Greenwich Papers in Political Economy

Was Pareto right? Do the US wealth and income distributions follow power laws?

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Year: 2023

No: GPERC92

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Acknowledgements: Financial support from the Hans Böckler Foundation through grant number 2021-544-2 is thankfully acknowledged. Special thanks for helpful comments and discussions goes to Jakob Kapeller. Furthermore, we thank Alejandro Gonzalez, Thorsten Heinrich, Marco Ranaldi, Ben Tippet and Jan Weber. All remaining errors are ours.

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1 Introduction

This paper revisits the question of whether the tails of the wealth and income distributions in the United States follow a power law. Why is this a worthwhile research question, more than 100 years after Pareto's (1896) influential work? We argue that identifying power laws in wealth and income data is relevant for economists for two reasons: First, power laws are indispensable tools for the analysis of income and wealth inequality. Belonging to the family of heavy-tailed distributions, they are able to describe populations characterized by extreme outliers several thousand times larger than the typical (i.e. median) unit of observation as is the case in income and wealth data. Examples of research areas which use and rely on power laws include the literature on Distributional National Accounts (DINA) which aims to establish consistency between survey based measures of national income or wealth with national aggregates (Piketty et al. 2018, Blanchet et al. 2021, Garbinti et al. 2018, Waltl 2022, Jenkins 2016, Chakraborty et al. 2019) as well as the closely related literature on correcting underreporting in survey data (Advani, Bangham & Leslie 2020, Bach et al. 2019, Eckerstorfer et al. 2016, Vermeulen 2016, 2018, Wildauer & Kapeller 2021, 2022, Kennickell 2024). Second, power laws for the income and wealth tail distributions are often used in research on taxation and tax policy. Examples include revenue modelling of wealth taxes (Advani, Hughson & Tarrant 2020, Kapeller et al. 2021, Tippet et al. 2021, Krenek & Schratzenstaller 2022), estimations of wealth taxation effects on greenhouse gas emissions (Apostel & O'Neill 2022) and theoretical models of (optimal) capital tax policy (Saez 2001, Saez & Stantcheva 2016, 2018). However, despite their popularity the empirical support for power laws in income and wealth data is mixed. Several studies using formal tests reject power law tails for wealth (income) data (Brzezinski 2014, Ogwang 2013, Chan et al. 2017, Teulings & Toussaint 2024). Conversely, a large body of literature in support of power laws in income and wealth data relies heavily on inaccurate, graphical methods (Levy & Solomon 1997, Dragulescu & Yakovenko 2001, Klass et al. 2006, Ding & Wang 2007, Klass et al. 2007, Jayadev 2008, Jagielski et al. 2017, Bach et al. 2019). This paper provides a reconciliation of both "camps" and aims to move the debate beyond the current stalemate.

We comprehensively assess the fit of alternatives to the (strict) power law hypothesis in the US income and wealth distribution. Our approach improves upon previous tests in two ways: Firstly, we use more and better data to test the power law hypothesis. The Survey of Consumer Finances (SCF) allows us to avoid measurement and quality problems from which a large part of the existing literature which uses rich lists is suffering. These lists of billionaires and millionaires are compiled by journalists for the entertainment of their readers. The details of how these lists are constructed are often poorly documented and different magazines can report wildly different results for the same households or individuals (Capehart 2014). In contrast, the SCF uses tax data to effectively oversample and address the problem of differential non-response bias (Avery et al. 1988, Bricker et al. 2016, Kennickell 2017, Osier 2016). At the same time, it provides a representative sample of the entire distribution which allows us to test which proportion of the tail can be modelled as a power law, i.e. we can test the length of the tail. This is a stark contrast to previous tests of the Pareto hypothesis for the US which relied on the *Forbes 400* rich list which represents only the top 0.0003% of US households. The amount and quality of information

the SCF provides also enables us to distinguish difficult to distinguish alternatives such as power laws and the LogNormal distribution. Wealth in the SCF beyond the 91^{st} percentile threshold spans three orders of magnitude which is essential for such a task.

Secondly, we adopt a comprehensive approach to testing the power law hypothesis. We show that strict power laws (i.e. type I Pareto distributions) exhibit a constant degree of concentration along their tail. For example, the share of the top 5% within the top 10% is the same as the share of the top 5% within the top 1%. However, real data might not display such patterns of stable concentration, which requires us to consider more flexible alternatives. Therefore, this paper identifies four distributional families for modelling wealth and income tails and assesses their fit: 1) thin-tailed distributions, 2) heavy-tailed distributions which are not Pareto, 3) strict power laws or type I Pareto distributions and 4) asymptotic power laws[1]. Furthermore, instead of graphical analysis we rely on formal statistical goodness of fit tests, which yield clear answers to the question of "Which distributions fit the data better?". Lastly, we also assess the pattern of within-tail concentration for each tested distribution. This is not just informative in its own right but also provides a useful lens through which to interpret and better understand why one distribution provides a superior fit to another.

In terms of methods, we rely on Maximum Likelihood estimators where feasible and in addition use a rank correction approach (Wildauer & Kapeller 2022), which allows us to directly address the explicit exclusion of the richest 400 US families from the SCF, which prevented previous uses of the SCF for this purpose. We also employ an elemental percentile method (EPM) estimator following Castillo & Hadi (1997), which only recently has been used in the wealth and income distribution literature (Kennickell 2024, Heck et al. 2020). The goodness of fit is assessed via Cramer-von-Mises tests which are suitable to use in conjunction with the SCF's complex survey weights (Wildauer & Kapeller 2021). Fits are compared in a pair-wise manner, using the SCF's replicate weights due to the undisclosed details of the SCF's sampling procedure which prevents us from adopting Clauset et al.'s (2009) approach to goodness of fit testing.

The main results of the paper are as follows: First, we find strong support for asymptotic Pareto tails for both the US income and wealth distribution above the 91^{st} percentile and particularly strong support for asymptotic Pareto tails beyond the 97^{st} percentile of the US wealth distribution. While we find that asymptotic Pareto tails fit the data better than strict Pareto tails, the latter still provide a good fit beyond the 99^{th} percentile and could be used as a second best option in situations where simplicity is key. Overall, our results are in line with the rejection of strict power laws in the existing literature. However, instead of abandoning power laws altogether on this basis, our results point to asymptotic power laws as the preferred alternative. In this sense Vilfredo Pareto was (asymptotically) right all along.

Second, we show that the pattern of inequality within the tail of the wealth distribution changes. Wealth becomes less concentrated within the tail between the 91^{st} and the 99^{th} per-

[1]Asymptotic power laws are distributions where the tail becomes more and more similar to a type I Pareto distribution as $x \to \infty$. See section 3.1 for formal definition and discussion. They are also known as asymptotic Pareto distributions, *thick-tailed* distributions, *fat-tailed* distributions or as *regularly varying distributions*.

centile and concentration stabilises beyond the 99^{th} percentile. For the distribution of income, the pattern of within tail concentration changes through time. While in waves 1989 to 1995 it is increasing, from wave 1998 onwards it exhibits an S-shape with a decreasing middle section. Furthermore, while asymptotic Pareto tails can accommodate monotonically increasing or decreasing concentration patterns, they are not compatible with a switch from one to the other. Therefore, ideally they should only be fitted beyond the 99^{th} percentiles of the US wealth and income distributions.

The rest of the paper is organised as follows: section 2 provides an overview over the related theoretical and empirical literature. Section 3 outlines our methodology and presents the data we are using. Section 4 presents our results and section 5 concludes.

2 Pareto tails in wealth and income data: A brief review

2.1 A theoretical starting point

Heavy-tailed wealth distributions in general and Pareto tails specifically emerge from very simple theoretical models. Probably the best known example is a stochastic growth process such as:

$$y_t = y_{t-1} r_t \tag{1}$$

where y_t can be thought of as the wealth of an agent at time t and r_t is the return achieved in that period, for example by investing in the stock market, which follows a distribution $F(r)$. If $y_0 = 1$ and r_t are i.i.d. and strictly positive, the distribution of y_t becomes LogNormal as $t \to \infty$. This result directly arises from the central limit theorem, as the transformed variable $x_t = \log(y_t)$ follows an additive process, leading to the emergence of a Gaussian distribution[2]. This result has been discussed in detail most recently by Nair et al. (2022, ch. 6) as well as several authors before (Newman 2005, Benhabib & Bisin 2018). By adding a lower barrier $b > 0$, which we can think of as a social safety net, we get a new process: $y_t = \max(y_{t-1} r_t, b)$. Alternatively, we can use an additive component, which we can think of as saved income from work (w_t, is i.i.d. and strictly non-negative): $y_t = y_{t-1} r_t + w_t$. Both extensions of equation 1 yield a distribution of y_t with an asymptotic Pareto tail as $t \to \infty$ (Nair et al. 2022). For a detailed overview for these stochastic growth models see Yakovenko & Rosser (2009). Altogether, simple theoretical models provide support for asymptotic Pareto and LogNormal tails in income and wealth distributions. Benhabib & Bisin (2018) provides a review of theoretical models of wealth accumulation based on optimizing agents and Axtell & Farmer (2022) review agent-based models, many of which exhibit heavy-tailed income and wealth distributions.

2.2 Testing the Power Law Hypothesis

The empirical literature on testing wealth and income data for power law tails is characterized by three general features: First, it heavily focuses on the type I Pareto distribution. Second, many contributions rely on rich lists compiled by journalists whose data sources and underlying

[2] As long as $\log(y_t)$ has finite variance.

methodology are often hard to verify. And third, graphical methods are still widely used despite their shortcomings (Cirillo 2013) and only more recent contributions rely on formal statistical tests. Table 1 provides an overview.

For the United States Levy & Solomon (1997) were the first to use *Forbes 400* data and conclude, based on graphical inspection of the data, that the type I Pareto distribution appears to be a good fit except for deviations at rounded levels of wealth. Klass et al. (2006) report similar results using essentially the same approach. Dragulescu & Yakovenko (2001) also argue in favour of a power law tail, however, they differ from the two previous studies in two important ways. First, they use the Inland Revenue's reconstructed wealth data, rather than rich lists and secondly, they only analyse the income distribution for 50 US states, but do not report results for wealth. Brzezinski (2014) uses goodness of fit tests rather than purely graphical methods and finds that a type I Pareto distribution is only supported in seven out of 24 years of US rich list data. Chan et al. (2017) also finds the type I Pareto distribution to be a poor fit to US rich list data over the 1988-2006 period. Capehart (2014) argues that the rejection of power laws in rich list data is (at least partially) due to measurement error. He shows that different magazines can report wildly different results for the same individuals. In addition, rich list authors frequently round monetary estimates of wealth, leading to what is called a heaping effect where several households worth 500 million can be observed, but none worth 476.23 million, e.g. Besides the heaping effect that was already described by Levy & Solomon (1997), other major issues concern the lists' completeness, consistency and selectivity especially at lower ranks, i.e. magazines tend to include families which they consider of interest to their readers. This is barely surprising since the purpose of compiling these rich lists for media outlets is hardly to create the most accurate listings of wealthy individuals and families, but rather to entertain their readers. Most recently, Charpentier & Flachaire (2022) argue that the type I distribution is highly sensitive to the choice of the cut-off as well as deviations from strict power law behaviour. They address this problem by using Beirlant et al.'s (2009) Extended Pareto Distribution, applying it to 2013 US wealth data. They report a better fit of the Extended and Generalised Pareto distributions relative to type I Pareto, however, their results are based on graphical inspection only.

It is not only for the US where the literature presents a mixed set of evidence on power law tails in the wealth and income distribution. Klass et al. (2007) for example use a global rich list and argue in favour of power laws while Brzezinski (2014) and Ogwang (2013) also use global rich list data but employ formal goodness of fit tests and reject the power law hypothesis. Bach et al. (2019) apply a graphical analysis of rich list data for Germany, France and Spain and argue in favour of power law tails in the wealth distribution of these countries. Jagielski et al. (2017) use tax data to analyse the Norwegian wealth and income distributions. They find a slope coefficient of $\alpha = 1.5 \pm 0.3$ for the richest 100 Norwegian individuals for 2010-2013 and, based on graphical inspection, argue in favour of power law tails. Jayadev (2008) is one of the few studies which uses survey data, in this case for India. He fits a type I Pareto distribution and finds shape parameter estimates of around 2 and argues in favour of a Pareto tail based on R^2 but does not employ formal goodness-of-fit tests. Sinha (2006) support power laws for the top of the wealth distribution using 2002-2004 rich list data for India. Ding & Wang (2007)

use Chinese rich list data (2003-2005) and obtain the same finding. In contrast, Ogwang (2011) uses data for the 100 wealthiest Canadians compiled by *Canadian Business* and finds mixed evidence for 1999-2008 data. Similarly Campolieti (2018) cannot distinguish the type I Pareto distribution from other heavy tailed distributions for 1999-2017 Canadian data. Lastly, Jenkins (2016) uses strict and asymptotic power law distributions on UK income tax data and argues in support of the asymptotic rather than the strict power law. His goodness of fit analysis is based on likelihood ratio tests and probability plots.

This brief review of the empirical literature highlights the amount of disagreement when it comes to the question of power laws in wealth and income data. One "camp" argues strongly in favour but relies heavily on graphical eyeballing of the data. The other "camp" uses formal goodness of fit tests to point out the often poor fit of strict power laws in the data.

3 Methodology and data

In this section we define key concepts and terminology, discuss how we use measures of aggregate as well as within-tail concentration to compare alternative tail distributions, motivate the choice of distributions which we fit to the data, discuss our approach to estimation and goodness of fit testing and present the data we use.

3.1 Thick-, fat-, and heavy-tailed distributions

This subsection provides the definition of *heavy-tails* used throughout the paper and clarifies how this definition relates to the families of *thick-tailed* distributions. Some authors refer to the latter as *fat-tailed* distributions, a terminology which we do not use.

We follow Stachurski & Sargent (2022) and Nair et al. (2022) and define *heavy-tailed* distributions as any distribution whose tail decays slower than the Exponential distribution:

$$\lim_{x\to\infty} \frac{\bar{F}(x)}{e^{-\lambda(x-\mu)}} = \infty. \tag{2}$$

where $\bar{F}(x)$ is the complimentary distribution function[3]. Therefore, the Exponential distribution and any distribution with a tail decaying faster than the Exponential distribution are defined as *light-tailed*. The limit of the ratio of two distribution functions also allows us to compare the tails' heaviness of any two distributions. In general, the distribution $\bar{F}_X(z)$ is heavier than $\bar{F}_Y(z)$ iff:

$$\lim_{z\to\infty} \frac{\bar{F}_X(z)}{\bar{F}_Y(z)} = \infty. \tag{3}$$

Within the group of *heavy-tailed* distributions, we distinguish the subset of strict power laws: the type I Pareto distribution. The label 'strict power law' stems from the fact that the complimentary distribution function follows a power law:

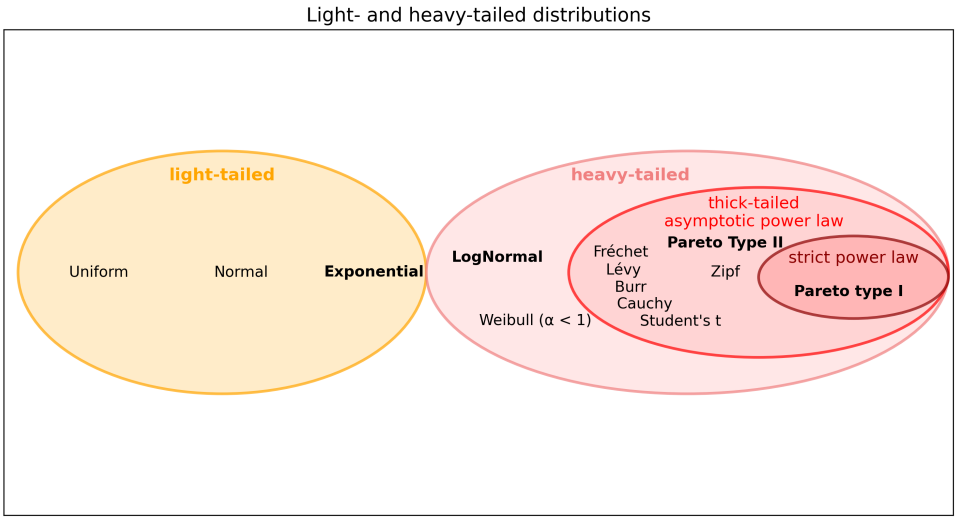
$$\bar{F}(x) = \left[\frac{x}{\sigma}\right]^{-\alpha} \tag{4}$$

[3]For brevity we use the term complimentary distribution function or even just distribution function to refer to the complimentary cumulative distribution function (CCDF). The CCDF is also called the survival function. It holds that $\bar{F}(x) = 1 - F(x) = Pr(X > x)$ where $F(x)$ is the cumulative distribution function or CDF.

Table 1: Summary of empirical papers testing power laws for wealth distributions

Paper	**y/w**	**Years**	**Country**	**Data**	**Tested**	**Method**	**Result**
Bach et al. 2019	w	2013, 2016	FR, ES, DE	HFCS, RL	Type I	Graphical	Type I supported
Brzezinski 2014	w	1988-2012	Global, US, CN	RL	Type I, LogN, Exp.	KS	65% of samples reject Type I
Capehart 2014	w	2013	US	RL	Type I	Mod. KS	Rej. questioned
Campolieti 2018	w	1999-2017	CA	RL	Type I, LogN, Weibull	KS	Inconclusive
Chan et al. 2017	w	1988-2016	US	RL	Type I, Beta Pareto	AD	Type I rejected, Beta supported
Charpentier & Flachaire 2022	w, y	2012, 2013	ZA, US	LIS, LWS	Type I, GPD, EPD	Graphical	EP or GPD preferred
Dragulescu & Yakovenko 2001	w+y	1994-1998, 1998	UK (w+y), US (y)	Tax	Type I, Exp.	Graphical	Type I supported
Jagielski et al. 2017	w+y	2010-2013	NO	Tax	Type I	Graphical	Type I supported
Jayadev 2008	w	1991, 2002	IN	Survey	Type I	R^2, Graphical	Type I supported
Jenkins 2016	y	96, 01, 07, 10	UK	Survey, tax	Type I & II	LR tests, PP plots	Type II preferred
Klass et al. 2006	w	1998-2003	US	RL	Type I	Graphical	Type I supported
Klass et al. 2007	w	1988-2003	Global	RL	Type I	Graphical	Type I supported
Levy & Solomon 1997	w	1996	US	RL	Type I	Graphical	Type I supported
Ding & You-Gui 2007	w	2003-2005	CN	RL	Type I		Type I supported
Ogwang 2011	w	1999-2008	CA	RL	Type I	KS, AD	inconclusive
Ogwang 2013	w	2000-2009	Global	RL	Type I	KS, AD, χ^2	PH rejected
Sinha 2006	w+y	2002-2004	IN	RL	Type I	Graphical	Type I supported
Teulings & Toussaint 2024	w	2001-2021	Global	RL	Type I, Weibull	Own test statistics	Truncated Weibull pref.

Figure 1: Light- and heavy-tailed distributions



where σ is the scale parameter and α is the shape parameter. Strict power laws are heavy-tailed distributions with an especially heavy tail. Their tail heaviness as well as their mathematical simplicity are among the key reasons why they are popular among researchers to model the income and wealth distributions. Furthermore, we distinguish the subset of asymptotic Pareto distributions or asymptotic power laws which includes all distributions with a distribution function of the form:

$$\bar{F}(x) = x^{-\alpha}\ell(x) \tag{5}$$

where $\ell(x)$ is a slowly varying function. As x tends to infinity, slowly varying functions remain asymptotically constant and therefore the tail behaves like a (strict) power law or type I Pareto distribution. Asymptotic power laws are sometimes referred to as *thick-tailed* distributions (e.g. see Benhabib & Bisin 2018), or as *fat-tailed* distributions or as *regularly varying distributions* (Nair et al. 2022). Examples for asymptotic power law distributions include the type II Pareto, the Burr type XIII and III, F and t, Frechet, inverse-gamma (including Levy), and the log-gamma distribution (Beirlant et al. 2004, Table 2.1). Asymptotic power laws are also especially heavy-tailed distributions, but not to the same extent that strict power laws are.

Figure 1 provides an overview over these classifications and several examples belonging to each distribution family. We clearly distinguish light and heavy-tailed distributions and identify strict and asymptotic power laws as especially heavy-tailed subsets within the larger family of heavy tailed distributions. It is important to note that the grouping in Figure 1 is based on the distributions' asymptotic properties as $x \to \infty$.[4] While this is useful in order to compare different distributions irrespective of specific parameter values, we are also interested in the resulting degree of concentration or how the pattern of concentration changes within the tail, for a specific parameterisation which we estimate from the data. For this purpose, we define two concentration measures in the next section.

[4]It means that a distribution (e.g. Pareto type I) which exhibits an asymptotically heavier tail than another (e.g. LogNormal) can have a parameterisation which results in a lower mean or lower within tail concentration.

3.2 Comparing the tails of income and wealth distributions: top shares and within shares

When choosing between competing models for the income or wealth distribution tails, the implied concentration is just as important as any formal goodness of fit test statistic. The population (or aggregate) wealth or income share is a possible measure of aggregate concentration and can be defined as follows: When fitting a distribution $\bar{F}(x)$ to wealth or income data beyond the p^{th} percentile, the share of total wealth/income belonging to households beyond the p^{th} percentile, is given by:

$$ts(p) = \frac{(1-p)N \cdot E[X]}{\text{bot}(p) + (1-p)N \cdot E[X]} \tag{6}$$

where N is the number of households in the population, $E[X]$ is the average wealth or income of the fitted distribution $\bar{F}(x)$ and $\text{bot}(p)$ is aggregate wealth/income of households below the p^{th} percentile which in our case is obtained from SCF data and independent of the fitted distribution. In other words, $ts(p)$ is fully determined by the mean of the fitted distribution once N and $\text{bot}(p)$ are obtained from the dataset to which $\bar{F}(x)$ is fitted to. Therefore, we can simply compare the estimated means of two fitted distributions instead of calculating $ts(p)$ although the latter can be more easily interpreted as a measure of the aggregate concentration implied by the fitted distributions.[5]

While the top share $ts(p)$ provides information about the distribution of income or wealth in the population, it does not provide any information about the distribution within the tail itself (e.g. within the top 1%). For this purpose, we define the within tail share $ws(q,p)$ as the share of wealth or income beyond the p^{th} percentile belonging to households beyond the q^{th} percentile within the first group. For example $ws(q = 0.95,\ p = 0.9)$ is the top 5% share within the top 10%. It can be obtained as:

$$ws(q,p) = \frac{(1-q)\ E[X|X > x(q+p-qp)]}{E[X|X > x(p)]}. \tag{7}$$

The within share, allows us to assess how the pattern of concentration within the tail of the income or wealth distribution changes as we look at higher and higher cut-offs (p). Increasing (decreasing) within shares represent increasing (decreasing) tail concentration. Formally: $\frac{\partial ws(q,p)}{\partial p} > 0$ implies increasing concentration or inequality within the tail and a negative partial derivative implies decreasing concentration within the tail. The mean excess function (Nair et al. 2022, p. 229) enables us to derive parametric expressions for $ws(q,p)$ for each of the distributions we fit to the data. The mean excess function is defined as:

$$m(x) = E[X - x|X > x] = \frac{\int_0^\infty \bar{F}(x+t)dt}{\bar{F}(x)}. \tag{8}$$

It follows that the conditional mean $cm(x) = E[X|X > x]$ and mean excess are related to each other as $cm(x) = m(x) + x$ and therefore:

$$ws(q,p) = (1-q)\frac{cm(q+p-qp)}{cm(p)}. \tag{9}$$

[5] If distribution $\bar{F}(x)$ is fitted to wealth/income data beyond p but we are interested in the aggregate share beyond the u^{th} percentile, we can obtain it by using conditional means: $ts(u) = \frac{(1-u)N \cdot E[X|X>x(u)]}{\text{bot}(p)+(1-p)N \cdot E[X]}$ where $u > p$ and $x(p)$ is the quantile function $x(p) = F^{-1}(X)$.

where $cm(p) = cm(x(p))$ and $x(p)$ is the quantile function $x(p) = F^{-1}(X)$.

Now we have two measures of concentration to compare different parametric distributions fitted to data beyond the p^{th} percentile of income and wealth data. First, $ts(p)$ provides a measure of aggregate concentration and second, the partial derivative of the within tail share $\frac{\partial ws(q,p)}{\partial p}$ provides a measure of how the concentration within the tail changes and indicates whether it is increasing or decreasing within the tail.

3.3 Motivating the choice of distributions

In section 3.1, we introduced the four families of distributions we focus on in our analysis. In this section, we motivate our selection of specific members from each family to fit to the data and compare against each other. We discuss the properties of each distribution, most importantly how the concentration changes within the tail of each distribution. Within tail concentration patterns are not only a way to distinguish and classify distributions but also allow us to check if a certain distribution is at all able to reproduce the patterns observed in the data.

We start with the Exponential distribution from the family of light-tailed distributions. Specifically, we use the shifted version of the latter with the complementary distribution function $\bar{F}(x; \lambda, \mu) = e^{-\lambda(x-\mu)}$ where μ is the location and λ the scale parameter and the support is $x > \mu$. As a light-tailed distribution, the Exponential distribution serves as the anti-thesis to our hypothesis that the US income and wealth distributions follow power laws. Nevertheless, it is the most heavy of the light-tailed distributions. Using the within top share $ws(q, p)$, we can show that the Exponential distribution always exhibits declining within tail concentration:

Proposition 1 If $\bar{F}(x) = e^{-\lambda(x-\mu)}$ then $ws(q, p) = (1 - q) - \frac{(1-q)ln(1-q)}{\lambda\mu+1-ln(1-p)}$ and $\frac{\partial ws(q,p)}{\partial p} < 0$.

Proof. See Appendix A.1.

Moving on to heavy-tailed distributions, we test the fit of the strict power law, i.e., the type I Pareto distribution due to its widespread use in applied work as discussed in sections 1 and 2. The type I Pareto distribution is special in two ways. Firstly, it is scale invariant which means its distribution function satisfies:

$$\bar{F}(cx) = f(c)\bar{F}(x) \tag{10}$$

where $c > 0$ is a constant and f is a continuous function. Nair et al. (2022, p. 34) show that only the type I Pareto distribution has this property. Using the within top share $ws(q, p)$, we can show that as a result of scale invariance, the concentration within the tail does not change as one moves along the tail:

Proposition 2 If $\bar{F}(x) = \left[\frac{x}{\sigma}\right]^{-\alpha}$ then $ws(q, p) = (1 - q)^{\frac{\alpha-1}{\alpha}}$ and $\frac{\partial ws(q,p)}{\partial p} = 0$.

Proof. See Appendix A.2.

Secondly, the type I Pareto distribution is not robust to an additive transformation: Assume X is type I Pareto with $X \sim P_I(\alpha, \sigma)$. A simple additive transformation $Y = X + a$ with $a > 0$ generates a new random variable Y which does not follow a type I Pareto distribution. Instead,

Y is distributed as type II Pareto: $Y \sim P_{II}(\alpha, \sigma, \mu = c)$ where $c = \sigma - a$.[6] This highlights the fragility of the type I Pareto distribution and potentially explains its poor performance when exposed to formal goodness of fit tests. Despite its fragility, it remains a very popular model for the tail of wealth and income distributions due to its simplicity and closed form solutions for its maximum likelihood estimator.

The third distribution we fit to US income and wealth data is the type II Pareto distribution. This choice is motivated from the so-called *peaks-over-threshold* (POT) result which is summarised in the words of Nair et al. (2022, p. 232): "[...] for [asymptotic power law] distributions [...], the sizes of the residual life, i.e., "peaks", beyond a (large) threshold are i.i.d. and well-approximated by a scaled version of a generalised Pareto distribution." In other words, if we assume that our (tail) sample of income or wealth data stems from an asymptotic power law distribution[7], we can model the tail of this distribution as a generalised Pareto distribution. For positive tail indices, this is the same as a type II Pareto distribution. The type II distribution is also more flexibel compared to the type I distribution, e.g., it is robust to linear transformations due to its separate location and scale parameters. Furthermore, the type II distribution allows for increasing as well as decreasing within tail concentration:

Proposition 3 If $\bar{F}(x) = \left[1 + \frac{x-\mu}{\sigma}\right]^{-\alpha}$, then $ws(q,p) = (1-q)^{\frac{\alpha-1}{\alpha}} \cdot \frac{\alpha\sigma+(\alpha-1)(1-p)^{1/\alpha}(1-q)^{1/\alpha}(\mu-\sigma)}{\alpha\sigma+(\alpha-1)(1-p)^{1/\alpha}(\mu-\sigma)}$ *and* $\frac{\partial ws(q,p)}{\partial p} > 0$ iff $\mu > \sigma$ and $\frac{\partial ws(q,p)}{\partial p} < 0$ iff $\mu < \sigma$.

Proof. See Appendix A.3.

Depending on the parameterisation, a type II distribution can yield larger or smaller expected values compared to a type I distribution even for the same tail index (α) and therefore the same (asymptotic) tail heaviness. As argued above, expected values uniquely define the aggregate top share ($ts(p)$) and therefore aggregate wealth concentration.

Proposition 4 If $\bar{F}_X(x; \alpha, \mu, \sigma) = \left[1 + \frac{x-\mu}{\sigma}\right]^{-\alpha}$ and $\bar{F}_Y(y; \alpha, \mu) = \left[\frac{y}{\mu}\right]^{-\alpha}$, then $E[X] > E[Y]$ iff $\sigma > \mu$ and $E[X] < E[Y]$ iff $\sigma < \mu$.

Proof. See Appendix A.4.

The additional flexibility allowing for increasing and decreasing concentration within the tail comes at the price of an additional parameter which requires more complicated estimation procedures which are discussed in the subsection 3.4.

Finally, we also test the fit of the LogNormal distribution to US wealth and income data. The motivation behind this choice is threefold. Firstly, its theoretical appeal, given that simple models of wealth accumulation yield LogNormal distributions (Sargan 1957). Secondly, the LogNormal serves as an example for a heavy-tailed distribution which is not thick-tailed or

[6] We start with the complementary distribution function $\bar{F}(x; \alpha, \sigma_X) = \left(\frac{x}{\sigma_X}\right)^{-\alpha}$ and then define $Y = a + X$. We obtain the distribution function for Y as $\bar{F}(y; \alpha, \sigma_X) = \left(\frac{x+a}{\sigma_X}\right)^{-\alpha}$ and if we define $a = \sigma_X - c$ we obtain a type II Pareto distribution: $\bar{F}(y; \alpha, \sigma_X, c) = \left(1 + \frac{x-c}{\sigma_X}\right)^{-\alpha} \sim P_{II}(\alpha, \sigma_X, \mu_Y = c)$.

[7] These include the Burr type XIII and III, F and t, Frechet, inverse-gamma (including Levy), and the log-gamma distributions

considered an asymptotic power law and thus a lighter heavy-tailed distribution. Thirdly, the LogNormal distribution remains a highly popular choice for parametric models of wealth and income data. While there are good reasons for testing its fit against the data, distinguishing the LogNormal from the asymptotic and strict power laws can require data spanning several orders of magnitude. The tail in the Survey of Consumer Finances wealth data spans three orders of magnitude which allows us to distinguish the two.

Proposition 5 If $X \sim LogN$ with $\bar{F}(x;\mu,\sigma,\delta) = \Phi\left(\frac{\mu - log(x-\delta)}{\sigma}\right)$, then $ws(q,p) = \frac{(E[X]-\delta)\Phi\left[\sigma - \Phi^{-1}(p+q-qp)\right]+(1-q)(1-p)\delta}{(E[X]-\delta)\Phi[\sigma-\Phi^{-1}(p)]+(1-p)\delta}$.

Proof. See Appendix A.5.

Unlike the three previous distributions, the LogNormal distribution allows for hump-shaped patterns of the within tail share, i.e. it can accommodate tails whose concentration is first increasing but after some point decreasing (see Appendix B). Table 2 provides an overview of all four distributions.

It is worth noting that Blanchet, Piketty & Fournier's (2022) concept of the inverted Pareto coefficient, defined as $b(p) = \frac{E[X|X>x(p)]}{x(p)}$, is closely related to the conditional mean function and the within tail share. Specifically, it holds that $b(p) = \frac{cm(p)}{x(p)}$. For comparison, the within tail share is defined as $ws(q,p) = (1-q)\frac{cm(q+p-qp)}{cm(p)}$. We can show for all analysed distributions, except for the LogNormal, that increasing (decreasing) within tail shares coincide with increasing (decreasing) inverted Pareto coefficients. See Appendix B for a detailed discussion. While Blanchet, Piketty & Fournier (2022) are interested in obtaining estimates of the entire distribution and simply assume a type II Pareto distribution beyond the last interpolation point, this paper tests this assumption of an asymptotic power law tail.

Table 2: Tested distributions

Dist	$\bar{F}(x)$	support	$E[X]$	$cm(p) = E[X\|X > x(p)]$
Exp	$e^{-\lambda(x-\mu)}$	$x > \mu$	$\mu + \frac{1}{\lambda}$	$\mu + \frac{1-ln(1-p)}{\lambda}$
Pareto I	$\left[\frac{x}{\sigma}\right]^{-\alpha}$	$x > \sigma$	$\frac{\alpha}{\alpha-1}\sigma$	$\frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}$
Pareto II	$\left[1+\frac{x-\mu}{\sigma}\right]^{-\alpha}$	$x > \mu$	$\mu + \frac{1}{\alpha-1}\sigma$	$\mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}$
LogN	$\Phi\left(\frac{\mu-\log(x-\delta)}{\sigma}\right)$	$x > \delta$	$\delta + \exp(\mu + \frac{\sigma^2}{2})$	$\frac{[E[X]-\delta]\Phi\left(\sigma+\Phi^{-1}(p)\right)}{1-p} + \delta$

Note: Φ is the CDF of the three parameter Standard Normal Distribution. $x(p)$ is the quantile function $x(p) = F^{-1}(X)$.

3.4 Estimation

This section briefly discusses how the four distributions are fitted to the data. First, we fit both the LogNormal and the Exponential distribution by quasi Maximum Likelihood where we use the complex survey weights (w_i) to expand the likelihood function for each observation:

$$\max_{\theta} \mathcal{L}(\theta|x_i, w_i) = \prod_{i=1}^{n} f(\theta|x_i)^{w_i} \tag{11}$$

where θ is the parameter vector and $f()$ the probability density function of either the LogNormal or the Exponential distribution. Since we are fitting all distributions to several different percentile cut-offs, the location parameter is set equal to the percentile threshold. See Tables A2 and A4 for the selected quantiles of the US wealth and income distribution. Second, we fit the type I Pareto distribution by means of a log-rank-log-wealth regression which allows us to take into account the SCF's exclusion of the richest 400 households on the *Forbes* list by applying a rank correction procedure (Wildauer & Kapeller 2022). We also incorporate Gabaix & Ibragimov's (2011) bias correction as described in Wildauer & Kapeller (2021) resulting in the following linear regression which we estimate by OLS:

$$\log(r_i) = c - \alpha \cdot \log(w_i) + \epsilon_i \tag{12}$$

where x_i is wealth (income) of the i^{th} largest observation and r_i is the corresponding population rank using the survey weights: $r_i = \left(\sum_{1 \leq j \leq i} w_j\right) - 0.5w_i + u$. We set $u = 400$ in order to correct for the exclusion of the 400 most affluent households from the SCF sampling frame. Lastly, the Pareto type II distribution has been avoided in wealth inequality (as well as other) research, presumably because it can be tricky to estimate: while Maximum Likelihood estimators for $\alpha < 1$ do not exist, they do for $1 < \alpha < 2$, but simulations have shown that their asymptotic efficiency requires sample sizes in excess of 500 observations (Hosking & Wallis 1987). To address this, we are employing an elemental percentile method (EPM) estimator following Castillo & Hadi (1997), which we refer to as the CH estimator from here on. This is a relatively recent development in the wealth and income distribution literature (Kennickell 2024, Heck et al. 2020). A key benefit compared to the well-known Hill estimator is that the CH approach is robust to location shifts. The underlying idea of the CH estimator is straightforward: compare a data-based estimate of $\bar{F}(x_i)$ with its theoretical counterpart. The distribution parameters are then chosen such that they minimise the difference between the theoretical distribution function and its empirical counterpart. We define the bias corrected empirical estimate of $F(x_i)$ as:

$$F(x_i)_{BC} = \frac{\left(\sum_{1 \leq j \leq i} w_j\right) - 0.5w_i + u}{N} \tag{13}$$

where $N = \sum_{i=1}^{n} w_i$ is the sum total of the complex survey weights, arranged in descending order (reversed order statistics)[8] and $u = 400$ is the correction for the omission of the Forbes 400 from the SCF sampling frame (Wildauer & Kapeller 2022). Using the shorthand notation $\log\left(\bar{F}(x_i)_{BC}\right) = C_i$, we can pick a pair of two data points (x_i, x_j) and by equating the empirical and theoretical survival function obtain a system of two equations in two variables $(\alpha,\ \sigma)$:

$$-\frac{1}{\alpha} C_i = \log\left(1 - \frac{x_i - \mu}{\sigma}\right) \tag{14}$$

$$-\frac{1}{\alpha} C_j = \log\left(1 - \frac{x_j - \mu}{\sigma}\right) \tag{15}$$

[8]The definition for the CDF as in equation 13 is equivalent to $F(x_i)_{AV} = \frac{\left(\sum_{1 \leq j \leq i} w_j\right) - 0.5w_i}{N}$. The latter is the formulation used by Gabaix & Ibragimov (2011).

where $\mu = \min(x)$ and we can readily eliminate α to obtain:

$$\frac{\log\left(1 - \frac{x_j - \mu}{\sigma}\right)}{C_j} = \frac{\log\left(1 - \frac{x_i - \mu}{\sigma}\right)}{C_i} \tag{16}$$

Castillo & Hadi (1997) show that for this equation a solution exists over the interval $(\sigma_0, 0)$ if $x_i < \frac{C_i X_j}{C_j}$ and over the interval (x_j, σ_0) if $x_i > \frac{C_i X_j}{C_j}$ where $\sigma_0 = \frac{X_j X_i (C_j - C_i)}{C_j X_i - C_i X_j}$. The solution can be found numerically by means of bisection. The resulting estimate $\hat{\sigma}$ can be used to obtain the second parameter estimate $\hat{\alpha} = C_i / \log\left(1 - \frac{x_i - \mu}{\hat{\sigma}}\right)$. This algorithm is then applied to all pairs (x_i, x_j) in the dataset which yields $\frac{n(n-1)}{2}$ estimates of $(\hat{\alpha},\ \hat{\sigma})$. The final estimates are the median values across all $\frac{n(n-1)}{2}$ pairs $(\hat{\alpha},\ \hat{\sigma})$.

The CH estimator for fitting the type II distribution is computationally demanding. For example our largest sample consists of $n = 1532$ observations (91^{st} percentile in wave 2019) and thus the CH estimator requires solving $\frac{n(n-1)}{2} = 1.17 \cdot 10^6$ nonlinear systems of equations for this single percentile-year cell. Repeating this exercise for 999 replicate weights and for 176 year-percentile cells yields 350 billion equations. Therefore, we relied on parallel computing (Python's joblib package) and more importantly on just-in-time (JIT) compilation (Python's numba package).

3.5 Goodness of fit testing

After fitting distributions to US wealth and income data, we assess their fit. Chu et al. (2019) reviews and compares dozens of goodness-of-fit tests for Pareto distributions. We follow their conclusion that the Kolmogorov-Smirnov (KS) family of tests (e.g., Clauset et al. 2009, Capehart 2014) exhibit the highest power and use the Cramér–von Mises (CvM) test. The crucial difference between CvM and KS is that the test statistic of the former is based on the difference between the empirical and theoretical CDF along the entire sample rather than just at a single data point which exhibits the most extreme difference.

Since the SCF does not publish the details of their sampling and, most importantly, their extensive oversampling procedure, we cannot obtain correct p-values for our goodness of fit tests[9]. Therefore, we simply calculate the CvM test statistic for each estimated distribution and compare them in a pair-wise manner. The distribution with the smaller CvM test statistic exhibits the better fit. In other words, we are assessing the *relative* goodness of fit of one distribution compared to another.

We take the SCF's complex survey weights into account and follow Wildauer & Kapeller (2021) and define the CvM test statistic as:

$$T = n\left[\frac{1}{3}F(x_1)^3 + \frac{1}{3}\sum_{i=1}^{n-1}\left[(U_i - F(x_i))^3 - (U_i - F(x_{i+1}))^3\right] + \frac{1}{3}\bar{F}(x_n)\right] \tag{17}$$

[9]SCF replicate weights represent the random variation in sampling from the true, *unknown* US wealth distribution. Obtaining a valid p-value for testing the null hypothesis that the US wealth distribution follows a Pareto tail would require assessing the random variation in sampling from a *Pareto* distribution. This information is not provided by replicate weights and because sampling details are not public, we cannot simulate it. Therefore, we cannot obtain valid p-values or critical values for our goodness of fit tests.

where $U_i = 1 - \frac{\sum_{1 \leq j \leq i} w_{j-1}}{N}$ is the empirical distribution function and x_i is the i^{th} largest value of the sample, w_i is the corresponding complex survey weight, n is the sample size and $F(x_i)$ is the distribution function under the null hypothesis. We then proceed in the following way: For example for wealth data beyond $p = 0.99$ in wave 2019, the Exponential and the type II Pareto distributions are fitted to the data. Two CvM test statistics are calculated based on the estimated parameters T_{exp} and T_{typeII}. If $T_{\text{typeII}} < T_{\text{exp}}$, we conclude that for the 99^{th} percentile 2019 cell the type II Pareto distribution exhibits a better fit to the data than the Exponential distribution.

3.6 Data

We use household wealth and income data from the Survey of Consumer Finances (SCF), starting with the 1989 wave. The SCF exhibits several unique features which make it especially suitable for our task at hand. Firstly, by heavily oversampling wealthy households using capital income tax information (Bhutta et al. 2020, Kennickell 2008), the SCF provides unique coverage of the tails of the US income and wealth distributions. Since the US does not administer a general net wealth tax, wealth estimates based on income tax capitalisation or estate tax data are the only alternative sources on the US wealth distribution. The SCF uses the former in its sample design and thus provides a data set of similar quality compared to purely tax-based estimates while being less reliant on choosing correct capitalisation or population multipliers. Secondly, in comparison to rich list or lower quality survey data, the SCF's coverage of the wealth distribution is unmatched. In this paper, we test for Pareto tails beyond the 91^{st} percentile which for net wealth starts between \$0.7 (1992) and \$1.4 million (2019). Net wealth of the most affluent observation ranges from \$310 million (1989) to \$1.9 billion (2019), all values in 2019 Dollars. This means the SCF's wealth data spans more than three orders of magnitude and covers the entire US population compared to one order of magnitude and merely 0.003% of the population covered by the *Forbes 400* list.

A key limitation of the SCF is that billionaires featured on the *Forbes 400* list are excluded from the public sample due to privacy concerns. This exclusion of the top tail of the wealth distribution from publicly available data is one of the reasons why researchers interested in the Pareto hypothesis explore and consider the *Forbes 400* list despite its much narrower scope. However, this problem can be dealt with simply and effectively by correcting the ranks (cumulative weights) of the survey (Wildauer & Kapeller 2022). Using this rank correction approach allows us to exploit the much richer information embedded in the SCF while being able to conduct unbiased tests of the Pareto hypothesis.

Hanna et al. (2018) provide an excellent overview over the details of the SCF in addition to the notes published by FED staff (Bricker et al. 2020). We use all five implicates of the data; final results are averages taken at the last step of the analysis. As the publicly available version of the SCF does not include the details of the sampling structure, we use the set of 999 replicate weights to assess the sampling-based variation in our results.

4 Results

In this section we first present the goodness of fit results for the income and wealth SCF data and then analyse the shape of the estimated distributions along the tail. We also asses the statistical significance of our results using the 999 sets of SCF replicate weights.[10] Finally we use our results to discuss the length of the tail, or in other words beyond which percentile a parametric distribution should be fitted to the data.

4.1 Testing for power laws in the US wealth and income distribution

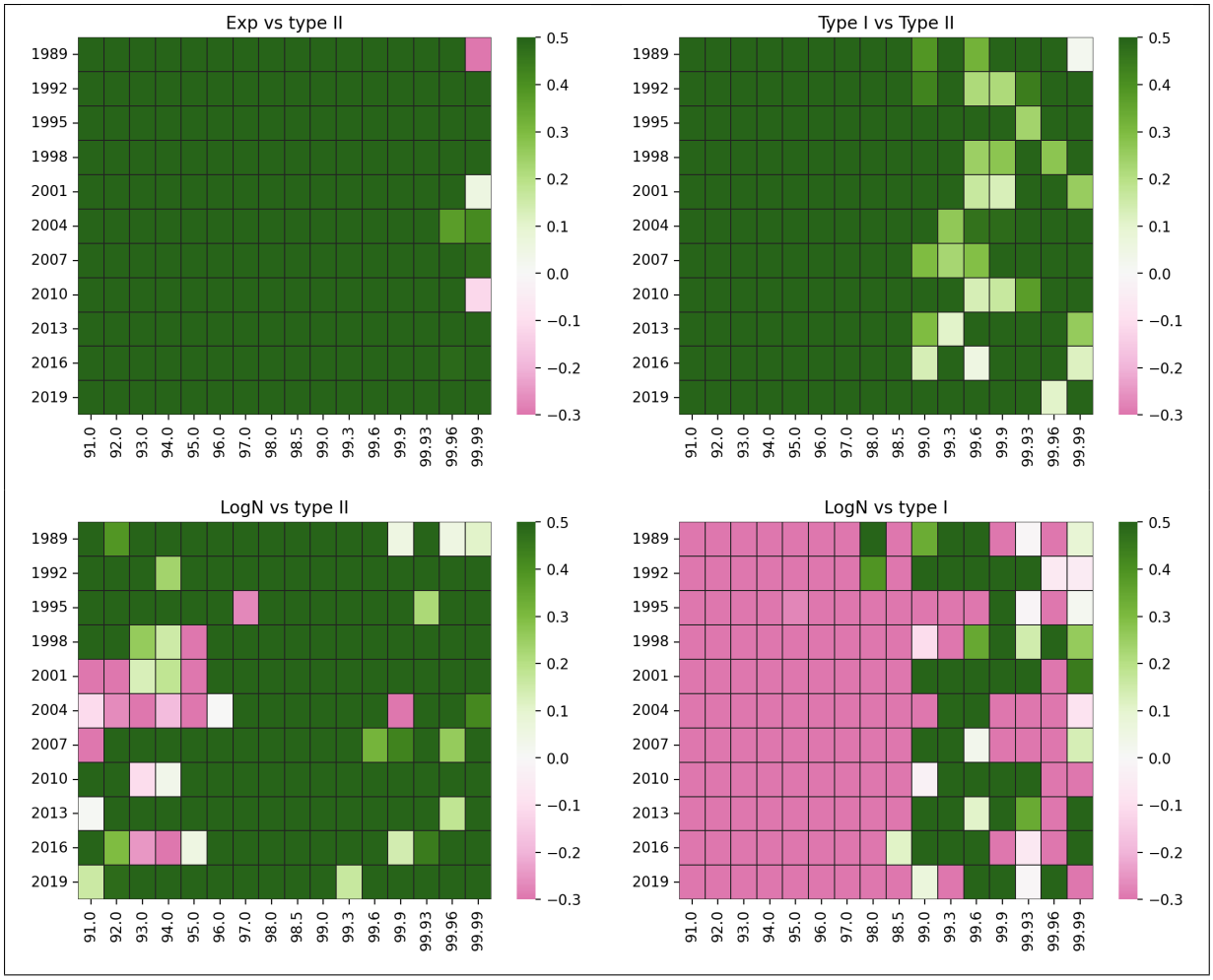
As we are comparing distributions in a pair-wise manner, our results are organised accordingly. Tables 3 and 4 present the differences of the CvM test statistics of four distribution pairs, for wealth and income respectively. Results are shown for all available years (y-axis) and percentile ranks (p) (x-axis) beyond which the distributions where fitted to the data. First, we discuss the fit of the type II Pareto distribution relative to the Exponential distribution (top left heat maps in Tables 3 and 4). Positive values in shades of green indicate that the CvM test statistic for the fitted type II Pareto distribution is smaller and thus exhibits a better fit than the CvM test statistic based on the Exponential distribution. The heat maps in both tables show that the type II Pareto provides a superior fit for all except two cutoff-year cells in the case of wealth and all but five in the case of income. We will show in section 4.3 that the 99.99^{th} percentile 2010 wealth cell is not statistically significant. This means the data strongly rejects the hypothesis that the tails of the income or wealth distribution follow an Exponential distribution.

Second, we compare the fit of the type I against the type II Pareto distribution for wealth and income data respectively (top right heat maps in of Tables 3 and 4). Positive values in shades of green indicate a smaller CvM test statistic and thus better fit for the type II distribution. In this pairing, the type II distribution exhibits a consistently superior fit relative to the type I distribution for wealth and income. Since the type I distribution is the only distribution exhibiting constant within tail concentration (see section 3.3), rejecting the type I distribution also means rejecting constant tail concentration, or put differently rejecting the scale invariance property of strict power laws. We will explore the patterns of within tail concentration of the US income and wealth distribution in more detail in the next subsection. Furthermore, this finding is consistent with the rejection of strict power laws in the existing literature (see section 2). Nevertheless, the dominance of the type II distribution is weaker beyond the 99^{th} percentile, which lends some support to modelling this section of the data as a type I distribution as a second best option if for example fitting a type II distribution is not feasible due to the more demanding estimation approach.

Next, the heat maps in the bottom left of Tables 3 and 4 compare the LogNormal distribution with the type II Pareto distribution. They report a consistently better fit of the type II distribution, especially beyond the 96^{th} percentile in the case of wealth data. For the income distribution, the type II distribution also consistently outperforms the LogNormal, especially

[10] See Kennickell & Woodburn (1999), and Hanna et al. (2018) for details on the use of the SCF's replicate weights.

Table 3: Relative fit of Pareto tails for US wealth data



Interpretation of top left panel "Exp vs type II": Cells in green (pink) indicate better (worse) fit of type II Pareto distribution compared to Exponential distribution. Each cell contains the difference in CvM test statistics $d = T_{\text{exp}} - T_{\text{typeII}}$. For a definition of the CvM test statistic see equation 17. Green (pink) values indicate $d > 0$ ($d < 0$). Equivalent interpretation of other distribution pairs. Estimated parameters are presented in Appendix D.

below the 99.6^{th} wealth percentile. We interpret these results as strong support in favour of an asymptotic Pareto tail in the US wealth and income distributions. Furthermore, these results show that the asymptotic Pareto tail stretches significantly beyond the richest 400 families on the *Forbes* list, not only for the wealth but also for the income distribution.

Finally, the bottom right of Table 3 shows that the type I distribution is consistently outperformed by the LogNormal below the 99^{th} wealth percentile. In contrast, the LogNormal distribution struggles to adequately describe the tail beyond the top 1% wealthiest households in the United States. This reinforces the observation from above that for the top 1% a type I Pareto model, can serve as a second best choice in case a type II model cannot be fitted. The bottom right of Table 4 presents the same analysis for the US income distribution. The results are inconclusive as neither the type I nor the LogNormal distribution provides a consistently better fit than the other distribution across years or thresholds. Therefore, our data on US household income does not allow us to distinguish between the type I and LogNormal

Table 4: Relative fit of Pareto tails for US income data

Interpretation of top left panel "Exp vs type II": Cells in green (pink) indicate better (worse) fit of type II Pareto distribution compared to the Exponential distribution. Each cell contains the difference in CvM test statistics $d = T_{\text{exp}} - T_{\text{typeII}}$. For a definition of the CvM test statistic see equation 17. Green (pink) values indicate $d > 0$ ($d < 0$). Equivalent interpretation of other distribution pairs. Estimated parameters are presented in Appendix E.

distribution.

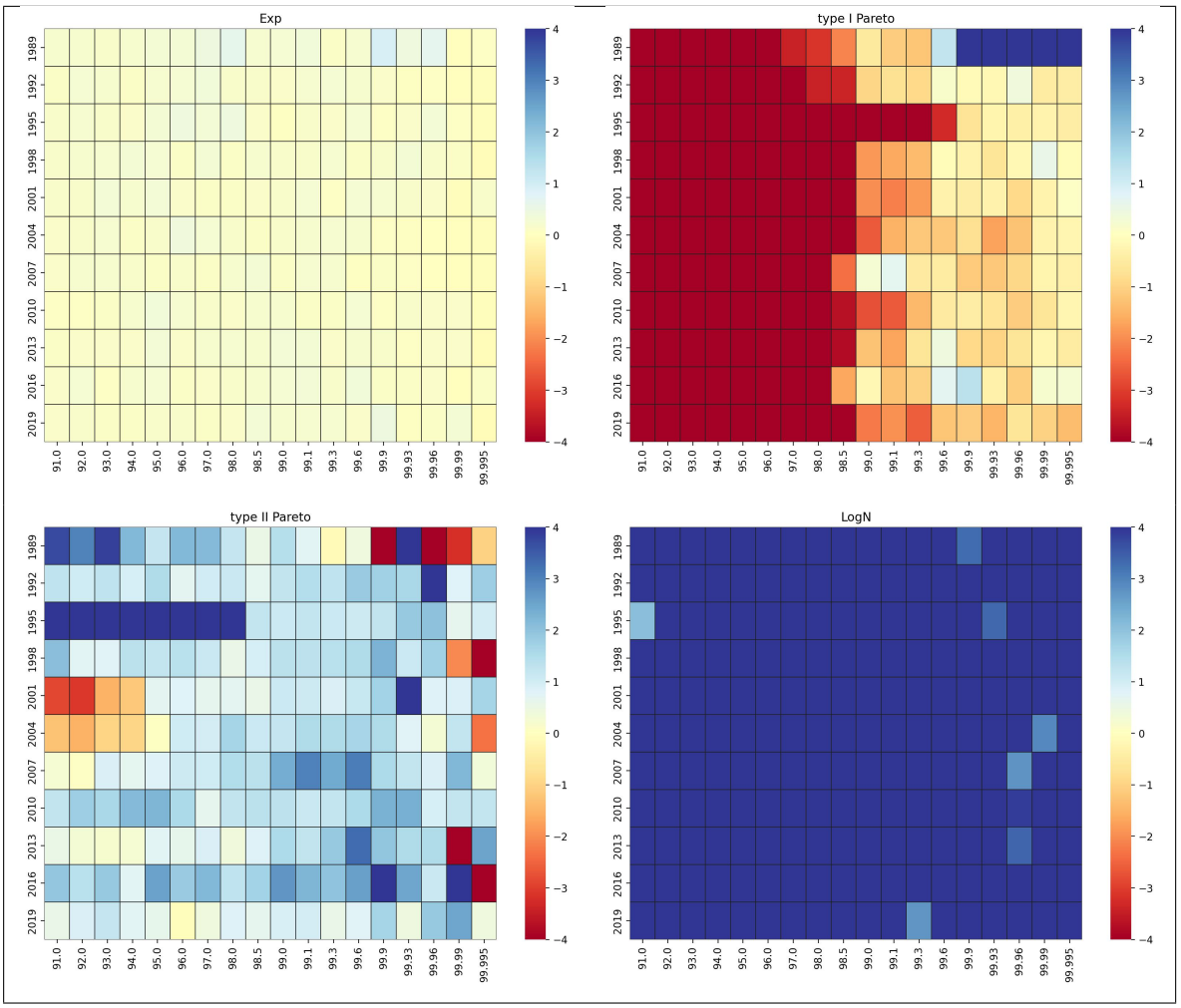
Overall, we find overwhelming support for asymptotic power laws in the US income and wealth distributions since the type II Pareto distribution yields consistently superior fits compared to the tested alternatives. It is worth emphasizing that a type II Pareto tail can emerge from any of the distributions within the family of asymptotic Pareto distributions which include the Burr types XIII and III, the F- and t-, as well as the Fréchet, inverse-Gamma (including Levy), and the LogGamma distributions. Our findings support the practice of modelling the tail of income and wealth distributions as power laws but highlights their asymptotic nature. For the right most part of the tail, such as the top 1%, a strict power law can be a feasible approximation, especially when a lack of data hinders fitting the more demanding type II model. Nevertheless, our results highlight that strict power laws are too fragile to serve as general tail models and are rejected in favour of their asymptotic cousins. In summary, our results support the hypothesis that the US income and wealth distributions exhibit thick tails (i.e. asymptotic

Pareto tails) rather than the alternatives of light or heavy tails[11].

4.2 Aggregate and within tail concentration

The previous section established the superior fit of the type II Pareto distribution to US wealth and income distribution tails. Next, we go beyond goodness of fit measures and compare the concentration of aggregate wealth (measured by top wealth shares, see equation 6) as well as the concentration within the tail (measured by within tail shares, see equation 9) based on the estimated distributions.

Table 5: Relative aggregate top wealth shares

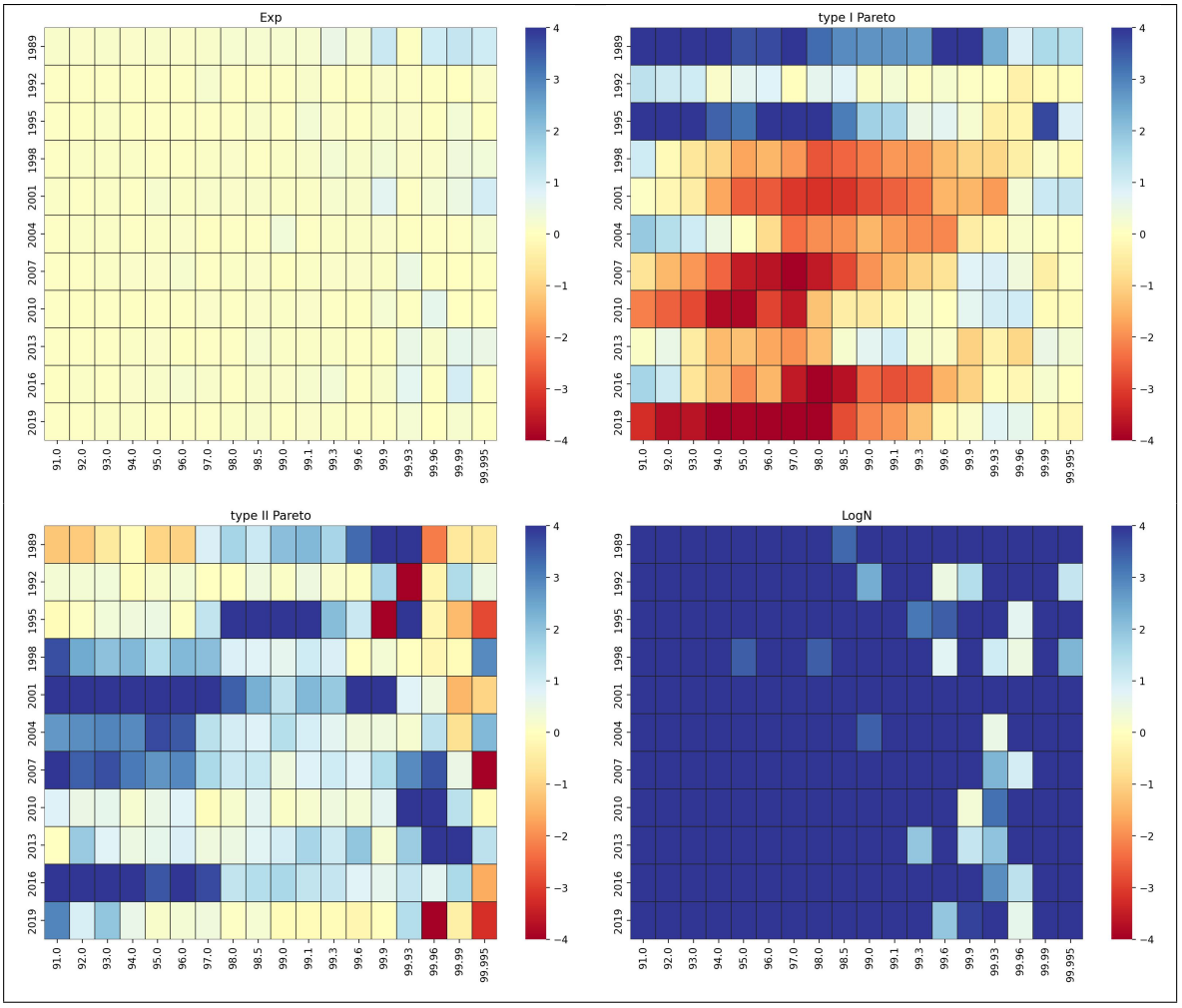


Each percentile-year cell shows the difference between the top $1-p\%$ wealth share based on the estimated distribution ($ts(p)$, see equation 6) and the top $1-p\%$ wealth share calculated based on the raw data. For example the 1995-99.3 cell for the type I Pareto distribution cell with a value of about -4 means that the top 0.7% share based on the type I Pareto distribution is 4 percentage points lower compared to raw SFC data.

Tables 5 and 6 present the differences in top wealth shares (ts(p), see equation 6) obtained from the estimated distributions relative to the raw SCF data. Blue (red) cells indicate higher (lower) top wealth shares compared to the raw data. Top shares based on raw SCF data are presented in Tables A6 and A7 in the Appendix. The heat maps in the top left of Tables 5 and 6

[11] Meaning heavy-tails which are not also thick-tailed, see figure 1.

Table 6: Relative aggregate top income shares

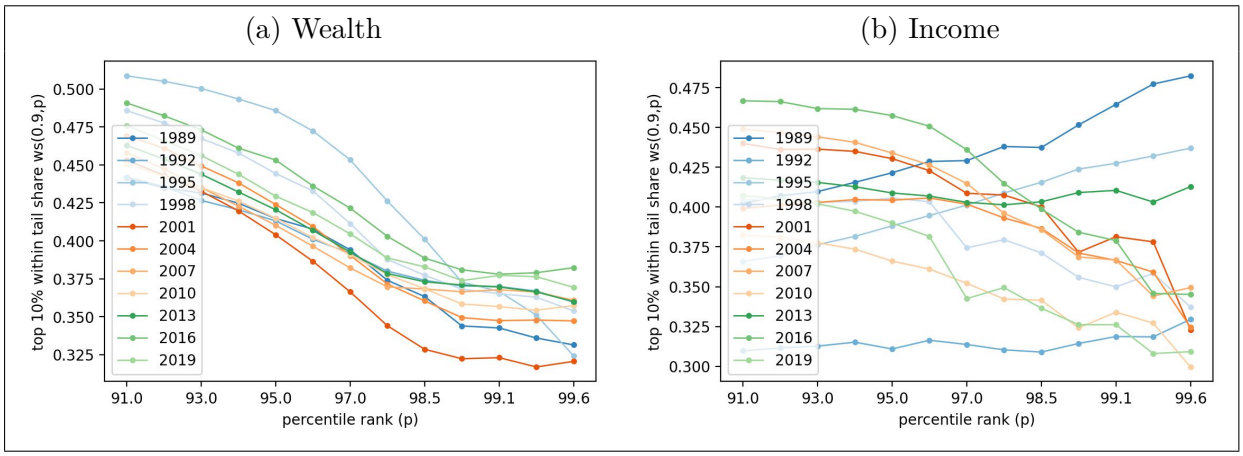


Each percentile-year cell shows the difference between the top $1-p\%$ income share based on the estimated distribution ($ts(p)$, see equation 6) and the top $1-p\%$ income share calculated based on the raw data. For example the 2019-98 cell for the type I Pareto distribution cell with a value of about -4 means that the top 2% share based on the type I Pareto distribution is 4 percentage points lower compared to raw SFC data.

show that the Exponential distribution consistently yields top shares very close to those obtained from raw survey data. In contrast, the type I distribution (top right heat map in both tables) consistently and substantially underestimates aggregate shares especially if fitted below the 99^{th} wealth percentile; further evidence against its usefulness below the rightmost tail of the distribution. The type II Pareto distribution (bottom left) yields slightly higher top wealth shares compared to the raw data, especially for wealth. Finally, the LogNormal distribution (bottom right of Tables 5 and 6) consistently and substantially overestimates aggregate concentration. These higher top wealth shares for the LogNormal emerge despite the fact that both the type I and type II Pareto distributions exhibit asymptotically heavier tails compared to the LogNormal distribution. What appears to be a paradox at first, is simply a reminder that tail heaviness is an asymptotic concept independent of specific parameter values of the respective distributions. However, when comparing distributions with specific parameterisations asymptotically lighter distributions can exhibit higher means and therefore higher top shares. Given that the type

II Pareto distribution yields a consistently superior fit to the data (see section 4.1), it is clear that the top shares implied by the LogNormal are too large and are not supported by the data. Along similar lines, it is worthwhile pointing out that the fitted type I Pareto distributions yield top shares which are substantially lower than the better fitting type II Pareto distribution. This means the fitted type I distributions underestimate aggregate wealth concentration up to the 99^{th} percentile. Within the top 1% of the US wealth data the difference between type I and type II Pareto distributions is considerably smaller. These results are an important reminder that, one should not only focus on asymptotic properties such as tail heaviness when comparing distributions in applied work.

Table 7: Top 10% within tail shares based on raw SCF data



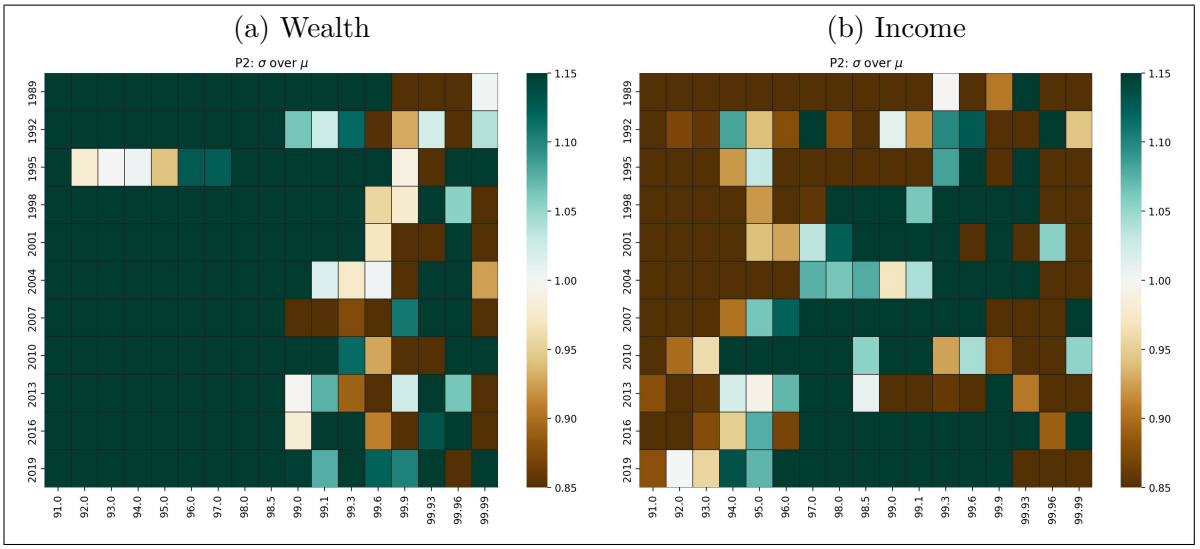
Plots of $ws(0.9, p)$, as defined in equation 7, for varying percentile cut-offs (p) based on raw SCF wealth and income data.

Table 7 shows the top 10% within tail share $ws(0.9, p)$, as defined in equation 7, for increasing percentile ranks p, estimated directly from the raw survey data. The tail of the wealth distribution exhibits a pattern of declining concentration up to the 99^{th} percentile and flattens out beyond. For the income distribution, the survey data exhibits a striking structural break: For the first three waves (1989, 1992 and 1995), the income distribution exhibits an increasing concentration pattern, but from 1998 onwards this changes to a decreasing pattern with the exception of 2013.[12] The patterns in Table 7 help to understand the poor fit and subsequent rejection of both the Exponential and the type I Pareto distribution. While the Exponential distribution is compatible with declining within tail concentration, it is incompatible with stable or increasing concentration. In addition, the type I Pareto distribution is only compatible with stable within tail concentration, explaining its poor performance below the 99^{th} percentile. In contrast, the type II distribution can exhibit declining or increasing within tail concentration depending on the parameterisation: $ws'(q, p) < 0$ if $\sigma < \mu$ and $ws'(q, p) > 0$ if $\sigma > \mu$ (see Proposition 3).

[12]It is important to keep in mind that using the raw data to estimate conditional means and therefore within tail shares, above increasing thresholds yields downward-biased estimators as $p \to 1$. This is due to the fact that in a finite sample the average above $x(p)$ approaches $x(p)$ as p approaches 1 due to the falling number of observations available above higher and higher thresholds. For this reason we only plot the results up to $p = 0.996$.

Table 8 displays the ratio σ/μ based on the estimated parameters of the type II Pareto distribution for each year-percentile cell. Values larger (smaller) than 1 in dark green (brown) indicate decreasing (increasing) within tail concentration. For wealth data, we can see that up to the 99^{th} percentile the fitted type II distributions exhibit decreasing within tail concentration (in line with the pattern in the raw survey data). This pattern is highly stable across time. Beyond the 99^{th} percentile more and more year-percentile cells exhibit increasing within tail concentration. For the distribution of income, the pattern of within tail concentration changes through time. While in waves 1989 to 1995 it is increasing (brown) over the entire range of percentile ranks, from wave 1998 onwards it exhibits a decreasing (green) "middle" section. Blanchet, Piketty & Fournier (2022) use interpolated inverted Pareto coefficients for the year 2010 to argue that US and French income distributions exhibit a pattern of monotonically increasing concentrations within their tails (for $p > 0.9$). [13] Importantly, the income concept in the SCF is different from DINA income; nevertheless, our results based on SCF wealth and income suggest that the pattern of within tail wealth inequality might not be smoothly U-shaped with increasing concentration towards the top as suggested by Blanchet, Piketty & Fournier (2022).

Table 8: Within tail concentration patterns for fitted type II Pareto distributions



Each percentile-year cell displays the ratio of the estimated scale and location parameters of the type II Pareto distribution: σ/μ. A ratio smaller than 1 indicates increasing (brown) and a ratio larger than 1 decreasing (green) within tail concentration for wealth and income. See proof of Proposition 3 in Appendix A.3.

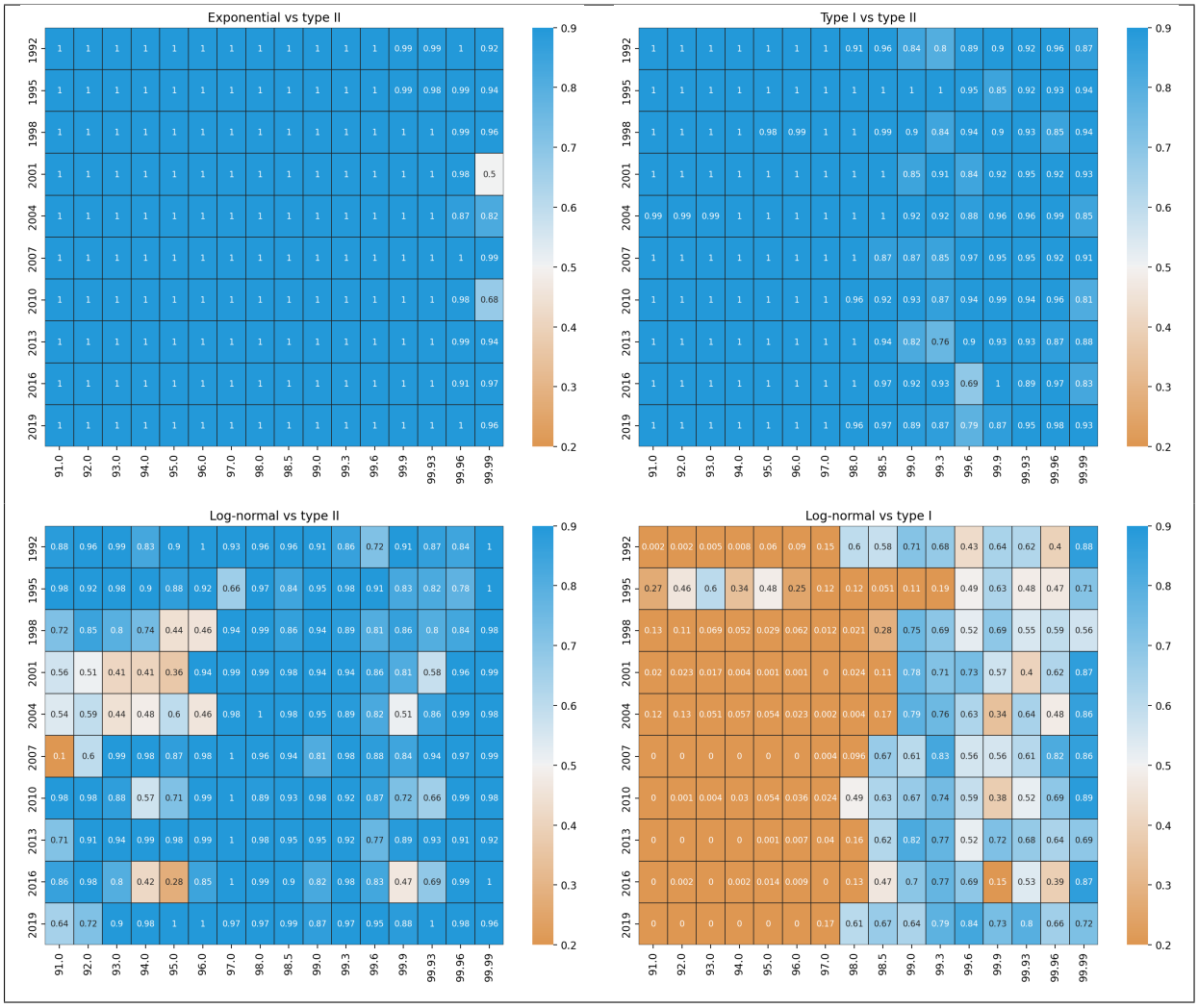
4.3 Statistical significance

The results in the previous sections are purely based on point estimates of the distributions' parameters. This section explores the statistical uncertainty in these results. One complication in doing this is the confidentiality of the SCF's sampling process which requires us to use the provided sets of replicate weights to account for the statistical uncertainty in our estimates. We

[13] As shown in Appendix A.3, for the type II Pareto distribution, declining inverted Pareto coefficients coincide with declining within tail shares and vice versa.

therefore repeated the goodness-of-fit tests discussed in section 4.1 for all 999 sets of replicate weights, resulting in 999 fitted parameters and CvM test statistics for each distribution and each percentile-year cell. This allows us to asses whether the results in section 4.1 are statistically significant in the sense that they hold up once the random variation in the data collection process is taken into account via the replicate weights.

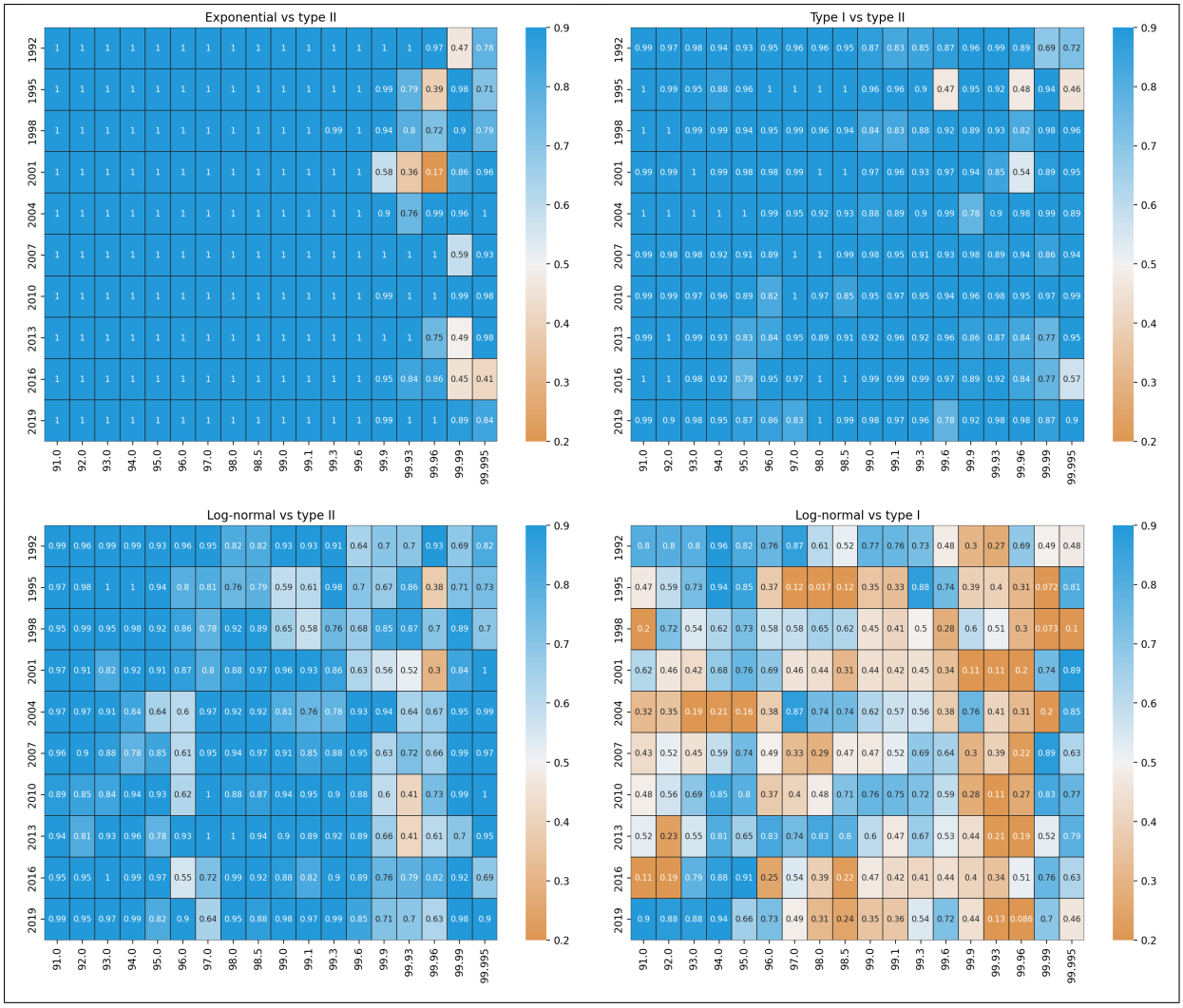
Table 9: Relative fit for US wealth data using replicate weights



Using the SCF's 999 sets of replicate weights, each distribution is fitted 999 times to the data in each percentile-year cell. Each cell reports the proportion of cases in which the type II Pareto distribution yields a smaller CvM test statistic and thus a better fit than the distribution it was compared to. For the heat map in the bottom right, the proportion of cases in which the type I Pareto distribution yields a better fit is reported. Shades of blue indicate shares in excess of 50%.

Tables 9 and 10 summarise the results for all four distribution pairs for the wealth and income data respectively. The heat maps in the upper left show that for most percentile-year cells, the type II distribution provides a far better fit than the exponential distribution for more than 99% of the 999 replicate weight sets, especially for wealth data and below the 99.9$^{\text{th}}$ percentile. There are some exceptions, especially for income data above the 99.9$^{\text{th}}$ percentile in the year 2001. The upper right heat maps in Tables 9 and 10 show that the type II distribution consistently outperforms the type I distribution for both income and wealth data. There are

Table 10: Relative fit for US wealth data using replicate weights



Using the SCF's 999 sets of replicate weights, each distribution is fitted 999 times to the data in each percentile-year cell. Each cell reports the proportion of cases in which the type II Pareto distribution yields a smaller CvM test statistic and thus a better fit than the distribution it was compared to. For the heat map in the bottom right, the proportion of cases in which the type I Pareto distribution yields a better fit. Shades of blue indicate shares in excess of 50%.

two important exceptions for the 99.6^{th} and the 99.96^{th} percentile cut offs in the year 1995 for income data.

The bottom two heat maps of Tables 9 and 10 compare the LogNormal distribution to type I and type II Pareto distributions. The type II distribution outperforms the LogNormal in all but one of the 120 cells beyond the 97^{th} percentile for wealth data. For income data, the type II distribution provides a better fit for more than 50% of the replicate weight sets in all but four percentile-year cells. The bottom right heatmap shows that in 57 out of the 70 percentile-year cells beyond the 99^{th} wealth percentile the type I distribution provides a better fit than the LogNormal distribution for more than 50% of the replicate weight sets. While clearly much worse than the fit of the type II distribution, the type I distribution nevertheless provides a decent fit for the top 1% of the US wealth distribution. For income, the situation is different. The LogNormal distribution provides a better fit than the type I Pareto distribution for more

than 50% of the replicate weight sets in 93 out of all 180 year-percentile cells. Conversely, in 87 year-percentile cells the type I distribution dominates. Therefore our data does not allow us to reliably establish which of these two distributions provides a better fit for the tail of the US income distribution.

Overall, the 999 sets of replicate weights, support our main findings: First, we find strong support for asymptotic Pareto tails for US wealth and income data. In particular, we find a superior fit of the type II Pareto distribution compared to the LogNormal beyond the 97^{th} wealth percentile and beyond the 91^{st} income percentile. Second, the type I Pareto provides an inferior fit compared to the type II distribution for almost all percentile-year cells and we reject the hypothesis of strict power laws for wealth and income data. Third, the type I distribution provides a better fit compared to the LogNormal distribution beyond the 99^{th} wealth percentile. Overall, these results support the notion that whenever possible the tails of the US income and wealth distributions should be modelled as an asymptotic power law.

4.4 How long is the power law tail?

After establishing that US wealth and income data follow an asymptotic power law in their tails, we turn to the question of the length of the asymptotic power law tail. In applied research, the choice of the cut-off point beyond which the tail of the distribution is parametrically modelled is crucial: Setting the cut-off point too low yields biased estimates while choosing it too high leaves the researcher with hardly any data and high standard errors. For many wealth and income surveys, Clauset et al.'s (2009) approach for choosing this cut-off is not viable because sampling from a synthetic dataset in the way it was done for the original survey is not possible because the complex sampling methods used in constructing these surveys are not disclosed due to privacy concerns.

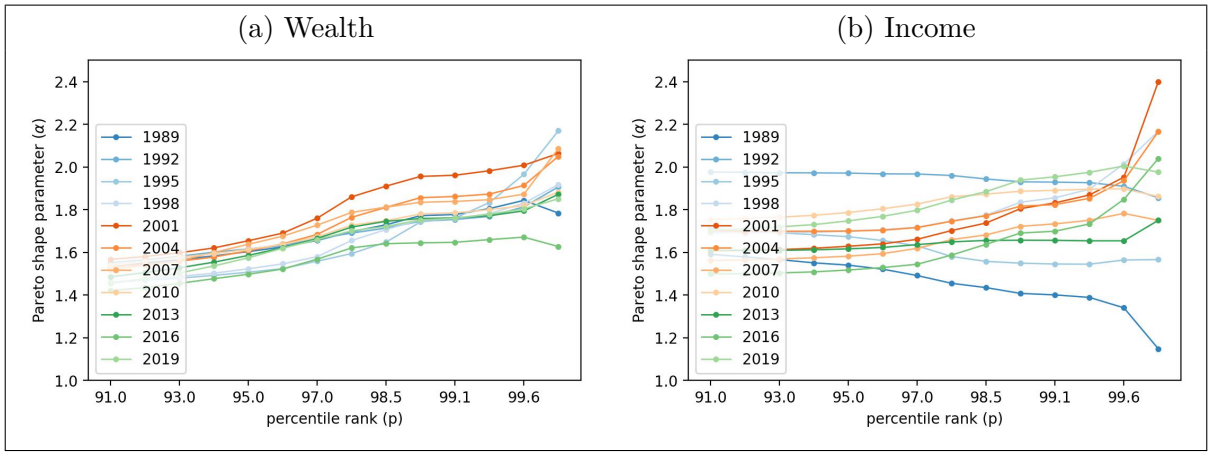
This leaves us with three alternative approaches: First, we inspect the estimated Pareto shape parameters (α) for a range of percentile cut-offs as depicted in Table 11. An appropriate cut-off is characterised by the stability of the estimated shape parameter beyond this cut-off with the caveat that estimates become more noisy as $p \to 1$ and fewer and fewer data points are available. The plots in Table 11 suggest a stabilisation beyond the 99^{th} percentile for wealth data and fairly stable estimates for the entire range of income data. [14]

Second, we can inspect the empirical estimates of the within tail shares in Table 7. While a type II Pareto distribution can exhibit increasing or decreasing within tail concentration, it cannot exhibit an inflection point where the pattern changes from one to the other. Therefore, monotonicity of the within tail concentration is a prerequisite for a type II Pareto tail. The stabilisation of within tail shares beyond the 99^{th} wealth percentile justifies the choice of the 99^{th} percentile as the tail cut-off. For income data the within tail shares in Table 7 exhibit monotonic patterns in all years. Increasing within tail shares in the earlier waves and falling within tail shares in later waves do not contradict a type II Pareto distribution as the underlying data generating process as it can accommodate both patterns. Therefore, within tail shares suggest

[14] The plots in Table 11 do not display estimates beyond the 99.9^{th} percentile as the sample size drops and volatility increases.

a cut-off as low as $p = 0.9$ is compatible with US income data. It is important to keep in mind that estimating within tail shares from SCF data directly is biased because of the omission of households on the *Forbes 400* list due to privacy concerns.

Table 11: Estimated shape parameters



Estimated Pareto shape parameters based on equation 12.: $\log(r_i) = c - \alpha \cdot \log(x_i) + \epsilon$ where r_i is the population rank of the i^{th} largest observation and x_i the corresponding wealth (income).

Finally, while the graphical methods discussed so far can be useful, a more formal approach is warranted. To that end, we can exploit the fact that the within tail concentration pattern for a type II Pareto distribution is determined by the ratio of its scale (σ) and location (μ) parameter (see Proposition 3 and proof A.3). Furthermore, a type II Pareto tail exhibits either monotonically increasing or decreasing within tail concentration. This means estimates obtained beyond an appropriate cut-off point should exhibit ratios which are consistently above or below 1. From Table 8 we can see that for the US wealth distribution, there is an inflection point at $p = 0.99$. Beyond this point the steady pattern of decreasing within shares (green cells), gives rise to much lighter colouring, suggesting ratios near unity. When $\sigma = \mu$ the type II distribution collapses to a type I Pareto and the within tail share is flat. Therefore, consistent with the previous graphical methods, this suggests choosing cut-offs above $p = 0.99$. This is confirmed when looking directly at the CvM test statistics. The minimum value in 9 out of 11 waves is beyond $p = 0.99$ and in 7 waves it is beyond $p = 0.9993$. For the US income distribution, the pattern visible in Table 8 is that from wave 1995 onwards the distributions show a pattern of consistently increasing within tail shares (brown) for low cut-offs, which then turn into a sequence of falling shares (green) before beginning to rise again. Given that the type II distribution can only exhibit monotonically increasing or decreasing within tail concentration, these results suggest choosing cut-offs around the second inflection point. The best fit according to CvM test statistics is achieved for $p = 0.98$ or beyond in 8 out of the 11 analysed waves.

5 Conclusion

Using data from the Survey of Consumer Finances (SCF), this paper rejects the hypothesis that the respective tails of the US income and wealth distributions follow strict power laws, instead

supporting the alternative hypothesis of asymptotic power laws. Our results are therefore in line with previous rejections of strict Pareto tails in wealth and income data (Brzezinski 2014, Ogwang 2013, Chan et al. 2017). Crucially, however, we show that rejecting strict power laws by no means implies a rejection of the heavy-tailed and even thick-tailed nature of income and wealth distributions. Our paper also sheds light on the shape and length of the tail distribution of US income and wealth data. The wealth distribution exhibits a pattern of decreasing concentration up to the 99^{th} percentile and flattens out beyond. This pattern is stable across our sample period ranging from 1992 to 2019. In the case of income, we find an S-shaped pattern of increasing within tail inequality interrupted by a decreasing middle section. A practical lesson from these changing patterns of tail concentration for parametric tail modelling is to only fit parametric distributions to data beyond inflection points. This means fitting asymptotic power laws only beyond the 99^{th} percentile of the wealth or income distribution. While more research is required to clarify the theoretical mechanisms behind these changes in tail concentration, our findings motivate a simple test for the plausibility of theoretical models of wealth accumulation: Is the model able to produce an asymptotic Pareto tail? And further, is the model able to reproduce the pattern of within tail concentration observed in the data?

Our results suggest that applied work should focus on asymptotic Pareto tails rather than strict Pareto distributions. However, further research is needed to compare the relative performance of available estimators[15] in practically relevant situations characterized by differential non-response and small sample sizes. Another open question is how parametric models compare with alternative approaches such as Generalised Pareto curves (Blanchet, Piketty & Fournier 2022), re-weighting as proposed by Blanchet, Flores & Morgan (2022) and re-scaling based on national account aggregates (Piketty et al. 2018) when attempting to correct small sample survey data which suffers from differential non-response as is the case for most countries for example in the European Central Bank's Household Finance and Consumption Survey (HFCS).

[15]Such as the Hill, QQ (Kratz & Resnick 1996), Maximum Likelihood, Rank Correction (Wildauer & Kapeller 2022), elemental percentile (Castillo & Hadi 1997), mean excess plots (Disslbacher et al. 2020) and adding rich list data (Vermeulen 2016)

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APPENDIX

A Proofs

A.1 Proof of proposition 1

If $\bar{F}(x) = e^{-\lambda(x-\mu)}$ we solve $m(x) = \frac{\int_0^\infty \bar{F}(x+t)dt}{\bar{F}(x)}$ to obtain $m(x) = e^{\lambda(x-\mu)} \int_0^\infty e^{-\lambda(x+t-\mu)}dt = \frac{1}{\lambda}$ and therefore $cm(x) = m(x)+x$. Next we use the quantile function $x(p) = \mu - \frac{\ln(1-p)}{\lambda}$ to obtain:

$$cm(p) = \mu + \frac{1 - \ln(1-p)}{\lambda}. \tag{A.1}$$

Next we can use $(1-q-p+qp) = (1-q)(1-p)$ and equation 9 to define the within tail share and the inverted Pareto coefficients as:

$$ws(q,p) = \frac{(1-q)cm(p+q-qp)}{cm(p)} = (1-q) - \frac{(1-q)\ln(1-q)}{\lambda\mu + 1 - \ln(1-p)} \tag{A.2}$$

$$b(p) = \frac{cm(p)}{x(p)} = 1 + \frac{1}{\lambda\mu - \ln(1-p)}. \tag{A.3}$$

Finally we obtain the derivatives as:

$$\frac{\partial ws(q,p)}{\partial p} = \frac{1-q}{1-p} \frac{\ln(1-q)}{\left[\lambda\mu + 1 - \ln(1-p)\right]^2} < 0 \tag{A.4}$$

$$\frac{\partial b(p)}{\partial p} = -\frac{1}{(1-p)\left[\lambda\mu - \ln(1-p)\right]^2} < 0 \tag{A.5}$$

Both derivatives being negative shows that both measures of tail concentration show that for an Exponential distribution the concentration within its tail decreases.

A.2 Proof of proposition 2

For $\bar{F}(x) = \left[\frac{x}{\sigma}\right]^{-\alpha}$ we solve $m(x) = \left[\frac{x}{\sigma}\right]^{\alpha} \int_0^\infty \left[\frac{x+t}{\sigma}\right]^{-\alpha} dt$ to obtain $m(x) = \frac{x}{\alpha-1}$ and $cm(x) = \frac{\alpha}{\alpha-1}x$. Together with the quantile function $x(p) = \frac{\sigma}{(1-p)^{1/\alpha}}$ we can define the within tail share and the inverted Pareto coefficient as:

$$ws(q,p) = \frac{(1-q)cm(p+q-qp)}{cm(p)} = (1-q)^{\frac{\alpha-1}{\alpha}} \tag{A.6}$$

$$b(p) = \frac{cm(p)}{x(p)} = \frac{\alpha}{\alpha-1}. \tag{A.7}$$

Therefore $\frac{\partial ws(q,p)}{\partial p} = \frac{\partial b(p)}{\partial p} = 0$ which proves proposition 2.

A.3 Proof of proposition 3

For $\bar{F}(x) = \left[1 + \frac{x-\mu}{\sigma}\right]^{-\alpha}$ we solve $m(x) = \left[\frac{\sigma+x-\mu}{\sigma}\right]^{\alpha} \int_0^\infty \left[\frac{\sigma+x+t-\mu}{\sigma}\right]^{-\alpha} dt$ to obtain $m(x) = \frac{1}{\alpha-1}x + \frac{\sigma-\mu}{\alpha-1}$ and $cm(x) = \frac{\alpha}{\alpha-1}x + \frac{\sigma-\mu}{\alpha-1}$. We use the quantile function $x(p) = \mu - \sigma + \frac{\sigma}{(1-p)^{1/\alpha}}$ to

obtain $cm(p) = \mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}$ and then define the within tail share and the inverted Pareto coefficient to obtain after simplification:

$$ws(q,p) = (1-q)\frac{cm(p+q-qp)}{cm(p)} \tag{A.8}$$

$$= (1-q)\frac{\mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}(1-q)^{-1/\alpha}}{\mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}} \tag{A.9}$$

$$= (1-q)^{\frac{\alpha-1}{\alpha}} \cdot \frac{\alpha\sigma + (\alpha-1)(1-p)^{1/\alpha}(1-q)^{1/\alpha}(\mu-\sigma)}{\alpha\sigma + (\alpha-1)(1-p)^{1/\alpha}(\mu-\sigma)} \tag{A.10}$$

$$b(p) = \frac{cm(p)}{x(p)} = \frac{\alpha}{\alpha-1} + \frac{\sigma-\mu}{(\alpha-1)(\mu-\sigma+\sigma(1-p)^{-1/\alpha})} \tag{A.11}$$

Both results demonstrated that the type I distribution is a special case of the type II Pareto distribution. Specifically if $\sigma = \mu$ we obtain the type I results $ws(q,p) = (1-q)^{\frac{\alpha-1}{\alpha}}$ and $b(p) = \frac{\alpha}{\alpha-1}$. For the within share we obtain:

$$\frac{\partial ws(q,p)}{\partial p} = (1-q)^{\frac{\alpha-1}{\alpha}}\frac{u'v - v'u}{v^2} \tag{A.12}$$

and $\frac{\partial ws(q,p)}{\partial p} > 0$ iff $\frac{u'}{u} > \frac{v'}{v}$ where $u = \mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}(1-q)^{-1/\alpha}$ and $v = \mu - \sigma + \frac{\alpha\sigma}{\alpha-1}(1-p)^{-1/\alpha}$. Since $u' = v'(1-q)^{-1/\alpha}$ the condition to determine the sign of the derivative simplifies to:

$$\frac{(1-q)^{-1/\alpha}v'}{u} > \frac{v'}{v} \tag{A.13}$$

$$\frac{(1-q)^{-1/\alpha}}{u} > \frac{1}{v} \tag{A.14}$$

$$(1-q)^{-1/\alpha}v > u \tag{A.15}$$

$$(1-q)^{-1/\alpha}(\mu-\sigma) > (\mu-\sigma). \tag{A.16}$$

Since the expression $(1-q)^{-1/\alpha} > 1$ for $q \in [0,1]$ the expression is only true iff $\mu > \sigma$ and therefore $\frac{\partial ws(q,p)}{\partial p} > 0$ iff $\mu > \sigma$ which proves proposition 3.

For the inverted Pareto coefficient we obtain a similar result:

$$\frac{\partial b(p)}{\partial p} = (1-p)^{\frac{\alpha+1}{\alpha}}\frac{\alpha}{\alpha-1}\frac{\mu-\sigma}{[\mu-\sigma+\sigma(1-p)^{-1/\alpha}]^2} \lessgtr 0 \tag{A.17}$$

The sign of $\frac{\partial b(p)}{\partial p}$ is directly determined by $\mu \lessgtr \sigma$ as all other terms are strictly positive: $\frac{\partial b(p)}{\partial p} > 0$ iff $\mu > \sigma$.

A.4 Proof of proposition 4

Random variable X is distributed as type II Pareto with $\bar{F}_X(x;\alpha,\mu,\sigma) = \left[1 + \frac{x-\mu}{\sigma}\right]^{-\alpha}$ and random variable Y is distributed as type I Pareto with $\bar{F}_Y(y;\alpha,\mu) = \left[\frac{y}{\mu}\right]^{-\alpha}$. Both distributions

exhibit the same shape (α) parameter and the scale parameter of X is the same as the location (μ) parameter for Y. Means are defined as (assuming $\alpha > 1$):

$$E[X] = \mu + \frac{\sigma}{\alpha - 1}, \tag{A.18}$$

$$E[Y] = \frac{\alpha}{\alpha - 1}\mu. \tag{A.19}$$

We can establish:

$$E[X] > E[Y] \tag{A.20}$$

$$\mu + \frac{\sigma}{\alpha - 1} > \frac{\alpha}{\alpha - 1}\mu \tag{A.21}$$

$$(\alpha - 1)\mu + \sigma > \alpha\mu \tag{A.22}$$

$$\sigma > \mu, \tag{A.23}$$

which proves proposition 4.

A.5 Proof of proposition 5

For the LogNormal distribution we have $\bar{F}(x; \mu, \sigma, \delta) = \Phi\left(\frac{\mu - \log(x-\delta)}{\sigma}\right)$ and $f(x; \mu, \sigma, \delta) = \frac{1}{\sigma(x-\delta)\sqrt{2\pi}} \cdot \exp\left(-\frac{[\ln(x-\delta)-\mu]^2}{2\sigma^2}\right)$. The conditional mean is defined as:

$$cm(k) = E[X|X > k] = \frac{\int_k^\infty x f(x) dx}{\bar{F}(x)} \tag{A.24}$$

We start by deriving $E[X - \delta|X > k]$[16] and solve $\int_k^\infty (x - \delta) f(x) dx$ by defining $y = \frac{\ln(x-\delta)-\mu}{\sigma}$ and $dt = \sigma\exp(\sigma y + \mu) dy$ and transform the integral:

$$\int_k^\infty (x - \delta) f(x) dx = \tag{A.25}$$

$$= \int_k^\infty \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{[\ln(x-\delta) - \mu]^2}{2\sigma^2}\right) dx \tag{A.26}$$

$$= \int_{y=\frac{\ln(k-\delta)-\mu}{\sigma}}^\infty \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{y^2}{2}\right) \sigma \cdot \exp(\sigma y + \mu) dy \tag{A.27}$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{y=\frac{\ln(k-\delta)-\mu}{\sigma}}^\infty \exp\left(-\frac{(y-\sigma)^2}{2}\right) dy. \tag{A.28}$$

Now define $v = y - \sigma$ and $dy = dv$ to obtain:

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{1}{\sqrt{2\pi}} \int_{v=\frac{\ln(k-\delta)-\mu-\sigma^2}{\sigma}}^\infty \exp\left(-\frac{v^2}{2}\right) dv \tag{A.29}$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \Phi(-v) \tag{A.30}$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right) \Phi\left(\frac{\mu + \sigma^2 - \ln(k-\delta)}{\sigma}\right) \tag{A.31}$$

[16] Starting with $E[X - \delta|X > k]$ instead of $E[X|X > k]$ directly simplifies the derivation because for the former the term $x - \delta$ cancels out when solveign the integral.

where $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$ is the CDF of the standard normal distribution. We can use this result together with the fact that $E[X|X > k] = E[X - \delta|X > k] + \delta$ to obtain:

$$cm(k) = \exp\left(\mu + \frac{\sigma^2}{2}\right) \frac{\Phi\left(\frac{\mu+\sigma^2-\ln(k-\delta)}{\sigma}\right)}{\Phi\left(\frac{\mu-\log(k-\delta)}{\sigma}\right)} + \delta. \tag{A.32}$$

With the quantile function $x(p) = \exp\left(\mu + \sigma\Phi^{-1}(p)\right) + \delta$ we can derive:

$$cm(p) = \frac{(E[X] - \delta)\,\Phi\left(\sigma - \Phi^{-1}(p)\right)}{1 - p} + \delta \tag{A.33}$$

and further the within tail share $ws(q, p) = \frac{(1-q)cm(p+q-qp)}{cm(p)}$:

$$ws(q, p) = \frac{(E[X] - \delta)\,\Phi\left[\sigma - \Phi^{-1}(p + q - qp)\right] + (1 - q)(1 - p)\delta}{(E[X] - \delta)\,\Phi\left[\sigma - \Phi^{-1}(p)\right] + (1 - p)\delta} \tag{A.34}$$

which proves proposition 5.

B Within tail concentration in the LogNormal distribution

As shown in section A, the within tail share for the LogNormal distribution is defined as:

$$ws(q, p) = \frac{(E[X] - \delta)\,\Phi\left[\sigma - \Phi^{-1}(p + q - qp)\right] + (1 - q)(1 - p)\delta}{(E[X] - \delta)\,\Phi\left[\sigma - \Phi^{-1}(p)\right] + (1 - p)\delta} \tag{B.1}$$

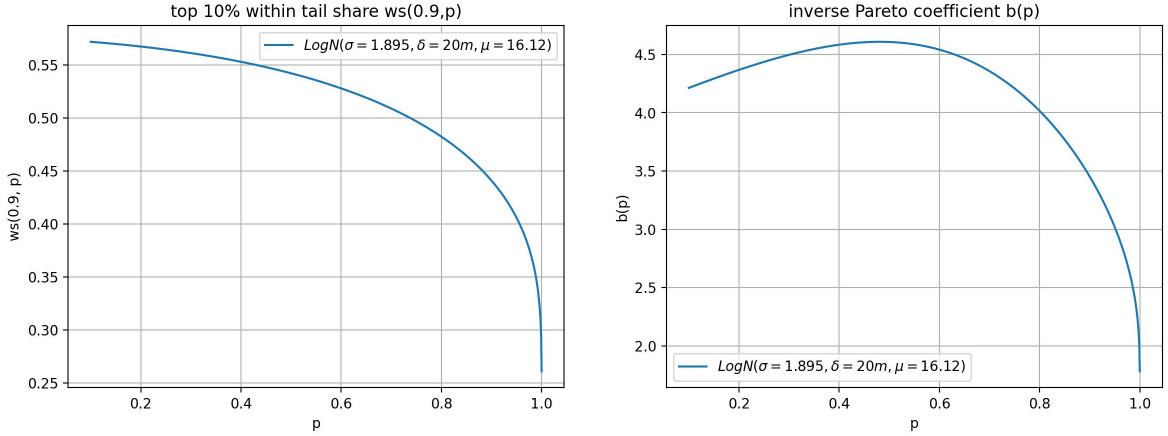
Furthermore we can define the inverted Pareto coefficient as:

$$b(p) = \frac{cm(p)}{x(p)} = \frac{\frac{E[X]-\delta}{1-p}\Phi\left(\sigma - \Phi^{-1}(p)\right) + \delta}{\exp\left(\mu + \sigma\Phi^{-1}(p)\right) + \delta}. \tag{B.2}$$

The LogNormal distribution is different from the type I and type II Pareto distribution and the Exponential distribution as it allows for parameterisations which do not depict monotonously increasing or decreasing concentration patterns within its tail as measured by the withing share, $ws(q, p)$, or the inverted Pareto coefficient $b(p)$. Table A1 demonstrates this point. It depicts both measures for the parameters obtained from the 2019 wave, implicate 1, fitted to observations with net wealth in excess of 20 million USD, corresponding to the 99.6^{th} percentile in the aggregate US wealth distribution. It can be seen that the top 10% within tail share falls monotonically, while the inverted Pareto coefficient first rises and falls afterwards.

The example in Table A1 highlights the fact that the within tail share and the inverted Pareto coefficient are not equivalent measures of tail concentration. While the former measures the share of the most affluent $(1 - q)$% households for increasing cut-offs, the latter simply measures the average in relation to those increasing thresholds. The possibility that within tail concentration is increasing in a certain part of the tail according to one measure, while it is falling according to the other, does not represent a contradiction but emphasizes that concentration can be measured in different ways. Different measures might be better suited for different tasks.

Table A1: Comparing within tail share and inverted Pareto coefficient for LogN



LogNormal fitted to SCF wealth data from 2019 wave, implicate 1, observations with wealth in excess of 20 million USD. Note that the percentile rank p in these plots does not refer to the rank in the aggregate distribution but to the rank within the tail. This means $p = 0.5$ in this plot refers to the median above 20 million USD.

C Descriptive Statistics

Table A2: Wealth-percentile cut-off values in million USD.

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	0.8	0.7	0.7	0.9	1.2	1.3	1.2	1.3	1.1	1.4	1.4
92.000	0.9	0.8	0.8	0.9	1.3	1.4	1.4	1.4	1.3	1.5	1.5
93.000	1.0	0.9	0.9	1.0	1.5	1.5	1.5	1.7	1.5	1.8	1.8
94.000	1.2	1.0	1.0	1.2	1.6	1.7	1.9	1.9	1.7	2.2	2.1
95.000	1.3	1.2	1.1	1.4	1.9	2.0	2.3	2.2	2.1	2.5	2.6
96.000	1.6	1.4	1.3	1.7	2.3	2.4	2.9	2.6	2.6	3.0	3.3
97.000	2.0	1.8	1.7	2.1	3.0	3.0	3.9	3.3	3.4	4.0	4.7
98.000	2.7	2.5	2.2	3.2	4.7	4.5	5.8	5.0	4.8	6.2	6.6
98.500	3.4	3.0	2.8	4.4	5.9	6.1	7.6	6.1	6.4	8.3	8.0
99.000	4.6	4.2	4.1	6.1	8.5	8.6	10.3	8.1	8.7	11.1	11.1
99.100	4.8	4.5	4.4	6.5	8.9	9.5	11.1	8.7	9.1	11.5	12.0
99.300	5.7	5.2	5.6	7.6	10.4	11.1	12.3	10.5	10.9	13.6	13.8
99.600	8.7	7.5	9.2	11.4	14.8	15.2	17.1	14.8	15.9	20.4	20.0
99.900	19.4	16.7	23.4	24.7	29.5	33.5	38.4	32.4	33.7	46.4	43.5
99.930	22.3	20.6	29.0	29.6	36.4	37.0	47.5	38.6	41.7	51.5	54.3
99.960	29.9	29.9	37.1	43.6	45.5	57.4	68.5	52.1	60.0	74.0	80.9
99.990	61.8	62.1	80.3	109.0	115.4	131.8	160.1	120.1	135.4	205.6	172.5
99.995	146.0	100.8	108.9	136.5	164.7	187.1	226.6	236.9	190.7	314.0	268.4

Table A3: Wealth - Observations beyond cut-off

year	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	923	1284	1412	1335	1363	1372	1457	1389	1442	1481	1532
92.000	878	1243	1356	1292	1323	1326	1406	1318	1371	1421	1470
93.000	835	1191	1301	1241	1276	1281	1358	1247	1301	1343	1399
94.000	793	1139	1233	1189	1217	1224	1277	1181	1226	1259	1319
95.000	749	1081	1168	1119	1144	1158	1192	1115	1141	1180	1229
96.000	698	1014	1093	1043	1073	1090	1105	1043	1044	1103	1122
97.000	636	919	1001	961	970	1009	998	951	934	982	1006
98.000	556	810	901	815	825	880	869	816	803	847	868
98.500	499	742	815	717	749	802	781	738	727	767	794
99.000	413	649	689	626	657	712	683	649	649	679	698
99.100	400	629	662	606	641	688	658	629	635	662	672
99.300	365	589	589	555	595	646	629	584	592	617	620
99.600	289	502	459	457	503	547	543	496	504	522	526
99.900	168	315	252	284	324	353	348	310	318	325	329
99.930	146	273	224	251	274	322	305	273	270	297	281
99.960	113	211	189	185	231	244	225	228	208	224	206
99.990	51	100	91	79	85	110	100	98	94	91	86
99.995	18	59	62	58	56	71	64	55	63	54	53

Table A4: Income-percentile cut-off values in million USD.

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	0.2	0.1	0.1	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
92.000	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
93.000	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2
94.000	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.3
95.000	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.2	0.3	0.3	0.3
96.000	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.3	0.3	0.3
97.000	0.3	0.2	0.3	0.3	0.3	0.3	0.4	0.3	0.3	0.4	0.4
98.000	0.3	0.3	0.3	0.3	0.5	0.4	0.5	0.5	0.5	0.5	0.5
98.500	0.4	0.4	0.3	0.4	0.6	0.5	0.6	0.6	0.6	0.7	0.7
99.000	0.5	0.4	0.4	0.6	0.7	0.7	0.8	0.7	0.8	0.9	0.9
99.100	0.5	0.4	0.4	0.6	0.8	0.7	0.9	0.8	0.8	1.0	0.9
99.300	0.6	0.5	0.5	0.7	1.0	0.8	1.1	0.9	0.9	1.2	1.1
99.600	0.9	0.7	0.7	1.1	1.5	1.1	1.6	1.2	1.3	1.9	1.6
99.900	1.8	1.2	1.5	2.3	3.6	3.0	3.9	2.7	2.8	4.3	3.3
99.930	2.7	1.5	1.8	2.8	4.2	3.9	4.7	3.3	3.8	6.3	4.2
99.960	3.6	1.8	3.4	4.3	8.4	5.3	6.0	4.1	5.1	8.5	5.5
99.990	16.1	6.0	7.1	8.1	14.7	10.2	12.5	7.4	17.3	14.6	9.7
99.995	16.2	7.5	8.2	9.8	14.9	13.5	27.3	12.7	23.5	27.5	15.7

Table A5: Income - Observations beyond cut-off

year	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	865	1185	1241	1251	1270	1313	1352	1358	1352	1392	1428
92.000	821	1138	1191	1205	1221	1268	1304	1286	1282	1328	1361
93.000	776	1087	1136	1154	1175	1221	1246	1217	1216	1260	1292
94.000	742	1043	1081	1100	1123	1169	1190	1146	1143	1191	1223
95.000	709	974	1018	1044	1058	1101	1134	1067	1064	1117	1137
96.000	655	917	937	970	993	1038	1065	970	984	1024	1047
97.000	591	857	855	883	908	970	976	878	887	943	952
98.000	524	743	740	764	791	869	854	746	778	836	840
98.500	480	681	680	684	720	798	790	660	698	738	761
99.000	414	613	610	578	610	700	684	577	611	647	664
99.100	397	592	588	554	575	682	655	557	588	633	635
99.300	367	543	555	503	525	639	605	514	554	579	587
99.600	280	453	434	386	404	553	502	424	479	459	473
99.900	144	267	232	213	205	328	269	244	301	250	280
99.930	103	223	202	189	174	268	231	198	241	175	227
99.960	76	178	137	124	72	202	184	169	186	126	169
99.990	9	43	48	53	40	90	99	95	62	62	88
99.995	8	27	37	37	39	70	53	60	47	30	52

Table A6: Raw SCF data: share of total wealth by year and cutoff, in %

Cutoff	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	64.920	64.920	66.010	66.710	67.570	67.440	69.670	72.360	73.100	75.230	74.710
92.000	62.680	62.760	63.910	64.720	65.440	65.290	67.790	70.060	71.050	73.180	72.770
93.000	60.130	60.340	61.590	62.520	63.020	62.920	65.730	67.440	68.730	70.960	70.570
94.000	57.280	57.620	58.990	60.030	60.360	60.350	63.350	64.370	66.060	68.260	68.000
95.000	54.000	54.350	56.020	57.120	57.300	57.360	60.310	60.770	62.850	65.030	64.900
96.000	50.100	50.480	52.580	53.620	53.740	53.780	56.520	56.770	58.920	61.340	60.950
97.000	45.180	45.740	48.440	49.430	49.290	49.500	51.630	51.740	53.850	56.700	55.710
98.000	38.980	39.340	42.930	43.740	42.850	43.500	44.750	44.690	46.820	49.920	48.250
98.500	35.130	35.270	39.570	39.570	38.320	39.170	39.890	39.960	42.010	45.110	43.370
99.000	29.850	29.980	34.800	33.770	32.120	33.190	33.540	34.010	35.350	38.450	37.170
99.100	28.650	28.690	33.580	32.410	30.620	31.650	31.910	32.610	33.840	36.930	35.560
99.300	25.950	25.740	30.820	29.220	27.270	28.270	28.600	29.340	30.510	33.570	32.190
99.600	20.430	20.240	24.840	23.210	20.770	22.020	22.400	22.830	23.970	26.760	25.510
99.900	10.270	11.100	12.970	12.440	10.350	11.600	12.290	12.190	13.110	14.640	13.890
99.930	8.720	9.440	10.820	10.600	8.640	9.830	10.470	10.390	11.180	12.720	12.110
99.960	6.770	7.270	8.050	8.210	6.660	7.650	8.080	8.160	8.630	10.230	9.420
99.990	3.810	3.820	3.860	3.890	3.110	3.450	3.740	4.070	4.250	5.180	4.870
99.995	2.520	2.670	2.550	2.730	1.880	2.240	2.490	2.770	2.910	3.310	3.660

Table A7: Raw SCF data: share of total income by year and cutoff, in %

Cutoff	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	40.374	35.526	37.398	39.345	43.634	40.702	45.366	42.513	45.067	48.070	44.688
92.000	38.425	33.461	35.458	37.434	41.764	38.663	43.517	40.537	43.084	46.143	42.729
93.000	36.301	31.214	33.337	35.447	39.779	36.537	41.534	38.433	40.965	44.154	40.599
94.000	34.014	28.869	31.119	33.323	37.625	34.273	39.403	36.167	38.666	41.965	38.317
95.000	31.525	26.343	28.709	31.000	35.266	31.761	37.026	33.667	36.183	39.541	35.769
96.000	28.790	23.511	26.009	28.394	32.664	29.023	34.347	30.819	33.363	36.713	32.862
97.000	25.597	20.296	22.776	25.338	29.531	26.017	31.302	27.518	30.074	33.452	29.437
98.000	21.791	16.569	18.939	21.664	25.656	22.285	27.305	23.312	25.896	29.491	25.206
98.500	19.583	14.297	16.750	19.441	23.166	19.916	24.645	20.454	23.156	26.837	22.502
99.000	16.918	11.679	14.328	16.596	19.977	16.836	21.268	17.107	19.707	23.316	19.081
99.100	16.243	11.007	13.681	15.915	19.197	16.252	20.368	16.262	18.859	22.436	18.193
99.300	14.872	9.760	12.548	14.281	17.359	14.721	18.446	14.512	17.020	20.392	16.324
99.600	12.339	7.439	10.265	11.453	13.815	11.775	14.649	11.127	13.576	16.549	12.537
99.900	7.640	3.671	6.073	5.908	7.428	6.248	7.841	5.544	8.063	8.958	6.224
99.930	7.099	3.108	5.422	5.120	6.567	5.287	6.349	4.747	6.862	7.051	5.028
99.960	5.953	2.451	4.486	3.863	4.464	3.821	5.119	3.334	5.605	5.712	3.876
99.990	1.014	1.220	2.220	1.772	1.265	1.710	2.820	1.936	2.499	2.621	1.987
99.995	1.003	0.661	1.772	1.249	0.644	1.070	1.911	1.409	1.366	1.682	1.461

D Estimated Distribution Parameters: Wealth

All parameter estimates reported in this section are the point estimates obtained after averaging the results across all five implicates of the SCF data.

D.1 Type II Pareto Distribution

Table A8: Estimated σ parameters for type II Pareto distribution in 1000 USD

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.2	1.1	0.8	1.1	1.6	1.6	2.5	2.4	2.1	2.2	2.9
92.000	1.4	1.4	0.8	1.4	1.9	1.8	3.1	2.7	2.4	3.0	3.4
93.000	1.4	1.5	0.9	1.6	2.2	2.2	4.1	2.8	2.8	3.5	4.2
94.000	1.7	1.7	1.0	2.0	2.9	2.9	4.5	3.1	3.4	3.4	5.1
95.000	2.1	1.8	1.1	2.3	3.8	3.7	4.8	3.8	4.1	4.2	6.1
96.000	2.2	2.5	1.5	2.9	5.3	4.6	6.0	4.7	4.9	5.9	7.6
97.000	2.7	2.7	1.9	4.6	7.9	6.8	7.3	6.6	6.3	8.8	7.6
98.000	4.3	3.2	3.3	6.7	8.7	10.2	9.8	6.1	8.6	10.0	8.9
98.500	5.2	4.3	4.8	6.6	11.1	9.8	8.8	7.8	8.2	10.0	12.0
99.000	5.5	4.4	7.2	7.2	10.0	10.6	8.1	10.6	8.6	10.9	12.7
99.100	6.9	4.6	9.0	7.6	11.6	9.7	8.1	11.5	9.8	13.7	12.9
99.300	10.3	5.7	10.8	9.0	13.6	10.8	10.8	11.7	9.7	16.5	18.8
99.600	10.8	6.4	16.7	10.8	14.3	15.2	13.1	13.7	11.2	18.6	22.5
99.900	9.6	15.5	22.8	24.1	24.0	27.0	42.5	24.7	34.4	20.0	47.8
99.930	11.7	20.4	21.5	55.9	22.6	74.8	61.5	32.8	48.4	57.8	79.7
99.960	15.3	13.0	46.1	46.1	87.2	230.8	102.9	126.3	63.3	195.5	63.3
99.990	57.1	62.4	95.4	35.2	81.3	121.3	101.7	201.1	105.8	131.6	250.5

Table A9: Estimated alpha parameters for type II Pareto distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.60	1.65	1.39	1.43	1.55	1.49	1.62	1.69	1.57	1.46	1.60
92.000	1.64	1.72	1.28	1.50	1.59	1.53	1.69	1.70	1.62	1.56	1.64
93.000	1.59	1.72	1.29	1.54	1.61	1.57	1.82	1.69	1.66	1.59	1.71
94.000	1.67	1.76	1.29	1.59	1.71	1.66	1.82	1.67	1.72	1.54	1.79
95.000	1.73	1.72	1.26	1.62	1.80	1.73	1.79	1.73	1.76	1.57	1.85
96.000	1.67	1.86	1.36	1.66	1.96	1.78	1.86	1.81	1.81	1.68	1.94
97.000	1.69	1.79	1.38	1.86	2.21	1.94	1.90	1.97	1.87	1.85	1.82
98.000	1.88	1.74	1.54	2.00	2.12	2.10	1.97	1.73	1.96	1.80	1.75
98.500	1.95	1.86	1.71	1.86	2.24	1.96	1.78	1.79	1.81	1.69	1.86
99.000	1.76	1.71	1.83	1.75	1.95	1.87	1.57	1.85	1.67	1.57	1.74
99.100	1.93	1.69	1.97	1.75	2.03	1.75	1.52	1.87	1.72	1.67	1.70
99.300	2.27	1.74	2.02	1.76	2.07	1.73	1.60	1.78	1.61	1.70	1.88
99.600	2.00	1.58	2.21	1.68	1.86	1.73	1.49	1.66	1.48	1.56	1.77
99.900	1.25	1.59	1.82	1.60	1.61	1.58	1.74	1.47	1.66	1.12	1.66
99.930	1.15	1.63	1.61	2.26	1.38	2.33	1.90	1.49	1.76	1.54	1.98
99.960	1.06	1.14	1.97	1.68	2.44	4.54	2.20	2.54	1.72	2.45	1.49
99.990	0.82	1.39	1.88	0.95	1.66	1.84	1.42	2.07	1.46	1.29	1.70

D.2 Type I Pareto Distribution

Table A10: Estimated alpha parameters for type I Pareto distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.54	1.58	1.46	1.47	1.58	1.54	1.51	1.55	1.49	1.42	1.46
92.000	1.55	1.59	1.47	1.48	1.59	1.56	1.53	1.57	1.51	1.43	1.48
93.000	1.57	1.60	1.48	1.50	1.61	1.57	1.55	1.59	1.54	1.45	1.51
94.000	1.58	1.62	1.50	1.51	1.63	1.59	1.58	1.61	1.56	1.47	1.54
95.000	1.60	1.64	1.51	1.53	1.66	1.62	1.62	1.63	1.59	1.49	1.57
96.000	1.62	1.66	1.53	1.56	1.70	1.65	1.66	1.66	1.63	1.52	1.62
97.000	1.64	1.68	1.56	1.59	1.77	1.70	1.71	1.70	1.67	1.57	1.66
98.000	1.67	1.71	1.60	1.66	1.87	1.78	1.76	1.74	1.73	1.62	1.70
98.500	1.69	1.73	1.65	1.71	1.92	1.83	1.79	1.76	1.76	1.64	1.72
99.000	1.72	1.75	1.73	1.74	1.97	1.87	1.80	1.79	1.78	1.65	1.75
99.100	1.72	1.76	1.75	1.75	1.97	1.88	1.81	1.80	1.78	1.66	1.75
99.300	1.74	1.77	1.81	1.76	1.99	1.89	1.81	1.82	1.79	1.67	1.77
99.600	1.76	1.78	1.93	1.79	2.02	1.92	1.83	1.83	1.81	1.68	1.79
99.900	1.58	1.83	2.13	1.83	2.07	2.03	1.99	1.88	1.90	1.65	1.84
99.930	1.52	1.85	2.16	1.87	2.12	2.08	2.05	1.91	1.95	1.66	1.89
99.960	1.43	1.86	2.20	1.94	2.21	2.34	2.21	2.00	2.04	1.78	1.95
99.990	1.39	2.08	2.55	2.08	2.85	2.77	2.70	2.34	2.29	2.00	2.30
99.995	3.79	2.49	2.85	2.10	3.04	3.14	3.16	3.12	2.52	2.12	3.21

D.3 Exponential Distribution

Table A11: Estimated scale parameter (μ) Exponential Distribution in millions

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.9	1.7	1.9	2.5	3.1	3.3	4.1	3.4	3.6	4.7	4.9
92.000	2.1	1.9	2.1	2.7	3.4	3.6	4.5	3.7	3.9	5.2	5.3
93.000	2.2	2.0	2.3	3.0	3.7	4.0	5.0	4.0	4.3	5.7	5.8
94.000	2.4	2.2	2.5	3.3	4.2	4.5	5.4	4.3	4.7	6.2	6.4
95.000	2.7	2.4	2.9	3.7	4.7	5.1	6.0	4.9	5.3	7.0	7.1
96.000	3.1	2.9	3.4	4.3	5.5	5.8	6.8	5.7	6.0	8.3	8.1
97.000	3.7	3.3	4.1	5.3	6.5	7.1	7.9	6.7	7.0	9.9	9.2
98.000	4.7	4.1	5.5	6.6	7.7	8.8	9.7	8.0	8.8	12.2	11.5
98.500	5.5	4.9	6.6	7.4	8.8	9.8	10.8	9.5	9.9	13.8	13.7
99.000	6.7	5.9	8.3	9.1	10.0	11.6	12.8	11.9	12.1	17.3	16.7
99.100	7.3	6.3	8.9	9.6	10.7	12.0	13.5	12.5	13.0	18.8	17.7
99.300	8.4	7.3	10.1	11.1	12.1	13.7	15.9	14.1	14.5	21.7	20.8
99.600	10.6	9.7	13.0	14.6	15.4	18.6	21.5	18.9	19.2	29.4	27.9
99.900	21.6	21.0	23.4	31.7	31.4	37.6	47.3	39.6	43.4	62.5	63.4
99.930	26.9	25.2	27.1	39.9	36.6	49.6	57.1	49.1	52.5	83.1	76.7
99.960	37.7	32.3	37.1	50.6	52.3	61.7	72.5	70.0	68.6	116.8	98.9
99.990	88.3	71.8	65.0	80.9	75.2	89.5	111.7	133.4	124.1	185.9	226.6
99.995	70.4	94.8	88.4	124.4	90.3	111.2	145.0	125.3	172.0	249.7	317.5

D.4 LogNormal Distribution

Table A12: Estimated shape parameter (σ) LogNormal Distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.74	1.88	1.64	1.95	1.82	1.87	1.87	1.65	1.76	1.83	1.80
92.000	1.66	1.64	1.77	1.84	1.77	1.86	1.91	1.72	1.82	1.69	1.79
93.000	1.95	1.72	1.75	1.75	1.97	1.80	1.63	1.69	1.82	1.69	1.77
94.000	1.75	1.56	1.85	1.79	1.88	1.82	1.63	1.82	1.88	1.75	1.67
95.000	1.64	1.73	1.92	1.67	1.79	1.80	1.73	1.86	1.69	1.90	1.69
96.000	1.83	1.61	1.77	2.09	1.89	1.84	1.83	1.79	1.71	1.78	1.58
97.000	1.79	1.62	1.78	1.73	1.67	1.78	1.69	1.64	1.82	1.76	1.73
98.000	2.19	1.94	2.12	1.64	1.74	1.59	1.65	1.76	1.65	1.85	1.84
98.500	1.65	1.61	1.79	1.61	1.71	1.68	1.65	1.74	1.72	1.66	1.89
99.000	1.94	2.04	1.72	1.71	1.92	1.67	1.95	1.82	1.93	2.04	1.74
99.100	2.28	1.85	1.65	1.75	1.63	1.86	2.10	2.19	1.99	2.02	2.20
99.300	2.27	2.16	1.81	1.67	1.96	1.71	1.94	1.74	2.02	1.91	1.46
99.600	1.80	2.11	1.68	1.67	1.73	2.06	1.83	1.99	1.90	2.29	1.73
99.900	1.72	2.31	2.34	2.04	1.81	1.96	1.75	2.44	2.04	2.19	2.24
99.930	2.34	2.33	1.70	2.19	2.55	2.41	1.91	3.07	1.94	1.99	1.84
99.960	1.81	1.95	1.96	2.06	1.88	1.87	1.63	1.67	1.72	1.75	1.96
99.990	2.34	2.44	2.14	3.27	2.15	1.93	2.72	2.35	2.39	2.69	2.86
99.995	3.90	2.83	2.61	3.44	3.40	2.65	2.68	2.73	2.81	2.28	2.89

Table A13: Estimated scale parameter (μ) LogNormal Distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	13.26	13.10	13.15	13.30	13.64	13.63	13.91	13.89	13.83	14.02	14.13
92.000	13.38	13.31	13.15	13.48	13.76	13.74	14.03	13.95	13.94	14.22	14.24
93.000	13.34	13.36	13.26	13.62	13.81	13.88	14.35	14.02	14.04	14.33	14.38
94.000	13.54	13.54	13.35	13.74	14.01	14.06	14.43	14.05	14.18	14.36	14.56
95.000	13.71	13.54	13.44	13.89	14.18	14.23	14.49	14.19	14.36	14.46	14.69
96.000	13.77	13.82	13.71	13.91	14.41	14.36	14.64	14.41	14.52	14.75	14.90
97.000	13.94	13.94	13.91	14.40	14.72	14.72	14.86	14.73	14.70	15.04	14.95
98.000	14.16	14.06	14.23	14.71	14.85	15.07	15.14	14.77	15.04	15.22	15.09
98.500	14.47	14.39	14.57	14.80	15.04	15.12	15.18	14.97	15.12	15.36	15.33
99.000	14.51	14.40	14.87	14.93	15.06	15.29	15.16	15.23	15.11	15.36	15.51
99.100	14.54	14.48	14.99	14.98	15.23	15.20	15.09	15.23	15.24	15.55	15.41
99.300	14.81	14.60	15.10	15.15	15.32	15.35	15.39	15.41	15.26	15.76	15.90
99.600	15.10	14.82	15.44	15.45	15.56	15.52	15.68	15.57	15.53	15.88	16.11
99.900	15.65	15.67	15.87	16.12	16.23	16.19	16.65	16.19	16.46	16.50	16.75
99.930	15.68	15.90	16.11	16.39	16.02	16.61	16.87	16.20	16.73	17.13	17.15
99.960	16.30	16.12	16.40	16.69	16.87	17.18	17.28	17.24	17.08	17.70	17.31
99.990	17.52	16.99	17.09	16.46	17.29	17.43	17.31	17.84	17.62	17.86	17.87
99.995	16.87	17.36	17.37	17.29	17.12	17.56	17.92	17.70	17.90	18.23	18.78

E Estimated Distribution Parameters: Income

All parameter estimates reported in this section are the point estimates obtained after averaging the results across all five implicates of the SCF data.

E.1 Type II Pareto Distribution

Table A14: Estimated σ parameters for type II Pareto distribution in 1000 USD

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.2
92.000	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.1	0.1	0.2
93.000	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.2	0.2	0.2	0.2
94.000	0.1	0.2	0.2	0.1	0.2	0.2	0.2	0.3	0.2	0.2	0.3
95.000	0.2	0.2	0.2	0.2	0.2	0.2	0.3	0.3	0.3	0.3	0.3
96.000	0.2	0.2	0.2	0.2	0.3	0.2	0.3	0.4	0.3	0.3	0.4
97.000	0.2	0.3	0.1	0.2	0.4	0.3	0.5	0.6	0.4	0.5	0.5
98.000	0.3	0.3	0.1	0.4	0.5	0.4	0.7	0.6	0.6	0.8	0.9
98.500	0.3	0.2	0.2	0.5	0.7	0.5	0.9	0.6	0.6	1.0	1.0
99.000	0.4	0.4	0.3	0.7	1.0	0.6	1.2	0.9	0.6	1.1	1.5
99.100	0.4	0.4	0.3	0.6	1.0	0.7	1.2	0.9	0.6	1.5	1.6
99.300	0.6	0.5	0.5	0.8	1.2	1.0	1.3	0.8	0.8	1.9	1.9
99.600	0.4	0.7	1.2	1.3	0.6	2.2	2.2	1.3	0.9	2.6	1.9
99.900	1.6	0.8	1.0	4.3	6.1	4.6	2.4	2.4	4.7	7.7	4.4
99.930	3.0	0.5	3.0	9.8	-37.4	6.1	3.1	1.0	3.4	8.6	1.8
99.960	-2.9	2.9	-13.0	-12.7	8.8	3.0	3.3	1.6	4.1	7.5	0.9
99.990	0.8	5.2	0.1	2.0	2.4	4.1	37.9	7.9	6.6	914.6	7.9
99.995	0.7	9.0	6.3	34.3	7.7	7.6	8.1	10.4	26.3	10.3	6.5

Table A15: Estimated alpha parameters for type II Pareto distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.64	1.88	1.61	1.41	1.35	1.47	1.35	1.56	1.54	1.28	1.50
92.000	1.63	1.91	1.57	1.48	1.33	1.47	1.39	1.59	1.43	1.26	1.59
93.000	1.56	1.89	1.59	1.49	1.30	1.45	1.38	1.62	1.50	1.36	1.55
94.000	1.53	2.09	1.72	1.49	1.34	1.45	1.42	1.78	1.58	1.39	1.67
95.000	1.62	1.97	1.78	1.56	1.41	1.42	1.48	1.88	1.56	1.45	1.66
96.000	1.58	1.90	1.63	1.48	1.40	1.45	1.50	1.83	1.59	1.34	1.71
97.000	1.42	2.19	1.42	1.49	1.46	1.61	1.64	2.18	1.72	1.47	1.80
98.000	1.43	1.92	1.23	1.69	1.51	1.63	1.72	1.98	1.77	1.72	2.08
98.500	1.46	1.71	1.21	1.70	1.61	1.65	1.76	1.83	1.64	1.69	2.05
99.000	1.38	1.96	1.28	1.80	1.77	1.59	1.87	2.03	1.52	1.64	2.28
99.100	1.36	1.84	1.29	1.67	1.69	1.63	1.78	1.98	1.43	1.77	2.36
99.300	1.47	2.02	1.48	1.76	1.71	1.76	1.73	1.80	1.52	1.87	2.40
99.600	1.12	2.10	1.85	2.02	0.96	2.20	1.92	1.88	1.38	1.90	2.16
99.900	0.83	1.35	1.03	2.47	2.01	2.34	1.42	1.74	1.92	2.29	2.37
99.930	1.03	0.84	1.43	4.01	-5.04	2.66	1.36	1.12	1.44	2.17	1.32
99.960	-0.03	1.66	-1.84	-2.76	2.97	1.49	1.24	1.08	1.21	1.85	0.91
99.990	0.46	1.83	0.23	0.92	0.58	1.10	2.67	1.35	1.12	54.69	1.22
99.995	0.42	1.66	0.49	2.50	0.85	0.98	0.90	1.15	1.38	1.11	0.87

E.2 Type I Pareto Distribution

Table A16: Estimated alpha parameters for type I Pareto distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	1.63	2.00	1.69	1.71	1.58	1.70	1.57	1.75	1.61	1.50	1.71
92.000	1.62	2.00	1.68	1.71	1.59	1.70	1.57	1.75	1.61	1.50	1.71
93.000	1.60	1.99	1.67	1.71	1.59	1.70	1.57	1.76	1.61	1.50	1.72
94.000	1.59	1.99	1.66	1.71	1.59	1.70	1.58	1.77	1.61	1.51	1.73
95.000	1.58	1.99	1.65	1.72	1.60	1.70	1.59	1.78	1.62	1.52	1.75
96.000	1.56	1.99	1.64	1.72	1.62	1.70	1.60	1.80	1.62	1.53	1.77
97.000	1.53	1.98	1.61	1.74	1.64	1.72	1.62	1.83	1.63	1.54	1.80
98.000	1.50	1.98	1.56	1.77	1.68	1.75	1.67	1.86	1.65	1.59	1.84
98.500	1.48	1.96	1.54	1.80	1.71	1.77	1.69	1.87	1.66	1.64	1.88
99.000	1.45	1.95	1.53	1.85	1.78	1.81	1.73	1.89	1.65	1.69	1.93
99.100	1.44	1.95	1.52	1.87	1.80	1.82	1.74	1.89	1.65	1.70	1.95
99.300	1.42	1.95	1.52	1.90	1.83	1.85	1.75	1.89	1.65	1.74	1.97
99.600	1.37	1.94	1.53	2.02	1.93	1.92	1.79	1.89	1.65	1.84	1.99
99.900	1.24	1.87	1.47	2.23	2.41	2.16	1.79	1.85	1.73	2.02	1.97
99.930	1.34	1.87	1.53	2.30	2.59	2.23	1.75	1.78	1.76	2.17	1.92
99.960	1.51	1.96	1.76	2.46	3.00	2.22	1.74	1.72	1.84	2.28	1.85
99.990	2.31	3.44	1.27	2.22	2.10	2.32	1.96	1.76	2.40	2.41	1.88
99.995	2.24	3.62	1.31	2.11	1.99	2.38	2.54	1.90	2.31	3.01	2.15

E.3 Exponential Distribution

Table A17: Estimated scale parameter (μ) Exponential Distribution in millions

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	0.2	0.1	0.2	0.2	0.3	0.3	0.3	0.3	0.3	0.4	0.3
92.000	0.2	0.2	0.2	0.2	0.3	0.3	0.4	0.3	0.3	0.4	0.4
93.000	0.3	0.2	0.2	0.3	0.4	0.3	0.4	0.3	0.4	0.5	0.4
94.000	0.3	0.2	0.2	0.3	0.4	0.3	0.5	0.4	0.4	0.5	0.5
95.000	0.3	0.2	0.3	0.3	0.5	0.4	0.5	0.4	0.4	0.6	0.5
96.000	0.4	0.2	0.3	0.4	0.5	0.4	0.6	0.5	0.5	0.7	0.6
97.000	0.4	0.3	0.3	0.5	0.7	0.5	0.8	0.6	0.6	0.9	0.7
98.000	0.6	0.3	0.4	0.6	0.9	0.7	1.0	0.7	0.8	1.2	0.9
98.500	0.7	0.4	0.5	0.7	1.0	0.8	1.2	0.8	0.9	1.4	1.1
99.000	1.0	0.5	0.7	0.9	1.3	1.0	1.4	1.0	1.2	1.7	1.4
99.100	1.1	0.5	0.8	1.0	1.4	1.1	1.5	1.1	1.2	1.9	1.5
99.300	1.3	0.6	1.0	1.2	1.6	1.3	1.8	1.2	1.5	2.2	1.7
99.600	1.8	0.8	1.4	1.6	2.1	1.9	2.5	1.7	2.1	2.9	2.2
99.900	6.1	2.0	3.8	4.1	5.2	3.7	5.1	4.2	5.5	6.5	5.5
99.930	6.9	2.5	5.0	5.0	6.2	4.1	6.9	4.9	7.0	8.1	7.1
99.960	10.8	3.9	6.8	6.9	5.6	5.3	9.3	8.7	10.0	9.7	10.1
99.990	31.8	7.8	17.5	18.9	18.8	12.6	25.0	23.9	19.6	24.9	35.2
99.995	32.0	15.6	32.1	36.3	31.7	22.5	31.3	40.6	37.6	33.9	60.1

E.4 LogNormal Distribution

Table A18: Estimated shape parameter (σ) LogNormal Distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	2.04	2.55	2.00	2.20	2.00	1.81	1.97	1.90	1.77	1.97	2.07
92.000	1.88	2.15	2.11	2.29	1.88	2.02	1.77	1.74	2.00	2.07	1.69
93.000	2.00	2.34	2.39	1.82	2.14	1.72	1.92	1.94	1.92	1.92	1.95
94.000	2.00	1.92	2.02	1.99	2.38	1.99	2.16	1.70	2.08	1.81	2.15
95.000	2.25	1.97	1.68	1.64	1.85	2.46	1.90	1.76	1.78	1.70	1.74
96.000	1.82	1.84	1.64	1.94	1.89	2.71	2.09	1.94	1.78	1.85	1.79
97.000	2.03	1.86	1.78	1.93	1.94	1.89	1.88	1.60	1.71	1.79	1.71
98.000	1.89	1.70	1.98	1.63	1.73	1.99	1.74	1.64	1.67	1.72	1.78
98.500	1.69	1.93	1.99	1.79	1.78	1.79	1.73	1.81	1.63	1.97	1.83
99.000	2.01	1.58	2.05	1.71	1.78	1.70	1.68	1.68	1.79	1.73	1.72
99.100	1.99	2.10	1.85	1.88	1.59	1.82	1.91	1.73	1.97	1.71	1.62
99.300	1.74	2.00	1.74	2.43	1.81	1.88	1.77	1.99	1.59	1.92	1.98
99.600	2.02	1.39	1.74	1.40	2.76	1.70	1.89	1.84	1.78	1.75	1.52
99.900	2.66	1.69	1.89	1.89	1.79	1.84	1.98	1.39	1.50	1.92	1.74
99.930	2.50	2.30	2.12	1.56	1.43	1.40	1.78	1.79	1.64	1.64	2.24
99.960	1.82	3.64	1.40	1.45	2.00	2.10	1.68	2.75	2.17	1.61	1.96
99.990	3.67	2.78	5.05	2.62	4.98	2.45	2.30	2.49	2.04	1.86	2.42
99.995	4.51	2.16	2.33	2.19	3.67	2.88	3.17	2.94	2.71	2.59	3.45

Table A19: Estimated scale parameter (μ) LogNormal Distribution

year cut	1989	1992	1995	1998	2001	2004	2007	2010	2013	2016	2019
91.000	10.92	10.46	10.72	10.77	11.24	11.13	11.30	11.20	11.32	11.39	11.36
92.000	11.02	10.68	10.72	10.91	11.34	11.15	11.46	11.32	11.26	11.42	11.58
93.000	11.03	10.65	10.75	11.16	11.33	11.30	11.51	11.37	11.43	11.66	11.57
94.000	11.09	11.00	11.03	11.20	11.40	11.31	11.58	11.62	11.55	11.81	11.67
95.000	11.19	11.01	11.28	11.48	11.76	11.25	11.83	11.75	11.74	12.02	11.89
96.000	11.47	11.10	11.35	11.48	11.88	11.30	11.94	11.75	11.89	12.04	12.04
97.000	11.51	11.43	11.32	11.69	12.12	11.93	12.29	12.20	12.17	12.38	12.25
98.000	11.82	11.55	11.39	12.14	12.46	12.17	12.62	12.30	12.47	12.81	12.58
98.500	12.11	11.53	11.58	12.27	12.68	12.40	12.81	12.33	12.57	12.90	12.67
99.000	12.29	11.97	11.93	12.56	12.97	12.62	13.08	12.66	12.71	13.16	12.99
99.100	12.37	11.84	12.11	12.51	13.05	12.67	13.05	12.68	12.64	13.31	13.06
99.300	12.69	12.09	12.45	12.49	13.16	12.90	13.20	12.68	13.03	13.44	13.12
99.600	12.79	12.56	12.88	13.23	12.92	13.44	13.56	13.06	13.27	13.80	13.45
99.900	14.18	13.20	13.75	13.88	14.50	14.05	14.18	13.97	14.49	14.60	14.15
99.930	14.23	13.04	14.16	14.21	15.05	14.32	14.40	13.71	14.60	14.83	13.81
99.960	15.47	13.21	14.80	14.47	14.44	14.14	14.80	13.93	14.87	14.97	13.94
99.990	14.07	13.99	12.84	14.39	10.71	14.90	15.97	15.28	15.77	16.02	15.45
99.995	13.57	15.04	16.29	15.65	15.40	15.46	15.55	15.64	16.23	15.80	15.68