u_3(t) \neq 0$, the values of J decrease significantly compared with other combinations. This indicates that the improvement of air quality is the most effective measure to reduce respiratory disease infection. Therefore, taking measures to reduce the concentration of air pollutants can improve air quality and play a critical role in the control of respiratory diseases related to air pollution. We also obtain a similar conclusion by calculating the mean value of J for the stochastic model (10), which is shown in Table 2.

Table 2: The values of $J(u)$ for the deterministic model (2) and stochastic model (10).

Combination of control variables	$J(u)$ of deterministic model	$J(u)$ of stochastic model
$u_1(t), u_2(t), u_3(t) \equiv 0$	$5.24 * 10^6$	$5.99 * 10^6$
$u_1(t) \neq 0, u_2(t), u_3(t) \equiv 0$	$3.28 * 10^6$	$3.27 * 10^6$
$u_2(t) \neq 0, u_1(t), u_3(t) \equiv 0$	$3.33 * 10^6$	$3.33 * 10^6$
$u_3(t) \neq 0, u_1(t), u_2(t) \equiv 0$	$7.56 * 10^4$	$7.69 * 10^4$
$u_1(t), u_2(t) \neq 0, u_3(t) \equiv 0$	$3.24 * 10^6$	$3.22 * 10^6$
$u_1(t), u_3(t) \neq 0, u_2(t) \equiv 0$	$7.56 * 10^4$	$7.43 * 10^4$
$u_2(t), u_3(t) \neq 0, u_1(t) \equiv 0$	$4.06 * 10^4$	$4.18 * 10^4$
$u_1(t), u_2(t), u_3(t) \neq 0$	$3.68 * 10^4$	$3.78 * 10^4$

6. Comparison with other studies

We developed a coupled model to study optimal control measures for the transmission of respiratory diseases caused by air pollution. In many studies addressing optimal control problems of infectious disease models, parameters are commonly used to characterize factors of interest [46, 47, 48]. In our research, instead of using a parameter coupled with disease transmission dynamics, we employ a differential equation to represent changes in air pollutants. This approach introduces diversity into our findings compared to other studies. While many studies solely focus on minimizing the number of infections and reducing economic costs, our model aims for comprehensive governance by incorporating optimization of air quality.

When studying the optimal control problem of the deterministic model, our numerical implementation results indicate that with total control measures, the disease becomes extinct, and the concentration of air pollutants shows a trend of periodic changes following a short-term decrease. In the case of incomplete control measures, the disease persists, and the number of infected cases exhibits periodic changes, driven by the periodicity of the air pollutant concentration. This optimal control outcome from our periodic model aligns with the findings of the fractional optimal control problem when studying a human respiratory syncytial virus surveillance system [49]. Typically, when analyzing cost-effectiveness, weight coefficients correspond to unit costs and are set to 1. However, we have improved our numerical analysis method by considering the different costs associated with various control measures. We have analyzed the results under different weight factor values for the three control measures. The optimal control solution can exhibit various phenomena, such as stability to the boundary value and periodic oscillations. In the case of the stochastic optimal control solution of the coupled model, periodic oscillations are observed under specific weight values.

Gani et al. also conducted research on the optimal control problem of deterministic and stochastic models to analyze an epidemic model incorporating media awareness programs and treatment for infectives [43]. Their findings indicate that the optimal control problem of the stochastic model generally aligns with the numerical solution of the deterministic model. Our results further corroborate the relationship between the deterministic and stochastic models.

7. Discussion

This work studied the optimal control problem for respiratory diseases induced by air pollution based on a two-dimensional coupling system. We considered the optimal control for two models (the deterministic model and the corresponding stochastic model). In the optimal control problem, three types of measures, including self preventive measures, treatment of disease, and reducing the emission of air pollutants, are included in the air pollution control and disease control. According to Pontryagin's minimum principle and the Hamiltonian-Jacobi-Bellman equation, the optimal control solutions for the deterministic model are obtained. The adjoint vectors are used to find an approximate numerical solution for analyzing the stochastic optimality system. The identification of the model and the estimation of parameters are implemented by the fitting analysis of the actual measured data and the control measures. The effectiveness of the interventions against air pollution were verified based on the model with control. On this basis, in order to find the optimal control measures, we presented the numerical simulation of optimal control strategies and the corresponding state variables for models with different weight factor values. We also showed the numerical results without control for comparison. The values of objective functionals under various combinations of three control measures were calculated to evaluate the effectiveness of different control measures.

Assuming reducing the number of infected cases is the only control objective, numerical results showed that the relevant respiratory diseases can be well controlled by taking the three control measures. The number of infected cases will reduce to zero. At the same time, the air quality will improve once the disease is under control. If the objective is to reduce the number of infected cases and improve the air quality, numerical results show that the diseases will be eliminated and the air pollution will be well controlled. In addition, considering the flexibility of control measures in the implementation process, we replaced the time-dependent control variable $u_3(t)$ with a function dependent on both the time and the value of AQI. The optimal control results of the improved model show that the periodic change of air pollutant concentration control is more in line with the actual situation. From the cost values under different control measures, it can be found that reducing the emission of air pollutants is a control measure that has the lowest overall cost and plays a significant role in the control of related respiratory diseases.

The comparison of the numerical results between the deterministic and stochastic models shows the advantage of each model. The optimal control for the deterministic model can reflect the control effect, but it cannot reverse the randomness in the process. The optimal solution of the control variable in the stochastic model oscillates between the upper and lower boundaries of 1 and 0. This indicates that the intensity of control measures changes dynamically, which should be the case in realistic situations when the control measures are implemented. Therefore, the stochastic optimal control model and the corresponding optimal control solution are closer to reality and can reflect the actual situation. There are a couple of limitations in our work. Firstly, the numerical simulation only shows the results of three representative groups with different weight coefficient values, which only present part of the many possible scenarios. Secondly, the value of the objective function is only for comparison between different control strategies because the implementation cost will be hard to determine. Further work on surveying, sampling, and assessment of the cost of each control measure is needed.

8. Declarations

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8.2. Competing Interest

The authors declare that they have no conflict of interest.

8.3. Code availability

None

8.4. Authors'Contribution

This work was initiated when the first author visited the University of Florida in 2021. S.T., S.H conceived and designed the study. S.H., Q.Z. carried out the theoretical analysis, S.H, S.T., L.R. performed numerical simulations. S.H., S.T., Q.Z., L.R. and R.A.C. participated in writing and reviewing the manuscript.

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