Optimal bike allocations in a competitive bike sharing market

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Abstract: This paper studies the bike allocation problem in a competitive bike sharing market. To overcome computational challenges, a continuum approximation (CA) approach is applied, where the allocation points and user demand are assumed to be continuously distributed in a two-dimensional region. Companies offering bike sharing service bear both allocation cost and bike depreciation cost while earning revenue from fare collection. The user's selection of bike service is affected by both walking distance and preference towards bike quality. The elasticity of the demand is considered in relation to the density of allocation points in the market. A leader-follower Stackelberg competition model is developed to derive the optimal allocation strategy for market leader. Two sets of numerical studies - one hypothetical case and one from a real case - are conducted to specify the impact of the parameters on model performance and illustrate how the proposed model can be applied to support the decision making.

Keywords: bike sharing, allocation strategy, demand elasticity, continuous approximation, Stackelberg competition

1. Introduction

Bike sharing service is experiencing a evolution from station-based system to the form of dockless (also called stationless) system. This innovative system offers the users great flexibility in the way that users can pick-up and drop off the bikes nearby their trip origins and destinations through the mobile phone application. Due to the one-way usage characteristics, bikes are easily unbalanced distributed in the market, which leads to either over-supply or under-supply of bikes in certain areas. This problem has been usually solved by bike sharing relocation models and algorithms in the operational stage. However, it is ideal if the spatial distribution of the bike allocation points at the initial stage could be properly determined so that the bike sharing service quality can be improved. As dockless bike sharing is an emerging service, there is only one or few limited operators in the market at the initial stage. Such conditions could easily lead to a monopoly or duopoly situation, wherein the market has been dominant by one or two primary operators. As the market leader, it is imperative to understand how to arrange the bike allocation points to capture the maximum users in a short time so that the patronage will maintain or at least not drop too much when a competitor enters the market.

This problem can be regarded as the resource allocation problem in the bike sharing system, which aims to find the optimal allocation strategy with minimum cost. It is critical to support the sustainable development of bike sharing system. Many bike sharing systems failed due to financial mismanagement or operation issues. Oversupplying bikes in the system to obtain market sharing has been banned in many countries as it not only creates wastes but also occupies street spaces. The optimal bike allocation strategy can significantly reduce the number of bikes, increase bike usage efficiency and enhance the customers' satisfaction, which can contribute to the sustainable development of bike sharing system. The operator could directly achieve the best operation performance using less cost while the society and customers can obtain reliable low-carbon transport services in a long term (Rojas-Rueda et al., 2012; Cavallaro et al., 2017; Otero et al., 2018).

To address this question, this paper proposes a continuous approximation (CA) approach to determine the bike allocation strategy in a competitive market, wherein the output is in the form of optimal density function of bike allocation points. The CA approach is ideal to support the dockless bike allocation problem, as the dockless bike sharing demand is scattered in space and the supply has to cover space in order to make profit. The allocation point location variable and demand variable are approximated by continuous density functions in CA approach, which can realistically reflect the demand and supply in bike sharing system, especially the new type of dockless bike sharing system. Moreover, to simplify the exploration of the general model, only two competitors are considered in this study. These two competitors represent two types of companies: market leader, which already exists in the market, and market follower, which plans to enter the market soon.

The contributions of this paper are three-fold: (1) a set of CA based game theory models is proposed to seek the optimal bike allocation density considering the market competition. To our best knowledge, this is the first attempt to adopt CA approach in solving bike sharing allocation problem. The CA approach is particularly useful for the service planning of the dockless bike sharing services, as the prohibitively huge number of candidate bike allocation spots due to the feature of dockless system would make the discrete model formulation less efficient to be solved. The continuum modelling approach developed in this study is never assumed to be a substitute, but a complement to the existing discrete approaches, which is more suitable to be applied in the initial stage of service planning. (2) A duopoly regime is considered, wherein a leading bike sharing service provision company enters the market first and may face the competition with a following player. The optimal strategy for market leader is proposed to maximise the profits through a game theoretical modelling formulation. (3) The demand elasticity associated with the density of allocation points by market leader is explicitly considered.

The remainder of this paper is organised as follows. Section 2 summarised the current research outcomes in bike sharing allocation and CA approach development. Section 3 defines the basic concepts in this problem using CA approach. Section 4 develops a Stackelberg competition model and applied a discretised method to solve it. Section 5 introduces two sets of numerical examples to illustrate the mode performance. Section 6 concludes the research findings and points out the future research direction.

2. Literature review

Bike sharing has been promoted by many cities to attract people to use sustainable transportation mode. Better understanding cycling behaviour can support the operator's deicison making process. Remarkable literature has been published using various methods, including behavioural modelling (Griswold et al., 2018; Arellana et al., 2020), game theory (Alsaleh and Sayed, 2022), and entropy (Meng et al., 2014; Poliziani et al., 2022). Among them, game theory approach can provide mathematical framework and could be replicated in other cases.

Recently, great attentions have been given to bike sharing systems in terms of planning and operation. At present, most of the existing studies focus on issues in station-based bike sharing system, such as station location selection (Frade and Ribeiro, 2015), demand prediction (Lin et al, 2018), inventory rebalancing (Schuijbroek et al., 2017), vehicle routing (Ho and Szeto, 2017), and pricing analysis (Zhang et al., 2019). Most of the existing studies that address the bike allocation problem focus on balancing the distribution of the bikes that are already existing in the network, which is essentially a vehicle routing problem. Correspondingly, the research outcomes are generally limited to support the operation of single-player bike sharing system (Tian et al., 2020; Liu et al., 2018). The competition among two companies has not yet been investigated.

Bike allocation position would affect the market competition significantly. Users prefer to get the bikes at the nearest location. Companies need to make bike allocation decisions in response to the time-varying demand in a shared market. There are only few papers that study the initial bike allocation problem in a competitive market. Zhang and Meng (2019) proposed a community structure-based bike allocation framework, wherein allocation strategies for both market leader and market follower were provided. Bikes were designed to be allocated at the points with highest weight in the network. Jiang et al. (2020) developed a multi-period two-stage stochastic model to optimise the investment and management strategies for bike-sharing companies under competition. The allocation strategy was given when the market achieved Nash equilibrium status. Both studies applied discrete approach, wherein they assumed that there was a finite set of potential allocation locations and users were generated from a finite number of demand nodes (Xie et al., 2016). The challenge of discrete allocation approach is that the models are NP-hard and are difficult to be solved, especially for large scale problems.

Continuous Approximation (CA) is an efficient approach to approximate the demand variables using continuous functions, which reduces the complexity of the problem associated with large-scale problems (Li et al., 2016; Tsao and Linh, 2018). Unlike the discrete approach that outputs relatively straightforward location, CA approach offers a density function or service area in terms of the study objectives. CA approach has been successfully applied in facility design problem under competition (Li and Ouyang, 2010). Wang and Ouyang (2013) presented game-theoretical models based on CA approach to optimise service facility location design under spatial competition and facility disruption risks. They also analysed the competition of two symmetric companies and showed how the competition may evolve into three types of Nash equilibria. Wang et al., (2017) further extended the model to solve the dynamic facility location problem for a large-scale growing market. Lei and Ouyang (2018) proposed a CA-based hybrid modelling framework for one-commodity pickup and delivery problem, wherein the interdependent local routing and system-level matching decisions were made simultaneously. Huq et al. (2020) applied CA approach to develop an inventory optimisation model that incorporated the issues of location, production, inventory, and transportation at the same time. Bergmann et al. (2020) extended a continuum approximation-based route distance estimation model by proposing adjustment factors for integrated pickup and delivery operations. A comprehensive review of this topic could be found in Ansari et al. (2018) and Yun et al. (2019).

Bike allocation problem has some similarities to facility design problem, which is to optimise the location to minimise the one-time investment for construction and maximise the coverage area for serving customers (Ouyang et al., 2015). Therefore, it is suitable to adopt CA approach in bike allocation problem. Nevertheless, bike allocation problem has its unique characteristics that necessitate a different definition. From the supply side, allocation cost is associated with not only the investment at the allocation points, but the bike depreciation cost associated with the number of bikes. From the demand side, users, on the one hand, prefer to get the bikes at the nearest place; on the other hand, might sacrifice the distance to a certain degree for a better service (e.g. new bikes, bikes with innovative functions). Meanwhile, the bike sharing demand is heavily affected by the provision of bikes nearby. The low price of bike sharing usage encourages people to try the service if plenty of bikes are available nearby (Meng et al., 2016). The elasticity in demand should not be neglected. These highlight the needs for an optimal bike allocation point design framework that addresses both the supply and demand aspects of bike sharing services at the same time.

3. Problem definition

A competitive bike sharing market includes two companies, which are denoted by company $i \in \{0,1\}$. A nomenclature table is given in the Appendix to list all variables and parameters used in this study. We assume that the two companies apply the same fare structure and employ the same type of bikes. The CA approach assumes that demand is continuously distributed in a two-dimensional region $(x,y)\in\Omega$. The users are supposed to have paid the deposits to both two companies and have the full information on the bikes' position and availability according the Global Position System (GPS) function. To balance the bike distribution, companies allocate bikes to certain allocation points in order to serve unsatisfied demand. Each bike allocation point of company i at location (x,y) has its service area $A_i(x,y)$ with a radius of $r_i(x,y)$, while the inverse function of service area $\left[A_i(x,y)\right]^{-1}$ is the allocation points density $D_i(x,y)$ in the neighbourhood of location (x,y), $D_i(x,y)=\left[A_i(x,y)\right]^{-1}$. We assume there is no capacity constraints for each allocation point. The distance from the user at location (x,y) to the nearest bike from company i is a random variable denoted by $Z_i(x,y)$, which satisfies the following cumulative distribution function:

$$P\left(Z_i(x,y)\le z\right)=\frac{\pi z^2}{A_i(x,y)}=\pi D_i(x,y)z^2,\forall z\in\left[0,1\Big/\sqrt{\pi D_i(x,y)}\right] \tag{1}$$

When a user at location (x,y) has two options from both companies, he/she may choose company i with the probability of $P\left(Z_i(x,y)\le Z_{1-i}(x,y)\right)$. To simplify the illustration, the location coordinate variable (x,y) will be omitted in the rest of paper (e.g. $Z_i(x,y)$ is abbreviated to Z_i), and subscript $1-i$ will be indicated by $-i$.

Proposition 1. If a user always selects the nearest bike, the probability of he/she will use the bike from company i is calculated by:

$$P\left(Z_i\le Z_{-i}\right)=\frac{D_i\left(2D_{\max}-D_{-i}\right)}{2D_{\max}{}^2} \tag{2}$$

Where $D_{\max}=\max\left\{D_i,D_{-i}\right\}$.

Proof. If a user selects the bike from company i, it means that the distance from the user to the bike from company i is shorter than it is from company $-i$. Then we have:

$$
\begin{aligned}
P\left(Z_i \le Z_{-i}\right) &= \int_0^{\infty} P\left(Z_i \le Z_{-i} \middle| Z_i = z\right) dF_{Z_i}\left(z\right) \\
&= \begin{cases} \int_0^{\sqrt{\frac{1}{\pi D_{-i}}}} \left(1-\pi D_{-i} z^2\right) 2\pi D_i z dz, & \textit{if } D_{-i} \ge D_i \\ \int_0^{\sqrt{\frac{1}{\pi D_i}}} \left(1-\pi D_{-i} z^2\right) 2\pi D_i z dz, & \textit{otherwise} \end{cases} \\
&= \begin{cases} \dfrac{D_i}{2D_{-i}}, & \textit{if } D_{-i} \ge D_i \\ 1-\dfrac{D_{-i}}{2D_i}, & \textit{otherwise} \end{cases} \\
&= \frac{D_i\left(2D_{\max} - D_{-i}\right)}{2D_{\max}{}^2}
\end{aligned}
\tag{3}
$$

This completes the proof.

When there is only one company in the market, users' searching area for getting a bike is the service area of the bike allocation position $A_i = \pi r_i^2$. When there is more than one company in the market, users will compare the services from different companies and have their selection preferences. In this case, distance is not the only criteria for selecting a bike sharing service. For example, user might like to walk a further distance to get a bike with better quality components (e.g. a new bike, bike with new function). We set this selection preference as a_i, which is expressed by:

$$
a_i = \left(\frac{r_i}{r_{-i}}\right)^2 = \phi\left[\sum_{j=1}^{J} \omega_j \left(\frac{\delta_{i,j}}{\delta_{-i,j}}\right)^{\xi}\right]^{\gamma}, \quad \xi = -1,1 \tag{4}
$$

where ϕ and γ are coefficient parameters, δ is the factor that affects the users' selection, ω_j is the weightage of the variable j, $\sum_j \omega_j = 1$; ξ is adjusting parameter. When the impact of $\delta_{i,j}$ on a_i is positive, $\xi = 1$; otherwise, $\xi = -1$. In reality, the value of ω_j could be estimated by regression model based on field survey data. To simplify the analysis, we only consider two typical factors in the analysis: ticket fare and feature of the bike. Ticket fare has a negative influence on selection preference and is relevant to other parameters' setting, while feature of the bike has a positive influence and is independent from other parameter settings. Therefore, Eq. (4) could be rewritten as

$$
a_i = \left(\frac{r_i}{r_{-i}}\right)^2 = \phi\left(\alpha\frac{p_{-i}}{p_i} + \beta\frac{s_i}{s_{-i}}\right)^{\gamma} \tag{5}
$$

where α and β are weight parameters, $\alpha + \beta = 1$; p is the ticket fare and s is the features of the bike.

Proposition 2. Considering the selection preference Eq.(5), the probability of a user selecting a bike from company i could be expressed as:

$$P\left(Z_i \le Z_{-i}\right) = \frac{a_i D_i \left(2a_{\max} D_{\max} - a_{-i} D_{-i}\right)}{2\left(a_{\max} D_{\max}\right)^2} \tag{6}$$

where $a_{\max} D_{\max} = \max\left\{a_{-i} D_{-i}, a_i D_i\right\}$.

Proof. Similar to the proof of Proposition 1, we can get

$$\begin{aligned} P\left(Z_i \le Z_{-i}\right) &= \int_0^{\infty} P\left(Z_i \le Z_{-i} \middle| Z_i = z\right) dF_{Z_i}(z) \\ &= \begin{cases} 2\pi D_i a_i \int_0^{\sqrt{\frac{1}{\pi a_{-i} D_{-i}}}} \left(1 - \pi D_{-i} a_{-i} z^2\right) z dz, & \text{if } a_{-i} D_{-i} \ge a_i D_i \\ 2\pi D_i a_i \int_0^{\sqrt{\frac{1}{\pi a_i D_i}}} \left(1 - \pi D_{-i} a_{-i} z^2\right) z dz, & \text{otherwise} \end{cases} \\ &= \frac{a_i D_i \left(2a_{\max} D_{\max} - a_{-i} D_{-i}\right)}{2\left(a_{\max} D_{\max}\right)^2} \end{aligned} \tag{7}$$

This completes the proof.

We assume that the revenue comes from the fare collection. A single use fare p_i is charged to the user when he/she uses the bike from company i. There are two cost components in bike allocation. One is the fixed allocation cost at allocation point (e.g. rental fee and management cost) and the other is the depreciation cost due to the usage of bikes. The profit of company i at location (x, y) is independent of any other allocation points. Thus, considering the demand density λ, the total profit of company i could be expressed by

$$\Pi_i = \int_{\Omega} \left\{P\left(Z_i \le Z_{-i}\right)\lambda p_i - K_i D_i - P\left(Z_i \le Z_{-i}\right)\lambda L_i\right\} d\Omega \tag{8}$$

where K_i is the fixed allocation cost for each allocation point that used by company i in the market, L_i is the depreciation cost per unit bike. The demand elasticity is considered in this study. We assume that the demand is positively affected by the density of allocation point of the dominant company. That is, arranging denser allocation points in the market would result in higher demand. Let Q represent the maximum potential demand density, the demand density λ could be expressed by:

$$\lambda = \frac{a_{\max} D_{\max} Q}{c + a_{\max} D_{\max}} \tag{9}$$

where c is a positive constant value to ensure $\lambda < Q$.

Proposition 3. For company $i \in \{0,1\}$, the expected profit from one unit of market area near location (x, y) for company i could be calculated by:

$$\pi_i\left(D_i,D_{-i}\right)=\frac{a_iD_i\left(2a_{\max}D_{\max}-a_{-i}D_{-i}\right)\left(p_i-L_i\right)Q}{2a_{\max}D_{\max}\left(c+a_{\max}D_{\max}\right)}-K_iD_i \tag{10}$$

Proof. The expected profit from one unit of market area is equal to the fare revenue minus the total cost, which is

$$\begin{aligned}\pi_i\left(D_i,D_{-i}\right)&=P\left(Z_i\le Z_{-i}\right)\lambda p_i-K_iD_i-P\left(Z_i\le Z_{-i}\right)\lambda L_i\\&=\begin{cases}\dfrac{\lambda a_iD_i}{2a_{-i}D_{-i}}\left(p_i-L_i\right)-K_iD_i, & if\ a_{-i}D_{-i}\ge a_iD_i\\ \lambda\left(1-\dfrac{a_{-i}D_{-i}}{2a_iD_i}\right)\left(p_i-L_i\right)-K_iD_i, & otherwise\end{cases}\\&=\begin{cases}\dfrac{a_iQ\left(p_i-L_i\right)D_i}{2\left(c+a_{-i}D_{-i}\right)}-K_iD_i, & if\ a_{-i}D_{-i}\ge a_iD_i\\ \dfrac{Q\left(p_i-L_i\right)\left(2a_iD_i-a_{-i}D_{-i}\right)}{2\left(c+a_iD_i\right)}-K_iD_i, & otherwise\end{cases}\\&=\frac{a_iD_i\left(2a_{\max}D_{\max}-a_{-i}D_{-i}\right)\left(p_i-L_i\right)Q}{2a_{\max}D_{\max}\left(c+a_{\max}D_{\max}\right)}-K_iD_i\end{aligned} \tag{11}$$

Proposition 4. Given the competitor's allocation plan D_{-i}, the payoff function of company i $\pi_i\left(D_i,D_{-i}\right)$ is a concave function with respect to D_i. Therefore, the best response exists and is unique for each company.

Proof. From Eq.(10), we can obtain

$$\pi_i\left(D_i,D_{-i}\right)=\begin{cases}\dfrac{a_iQ\left(p_i-L_i\right)D_i}{2\left(c+a_{-i}D_{-i}\right)}-K_iD_i, & if\ a_{-i}D_{-i}\ge a_iD_i\\ \dfrac{Q\left(p_i-L_i\right)\left(2a_iD_i-a_{-i}D_{-i}\right)}{2\left(c+a_iD_i\right)}-K_iD_i, & otherwise\end{cases} \tag{12}$$

Denote

$$\tilde{\pi}_i\left(D_i,D_{-i}\right)=\left[\frac{a_iQ\left(p_i-L_i\right)}{2\left(c+a_{-i}D_{-i}\right)}-K_i\right]D_i,\quad a_{-i}D_{-i}\ge a_iD_i \tag{13}$$

and

$$\bar{\pi}_i\left(D_i,D_{-i}\right)=\frac{Q\left(p_i-L_i\right)\left(2a_iD_i-a_{-i}D_{-i}\right)}{2\left(c+a_iD_i\right)}-K_iD_i,\quad a_{-i}D_{-i}\le a_iD_i \tag{14}$$

Based on Eq.(13), one can derive D_i is a monotonic function on $\left[0,a_{-i}D_{-i}/a_i\right]$.

From Eq.(14), we can get

$$\frac{\partial\bar{\pi}_i}{\partial D_i}=\frac{a_iQ\left(p_i-L_i\right)\left(2c+a_{-i}D_{-i}\right)}{2\left(c+a_iD_i\right)^2}-K_i \tag{15}$$

and

$$\frac{\partial^2 \bar{\pi}_i}{\partial D_i^2} = -\frac{a_i^2 Q(p_i - L_i)(2c + a_{-i}D_{-i})}{(c + a_i D_i)^3} \tag{16}$$

As fare is greater than depreciation cost $p_i > L_i$, we can get $\frac{\partial^2 \bar{\pi}_i}{\partial D_i^2} < 0$. Then $\bar{\pi}_i(D_i, D_{-i})$ is a concave function with respect to D_i in $[a_{-i}D_{-i}/a_i, +\infty)$ and has its maximum value. As $\tilde{\pi}_i\big|_{a_iD_i=a_{-i}D_{-i}} = \bar{\pi}_i\big|_{a_iD_i=a_{-i}D_{-i}}$, $\frac{\partial \tilde{\pi}_i}{\partial D_i}\Big|_{a_iD_i=a_{-i}D_{-i}} = \frac{\partial \bar{\pi}_i}{\partial D_i}\Big|_{a_iD_i=a_{-i}D_{-i}}$, $\pi_i(D_i, D_{-i})$ is a continuous function. Thus, $\pi_i(D_i, D_{-i})$ is a concave function $[0, +\infty)$.

Proposition 5. The best allocation plan of company i given the competitor's allocation plan D_{-i} is

$$\varphi_i(D_{-i}) = \begin{cases} \sqrt{\frac{Q(p_i - L_i)(2c + a_{-i}D_{-i})}{2a_iK_i}} - \frac{c}{a_i}, & \text{if } \Delta_i(D_{-i}) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

where $\Delta_i(D_{-i}) = \frac{a_i Q(p_i - L_i)}{2(c + a_{-i}D_{-i})} - K_i$.

Proof. when $\Delta_i(D_{-i}) > 0$, D_i is monotonically increasing in $[0, a_{-i}D_{-i}/a_i]$, and a concave function in $[a_{-i}D_{-i}/a_i, +\infty)$. The optimal solution must exist in $[a_{-i}D_{-i}/a_i, +\infty)$. The extremal point satisfies the condition that is $\frac{\partial \pi_i}{\partial D_i} = \frac{a_i Q(p_i - L_i)}{2(c + a_i D_i)^2}(2c + a_{-i}D_{-i}) - K_i = 0$. Therefore, we can get

$$\hat{D}_i = \sqrt{\frac{Q(p_i - L_i)(2c + a_{-i}D_{-i})}{2a_iK_i}} - \frac{c}{a_i} \tag{18}$$

when $\Delta_i(D_{-i}) \le 0$, D_i is monotonically decreasing. Thus, the optimal solution is $\varphi_i(D_{-i}) = 0$.

4. Stackelberg competition

Assuming that one of the companies entered the market first, when the other company enters the market, it would be a typical Stackelberg leader-follower competition in bike sharing systems (Si et al., 2019). Company 0 is assumed as the leader while company 1 is set as the follower to simplify the analysis. To get the maximum profit for company 0, a bi-level optimization model is proposed with consideration of the allocation strategy of company 1 using the continuum approximation approach for the neighbourhood of each location $(x, y) \in \Omega$:

$$\max_{D_0 \ge 0} \pi_0\left(D_0, \hat{D}_1\right) \tag{19}$$

$$s.t. \quad \hat{D}_1 = \arg\max_{D_0 \ge 0} \pi_1\left(D_1, D_0\right) \tag{20}$$

From Eq.(17), Eq.(19) is equivalent to

$$\max_{D_0 \ge 0} \pi_0\left(D_0, \varphi_1\left(D_0\right)\right) \tag{21}$$

The leading company may have two differential strategies based on $\varphi_i\left(D_{-i}\right)$: the first strategy is to prevent the follower from entering the market and establish monopoly, that is

$$\max_{D_0 \ge \Delta_1^{-1}(0)} \pi_0\left(D_0, 0\right) \tag{22}$$

The second strategy would allow the follower to enter the market, that is

$$\max_{0 \le D_0 \le \Delta_1^{-1}(0)} \pi_0\left(D_0, \hat{D}_1\right) \tag{23}$$

We have the following results for this Stackelberg leader-follower competition.

Proposition 6. For the first strategy to prevent the follower from entering the market, the optimal solution is

$$D_0^{+} = \max\left\{\Delta_1^{-1}\left(0\right), \hat{D}_0\right\} \tag{24}$$

Where $\hat{D}_0 = \sqrt{\frac{cQ\left(p_0 - L_0\right)}{a_0 K_0}} - \frac{c}{a_0}$, $\Delta_1^{-1}\left(0\right) = \frac{a_1 Q\left(p_1 - L_1\right) - 2cK_1}{2a_0 K_1}$

Proof. From Eq.(22), we can get

$$\pi_0\left(D_0, 0\right) = \frac{a_0 Q D_0}{c + a_0 D_0}\left(p_0 - L_0\right) - K_0 D_0 \tag{25}$$

Taking the deviation of Eq.(25), it is

$$\hat{D}_0 = \sqrt{\frac{cQ\left(p_0 - L_0\right)}{a_0 K_0}} - \frac{c}{a_0} \tag{26}$$

Meanwhile, we know $D_0 \ge \Delta_1^{-1}\left(0\right)$. Thus, it can be obtained that

$$D_0^{+} = \max\left\{\Delta_1^{-1}\left(0\right), \hat{D}_0\right\} \tag{27}$$

Proposition 7. For the second strategy to allow the follower to enter the market, the optimal solution is

$$D_0^- = \min\left\{\hat{D}_0, \Delta_1^{-1}(0)\right\} \tag{28}$$

where $\hat{D}_0 = \arg\max\left\{\sqrt{\frac{\left[a_0(p_0-L_0)D_0\right]^2 QK_1}{2a_1(p_1-L_1)(2c+a_0D_0)}} - K_0D_0\right\}$

Proof. Based on Eq.(17), we can get

$$\hat{D}_1 = \sqrt{\frac{Q(p_1-L_1)(2c+a_0D_0)}{2a_1K_1}} - \frac{c}{a_1} \tag{29}$$

Taking $\pi_0(D_0, D_1) = \left[\frac{a_0Q(p_0-L_0)}{2(c+a_1D_1)} - K_0\right]D_0$ into Eq.(29), it obtains:

$$\hat{D}_0 = \arg\max\left\{\sqrt{\frac{\left[a_0(p_0-L_0)D_0\right]^2 QK_1}{2a_1(p_1-L_1)(2c+a_0D_0)}} - K_0D_0\right\} \tag{30}$$

Meanwhile, as $0 \le D_0 \le \Delta_1^{-1}(0)$, we can conclude $D_0^- = \min\left\{\hat{D}_0, \Delta_1^{-1}(0)\right\}$.

After market leader makes the decision on which strategy to take, its primary objective is to acquire higher profits. If the leader prevents the follower from entering the market, the total profits should be higher than allowing it entering the market, $\pi_0(D_0^+, 0) > \pi_0(D_0^-, \hat{D}_1)$. Thus, the optimal solution to Eq.(19) is

$$D_0^* = \begin{cases} D_0^+, & if\ \pi_0(D_0^+, 0) > \pi_0(D_0^-, \hat{D}_1) \\ D_0^-, & otherwise \end{cases} \tag{31}$$

The output of the proposed CA model is a continuous density function of allocation points. In order to support the practical application, we adopt a discretised idea to obtain the solution in the market. The study region Ω is firstly divided into certain number of small square cells. The cell size is set as the acceptable walking distance to access the bike services (e.g. less than 50 meters). The solution to the entire market is then converted to get the solution to each cell. Newton's method is applied to find the maximum of Eq. (30), in another word to find the roots of the gradient. Other equations can be calculated directly.

5. Numerical studies

Two sets of numerical studies are used to illustrate the model performance: one is the ideal case with hypothetical data and the other is the real network in Singapore. The solution procedures are coded in Matlab R2019b and run on a laptop with 2.60 GHz and 16GB memory.

5.1 Hypothetical case

A series of hypothetical cases is tested on a 4 × 4 square market region to investigate the influence of parameter setting on the optimal solutions. Suppose the constant parameters are: $Q=2$, $p_0=p_1=5$, $s_0=s_1=1$, $\alpha=\beta=0.5$, $L_0=L_1=0.1$, $c=0.1$, $\phi=1$, $\gamma=1$, $\varepsilon=0.001$, and the location-dependent parameters are $K_0=x$, $K_1=4-x$. For illustration purpose, the fixed operation cost K_0 and K_1 at allocation points are set to be the function with respect to x while holding the value of y constant. The plots of the input functions of K_0 and K_1 are shown in Fig.1.

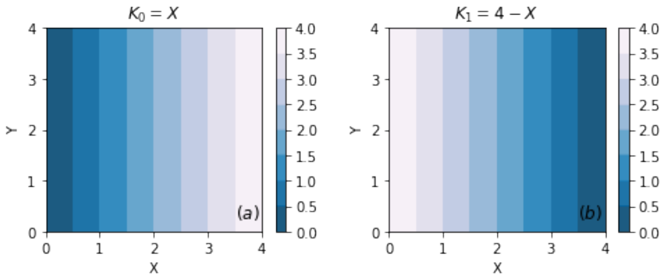

Figure 1 Spatial distribution of the fixed allocation cost by company leader (a) and follower (b)

The influence of K_0 and K_1 on the optimal allocation points density, total profit, total fixed operation cost and bike density are shown in Fig. 2. The bike density reflects the usage. A regular pattern could be detected in four areas in the market.

(1) In the area where K_0 □ K_1 (e.g. $x \in [0, 0.5]$ in Fig.2), with the increase of K_0 (see Fig.1a), market leader can still maintain the monopoly position under the pressure of increased K_0. It is because the market leader can reduce the density of allocation points to save the cost, as the market leader has obvious advantage on K at this stage than the market follower. Though the total revenue will decrease (see Fig. 2c), the number of users does not drastically change (see Fig. 2d).

(2) In the area where $K_0 < K_1$ (e.g. $x \in (0.5, 2]$ in Fig.2), the market leader has to increase the density of allocation points (see Fig.2a) to grasp the market share. The total fixed operation cost increases significantly (see Fig. 2c), while the total profits continue to decrease (see Fig.2b) as the total patronage does not increase accordingly (see Fig.2d).

(3) In the area where $K_0 > K_1$ (e.g. $x \in (2, 2.6]$ in Fig.2) (see Fig.1a), the market leader can still maintain the existing number of users (see Fig.2d) by increasing the density of the allocation points (see Fig.2a) to deter the entry of the market follower. Though the total profit at this stage is greater than that at the market sharing stage after follower's entry, the total profit decreases significantly due to the increasing allocation cost (see Fig.2c).

(4) In the area where K_0 □ K_1(e.g. $x \in (2.6, 4]$ in Fig.2), the market leader has to give up the preventing strategy, as it cannot increase the total profits (see Fig.2b). However, the

market leader can still achieve a positive profit while allocating certain number of bikes. This is different from market follower's situation, as the market leader has its first mover advantage.

When we sum up the total results from the market leader and follower, we can notice that different values of K do not affect the total number of users. However, similar setting of K by the market leader and follower will lead to more fierce competition, which will increase the total cost and reduce the total profits. The critical point is at the position where the market leader has to give up the preventing strategy. Moreover, due to the first mover advantage, $K_0 > K_1$ at the critical point.

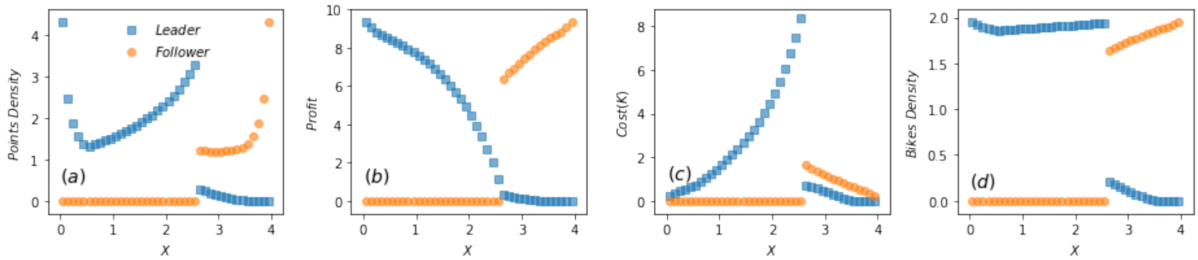

Figure 2 The resultant optimal allocation points density (a), total profit (b), total fixed operation cost (c) and bikes density (d) for company leader and follower

Fig.3 shows the impact of bike feature parameter s on model performance. It could be noted that the lines are not smoothing, which is due to the space discretisation in solution procedure. Taking Fig. 3a as an example, based on the analysis in Fig.2, we know that the market leader has to give up the preventing strategy at the critical point along with the change of K. In reality, the position of this critical point moves smoothly with change of s. In our numerical study, the space discretisation process lets the critical points become the centre point in a cell in the centre of the network, which leads to the movement by leaps and bounds. When s has a minor change, the location of the critical point remains in the same cell. However, the increase of s will decrease the density of allocation points slowly. When there is a significant change of s, the critical point will move to other cells in the network. From Fig.2, the value of the critical points has a huge change, which leads to the obvious kink. If we take a smoothing approach to the adjacent values (e.g. taking average of three adjacent values), a dotted red line in Figure 3(a) can be obtained, which is in line with the trend. Moreover, the influence caused by the increased s is similar to both the market leader and follower, that is the raising of the number of users and profits while squeezing the competitor's market space (a smaller number of users and profits), and vice versa.

The response strategy and the effects caused by the strategy are different, depending on the different market status of the market leader and follower. For example, the increase of s will lead to a decrease in the number of allocation points. However, the allocation cost for the market leader K_0 does not reduce, while the allocation cost for market follower K_1 goes up. It is because they will reduce the density of allocation points in the advantageous area (with smaller K), meanwhile, expand the allocation in the weakness area to attract more users. When the competitor levels up s, the response strategy of market leader is to increase the number of allocation points to maintain the dominant status while the market follower is to reduce the number of allocation points.

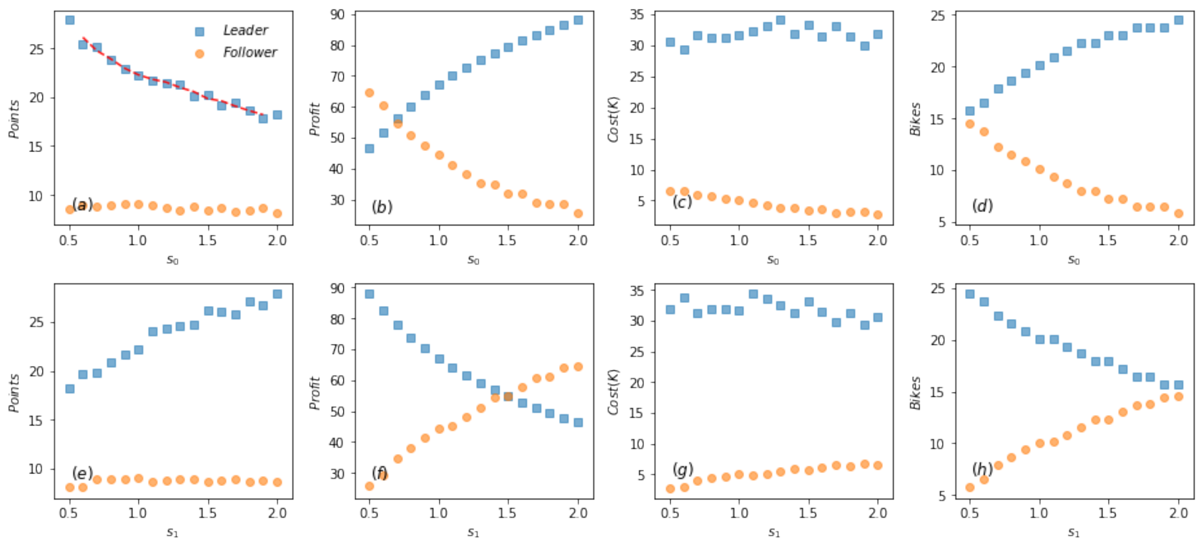

Figure 3 Impact of bike feature parameter s

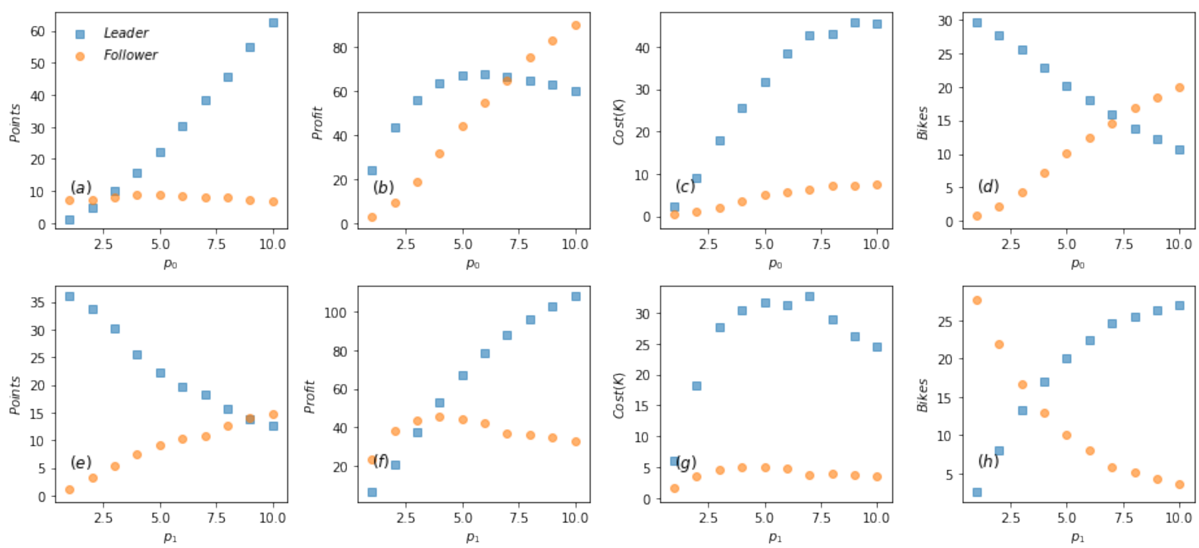

Figure 4 Impact of fare parameter p $(\gamma = 3)$

Figure 4 investigates the impact of fare parameter p on the model performance. The influences of fare changes on the ridership and response strategy for both the market and follower are the same. The increasing of p will reduce the number of users gradually. The total profit will increase and then taper. Because once the total profit increases, the company will then increase the density of allocation points to attract more users or detain existing users. The only difference between the market leader and follower is that the response of the market follower to the rising fare by the market leader is to first increase and then reduce the number of allocation points.

Figure 5 shows the influence of the depreciation fee L on model performance. The increase of L has negative affect on all aspects. The cut in total profits will reduce the total number of optimal allocation points. These negative impacts will give more space to competitor to attract the users and get more profits. The growth trend of two companies are totally opposite to each other. Thus, to control bike depreciation in operation is a key issue for operators.

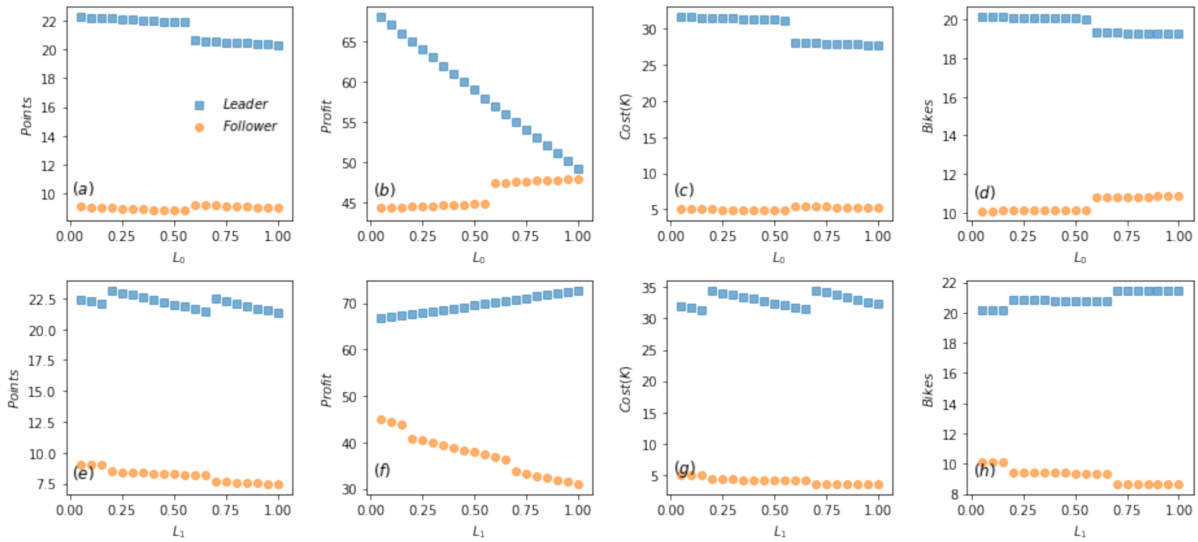

Figure 5 Impact of depreciation parameter L

Figure 6 illustrates the impact of weight parameter α and β on the model performance. In Figs.(a)-(d), the parameters' setting is: $p_0 = 6, p_1 = 5\left(p_0 > p_1\right)$, $s_0 = 1, s_1 = 1.2\left(s_0 < s_1\right)$. It is a condition that the market leader provides cheap service with poor quality bikes. The market leader has more clear-cut advantage, when users are more sensitive to fare price. With a greater fare sensitivity, the market leader can reduce the density of allocation points, but receive higher profits with more users. In Figs.(e)-(h), the parameters' setting is: $p_0 = 5, p_1 = 6\left(p_0 < p_1\right)$, $s_0 = 1.2, s_1 = 1\left(s_0 > s_1\right)$. In this situation, the market leader provides expensive service with high quality bikes. The market follower has its advantage when users are more sensitive to fare price. With a greater of fare sensitivity, the market leader has to increase the number of allocation points to prevent the loss in users, while the total profit reduces.

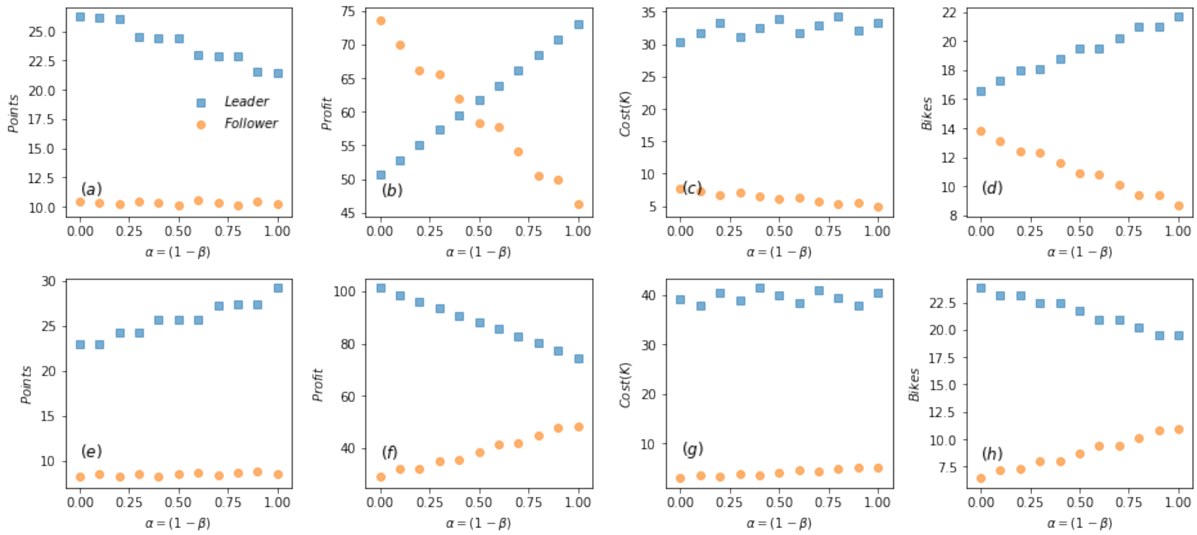

Figure 6 Impact of weight parameter α and β

5.2 Real network in Singapore

A real network located in Jurong Lake District in Singapore is selected to illustrate how the model can be used in practice in different scenarios. Jurong Lake District has been planned to be the second central business district in Singapore with around 472 hectares in size, three major expressways and three metro stations. The mixed land-use pattern, especially associated with high population density, enables the generation of short-distance trips that are suitable to develop bike sharing schemes. The potential demand density has been investigated by a two-stage field survey by Zhang et al. (2017), while the fixed allocation cost is determined by the land use characteristics as defined by Singapore Master Plan. The inputs of this network to our model is shown in Fig.8. Five cases are conducted to discuss allocation strategy in different market scenarios, wherein the first case is the benchmark case. Table 1 summarises the inputs and outputs for all five cases.

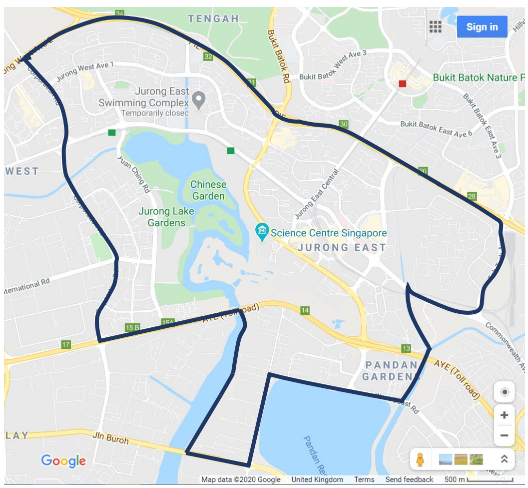

Figure 7 Jurong Lake District in Singapore

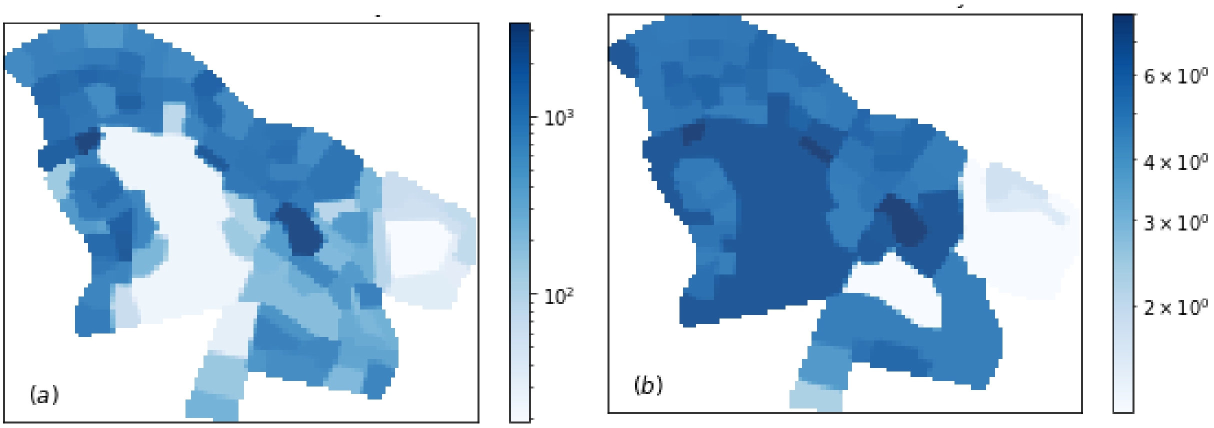

Figure 8 Input value: (a) potential demand density and (b) allocation cost density

Case I (Benchmark case): The results generated for the market leader in the benchmark network are shown in Fig.9, including the density distribution of allocation points, profits, allocation cost and bike depreciation cost. It could be found that the results match the real situation closely, as the areas with high density distribution are mainly metros stations, office and residential buildings with high ratios, and shopping malls. The densest area located in the central right part is the centre in Jurong Lake District, which covers one public transport interchange station, one hospital, one business centre with four shopping malls and one government agency building. From Figure 9(a), we obtained that the optimal number of allocation points is 581. To practically find out the location of these points and service area, we adopt an automated algorithm proposed by Ouyang and Dagazo (2006) via a Voronoi diagram to illustrate the optimal locations for the market leader to allocate bikes, as shown in Figure 10.

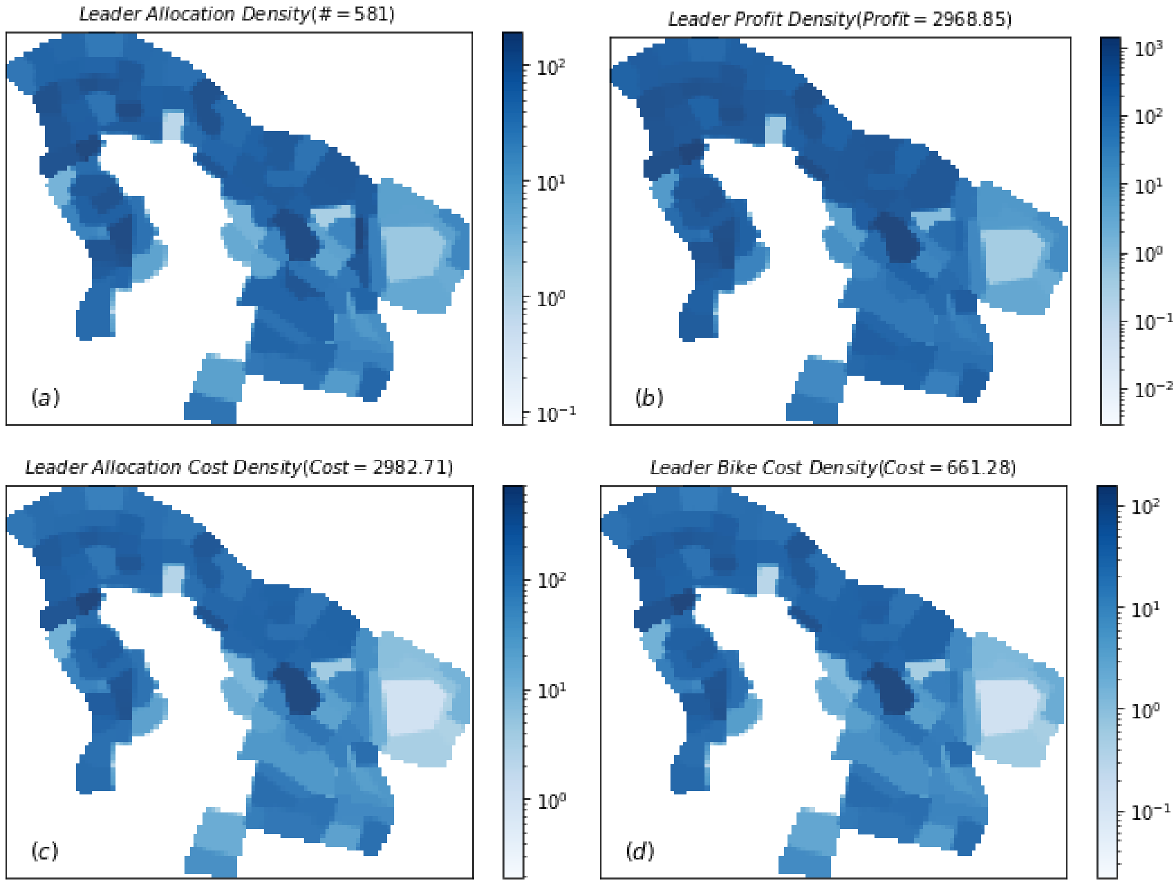

Figure 9 Result from Case I

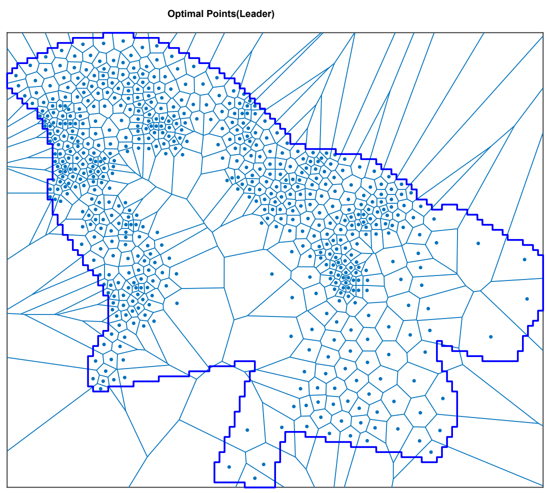

Figure 10 Optimal bike allocation points for market leader

Case II (Follower with more attractiveness): As we have discussed in the hypothetical case, if market follower could provide better service in terms of lower price and higher quality bike, it might force the market leader to give up the preventing strategy. However, though the comprehensive advantage of the market follower is not obvious than the market leader to switch the market position, it still can affect the market leader's decision making. As listed in Table 1, when the market follower's bike feature s_1 improves from 1 to 1.5, market lead has to increase the bike allocation points from 581 to 897 to maintain the dominant position in the market. The allocation cost increases significantly (from 2983 to 4596), while the increase of number of users is very limited (from 6613 to 6808). From users' point of view, more bike allocation points ensure more convenient service. But for company, the total profit becomes less (from 2969 to 1531).

Case III (Follower with half of allocation cost and half of bike depreciation cost): When the market follower starts to enter the market, all costs are half than the market leader. The market leader cannot maintain the monopoly position and lose most of the market share. However, due to the first mover advantage, the market leader can still get positive profits.

Case IV (Follower with newly launched bikes): With the technique development, the market follower might purchase newly launched bikes rather to attract users. The fare ticket, allocation cost and bike depreciation cost are different with the benchmark case. Assume fare ticket and cost components all increase, the influence by adopting newly launched bikes is similar to it in Case III, wherein the market leader cannot maintain the monopoly position but has positive profits.

Case V (Users prefer bike quality rather than fare in decision-making): As the market becomes mature, users' sensitivity towards fare ticket might decrease while the sensitivity towards bike quality might increase. Users are willing to pay higher price to the bikes with good feature. Assume the market follower can provide better bikes, it could grasp the market from the market leader with higher profits.

Table 1 Comparison of system-wide results

Case	Input		# Points	Profit	Allocation cost	Bike cost	Bikes
I	$p_0 = p_1 = 1.0$ $s_0 = s_1 = 1.0$ $\alpha = \beta = 0.5$	$K_0 = K_1$ $L_0 = 0.1$ $L_1 = 0.1$	L: 581 F: 0 T: 581	L: 2969 F: 0 T: 2969	L: 2983 F: 0 T: 2983	L: 661 F: 0 T: 661	L: 6613 F: 0 T: 6613
II	$p_0 = p_1 = 1.0$ $s_0 = 1.0$ $s_1 = 1.5$	$\alpha = \beta = 0.5$ $K_0 = K_1$ $L_0 = L_1 = 0.1$	L: 897 F: 0 T: 897	L: 1531 F: 0 T:1531	L: 4596 F: 0 T: 4596	L: 681 F: 0 T: 681	L: 6808 F: 0 T: 6808
III	$p_0 = p_1 = 1.0$ $s_0 = s_1 = 1.0$ $\alpha = \beta = 0.5$	$K_1 = 0.5K_0$ $L_0 = 0.1$ $L_1 = 0.05$	L: 63 F: 505 T: 568	L: 76 F: 4515 T: 4591	L: 325 F: 1275 T: 1560	L: 45 F: 305 T: 350	L: 446 F: 6098 T: 6544
IV	$p_0 = 1.0$ $p_1 = 2.0$ $s_0 = 1.0$ $s_1 = 2.5$	$\alpha = \beta = 0.5$ $K_1 = 1.2K_0$ $L_0 = 0.1$ $L_1 = 0.2$	L: 73 F: 394 T: 467	L: 102 F: 8687 T: 8789	L: 374 F: 2390 T: 2764	L: 53 F: 1231 T: 1284	L: 530 F: 6154 T: 6684

V	$p_0 = 1.0$ $p_1 = 1.0$ $s_0 = 1.0$ $s_1 = 1.8$	$\alpha = 0.2$ $\beta = 0.8$ $K_0 = K_1$ $L_0 = L_1 = 0.1$	L: 63 F: 244 T: 307	L: 74 F: 4195 T: 4269	L: 326 F: 1238 T: 1564	L: 44 F: 604 T: 648	L: 444 F: 6037 T: 6481

Note: L= leader, F=follower; T=Total. Cost is measured in Singapore dollar (S$).

6. Conclusion

In this paper, we develop an CA-based Stackelberg competition model to seek the optimal location design of bike allocation points. From the supply side, the company aims to minimise the total allocation cost and bike depreciation cost. From the demand side, the elasticity of the demand with respect to the density of the allocation points is considered. Users' decision making is affected by both distance to bikes and quality of bikes. In Stackelberg competition, market leader has its first mover advantage. We provide the optimal response for market leader in anticipation of market follower's design strategy. A discretised solution framework is given to support the location selection in real application. The parameters' impact and model performance are studied through two sets of numerical networks.

Some remarkable results are obtained. Due to the first mover advantage, market leader is likely to maintain its monopoly position in the area wherein the allocation cost is lower than that of the market follower. However, market leader might sacrifice the total profits to prevent the entry of market follower, if market follower can reduce the cost or provide better quality bike. Market leader has to give up the monopoly position when the allocation cost is sufficiently lower, or the service is superiorly attractive. The optimal location of bike allocation points is affected by the fare structure, allocation cost, bike depreciation cost, users' sensitivity towards fare ticket and the quality of bike.

Nevertheless, the model has its limitation of simplifying the actual situation with ideal assumptions, e.g., homogeneous location, static demand level and restriction of two companies only. Future research could be conducted to relax the assumptions and can be extended to the following directions. The user preference towards bike quality can be specified with considering users with different sensitivity level and at different locations. On the basis of this model, the optimal fare can be designed. Generalised model could be included more operators in the market. Finally, a comprehensive validation of the model parameters (e.g. a) can be conducted to support the applications in other places.

Acknowledgments

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Appendix - Nomenclature

Ω	Total domain
x, y	Coordinates of a location
i	Bike sharing company, $i \in \{0,1\}$
r	Radius
A	Service area
D	Allocation point density
z	The distance from the user to the nearest bike
a	Selection preference
ϕ, γ	Coefficient parameters
δ	The factor that affects the users' selection
ω_j	The weightage of the variable j
ξ	Adjusting parameter
α, β	Weight parameter
p	Ticket fare
s	Features of the bike
λ	Demand density
Q	Maximum potential demand density
K	Fixed allocation cost
L	Depreciation cost per unit bike

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