Robustness Properties of Distributed Configurations in Multi-Agent Systems ⋆

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Abstract: This paper considers a distributed LQR design framework for a multi-agent network consisting of identical dynamically decoupled agents. A systematic method is presented for computing the performance loss of various distributed control configurations relative to the performance of the optimal centralized controller. Necessary and sufficient conditions have been derived for which a distributed control configuration pattern arising from the optimal centralizing solution does not entail loss of performance if the initial vector lies in a certain subspace of state-space which is identified. It was shown that these conditions are always satisfied for systems with communication/control networks corresponding to complete graphs with a single link removed. A procedure is extended for analyzing the performance loss of an arbitrary distributed configuration which is illustrated by an exhaustive analysis of a network consisting of agents described by second-order integrator dynamics. Presented results are useful for quantifying performance loss due to decentralization and for designing optimal or near-optimal distributed control schemes.

Keywords: Distributed control, Linear quadratic regulator, Multi-agent systems, Robustness, Performance cost.

1. INTRODUCTION

Distributed coordination of multi-agent systems has received considerable attention in recent years. This problem has broad applications in many fields, e.g. unmanned aerial vehicles, distributed sensor networks and congestion control in communication networks (see Ren and Beard (2010), Jadbabaie et al. (2003), Cortes et al. (2004) and Paganini et al. (2001)). Typically in this type of systems, agents interact with each other in a distributed manner through local information exchange in order to achieve a common objective.

Literature tends to favour distributed control of multi-agent systems due to its advantages over centralized and decentralized control methods which become infeasible or unpractical as the number of agents and distance between them increases. An overview of these three control methods in multi-agent systems is given in Massioni and Verhaegen (2009). Often the problem of controlling the multi-agent system is combined with the graph theory where the information exchange is represented in terms of a graph, see e.g. Langbort et al. (2004) and Lin et al. (2007). Further, Fax and Murray (2004) proposed the graph theory based method for analysis of a formation of interacting and cooperating identical agents. The authors analyzed necessary and sufficient stability conditions for a given undirected communication topology. This framework was extended in Popov and Werner (2009) to the robust formation control method which is applicable to an arbitrary communication topology.

In general, optimal control of agents formations is a topic of considerable interest to the control community. It can be carried out to minimize the relative formation errors (Zhang and Hu (2007)), energy expenditure (Bhatt et al. (2009)), etc. Linear Quadratic Regulator (LQR) has been widely used in vast variety of scenarios due to its guaranteed robustness properties. For example, in Rogge et al. (2010) LQR-based method is proposed as a solution to the consensus problem over a ring network. Similarly, in Cao and Ren (2010) LQR theory has been successfully applied to control of multi-agent systems with single-integrator dynamics. Furthermore, the problem of controlling a formation of interacting and cooperating systems by employing a distributed LQR design was considered in Huang et al. (2010). Distributed LQR framework was also used to control a collection of identical dynamically coupled systems in Deshpande et al. (2011) and Borrelli and Keviczky (2008). Proposed control laws guarantee a certain level of performance in terms of LQR cost at network level, but they use different LQR cost functions. In Deshpande et al. (2011) the solution is depended on the total number of agents, while in Borrelli and Keviczky (2008) the solution is derived as a function of the maximum vertex degree.

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In the present work we use the distributed LQR design strategy for dynamically decoupled multi-agent systems introduced in Borrelli and Keviczky (2008). By solving a simple local LQR problem whose size is limited by the maximum vertex degree, a stabilizing distributed controller can be found. The aim of this paper is to compare the family of the distributed suboptimal controllers that has been introduced in Borrelli and Keviczky (2008) with the optimal centralized controller. It is shown that for any distributed control configuration which differs from a complete graph by a single link, there is no performance loss if the initial vector lies in a certain subspace of state-space. Additionally, the near-optimal schemes can be identified. A procedure is extended for analyzing the performance loss of an arbitrary distributed configuration which is illustrated by an example in which individual agents are described by second-order integrator dynamics. The results presented allow the application of the method described in Borrelli and Keviczky (2008) to decentralized control schemes optimized with respect to the structure.

The remainder of this paper is organized as follows. Section 2 defines the notations used in paper, which is followed by a brief summary of relevant results from graph theory. In Section 3 two different LQR control designs, centralized and distributed, are proposed. The main results of the paper are described in Section 4. Distributed control configurations that do not entail loss of performance relative to the LQR optimal centralized controller are identified. Also, the method for finding near-optimal distributed configurations for an arbitrary network is presented. These results are illustrated in Section 5 by an exhaustive analysis of a network consisting of five agents. Finally, the paper's conclusions appear in Section 6.

2. PRELIMINARIES

The following notation will be used through the paper: I_n denotes the $n \times n$ identity matrix; A^T and $\boldsymbol{a}^T$ are, correspondingly, the transpose of matrix A and the transpose of column vector $\boldsymbol{a} = [a_1, \ldots, a_n]^T$; $\mathcal{S}(A) = \{\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)\}$ denotes the spectrum of matrix $A = A^T \in \mathbb{R}^{n\times n}$, where for a real spectrum of A the eigenvalues, $\lambda_i(A)$ for $i = 1, \ldots, n$, are indexed in decreasing order; $A \otimes B$ denotes the Kronecker product of $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$.

Definition 1. A matrix $A \in \mathbb{R}^{n\times n}$ is called *stable* or *Hurwitz* if all its eigenvalues have negative real part, i.e. $\mathcal{S}(A) \subseteq \mathbb{C}_-$.

Definition 2. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$. The matrix A is *similar* to B if there is an invertible matrix $P \in \mathbb{R}^{n\times n}$, such that $A = P^{-1}BP$.

Next, we present some concepts and basic results on graph theory, which are necessary for the development of the paper.

A multi-agent system is represented by an *undirected* graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of nodes (or vertices), $\mathcal{V} = \{1, 2, \ldots, N\}$, and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, $\mathcal{E} \subseteq \big\{(i,j) : i,j \in \mathcal{V}, j \neq i\big\}$. Communication between agents is bidirectional and agents i and j are said to be neighbours if $(i,j) \in \mathcal{E}$. Each node has associated *degree* or *valency*, d_i for $i = 1, 2, \ldots, N$, which represents the number of neighbours of agent i. Any undirected graph can be represented by its *adjacency matrix*, $\mathbf{A}(\mathcal{G})$. Let $\mathbf{A}_{i,j} \in \mathbb{R}$ be the $(i,\, j)$ element of adjacency matrix. Then $\mathbf{A}_{i,i} = 0$ for $i = 1, 2, \ldots, N$ as we assume that there is no edge from node to itself, and

$$\mathbf{A}_{i,j} = \begin{cases} 0 \;\; if \;\; (i,j) \notin \mathcal{E} \;\; \forall i,j = 1,2,\ldots,N, \;\; i \neq j, \\ 1 \;\; if \;\; (i,j) \in \mathcal{E} \;\; \forall i,j = 1,2,\ldots,N, \;\; i \neq j. \end{cases}$$

An undirected graph is said to be *complete* if there is an edge between every pair of nodes. Then all nodes will have the same degree, $d = N - 1$, where N is the number of nodes (agents).

Definition 3. (Borrelli and Keviczky (2008)) The class of matrices denoted as $\mathcal{K}^{N_d}_{n,m}(\mathcal{G})$ is defined as

$$\begin{aligned}\mathcal{K}^{N_d}_{n,m}(\mathcal{G}) = \{&M \in \mathbb{R}^{nN_d \times mN_d} \mid M_{ij} = 0 \;\text{ if }\; (i,j) \notin \mathbf{A}, \\ &(M_{ij} = M[(i-1)n+1 : in, (j-1)m+1 : jm] \\ &\text{if }\; (i,j) \in \mathbf{A} \;\text{ where }\; i,j = 1,2,\ldots,N_d\}.\end{aligned}$$

3. PROBLEM FORMULATION

3.1 Centralized LQR Control of Multi-Agent Systems

Let consider the linear continuous-time system of ith agent whose dynamics can be described as

$$\dot{\boldsymbol{x}}_{\boldsymbol{i}}(t) = A\boldsymbol{x_i} + B\boldsymbol{u_i}, \;\; \boldsymbol{x}(0) = \boldsymbol{x_{i_0}} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $\boldsymbol{x_i}(t) \in \mathbb{R}^n$, $\boldsymbol{u_i}(t) \in \mathbb{R}^m$ are the state and input vectors of the system at time t, respectively. Then, the collective dynamics of N identical and decoupled systems, indexed as $1, 2, \ldots, N$, is given by

$$\dot{\boldsymbol{x}}(t) = A_a\boldsymbol{x} + B_a\boldsymbol{u}, \;\; \boldsymbol{x}(0) = \boldsymbol{x_0} \tag{2}$$

where the column vectors $\boldsymbol{x}(t) = [\boldsymbol{x_1}^T(t), \ldots, \boldsymbol{x_N}^T(t)]^T$ and $\boldsymbol{u}(t) = [\boldsymbol{u_1}^T(t), \ldots, \boldsymbol{u_N}^T(t)]^T$ collect the states and inputs of the N systems, while $A_a = I_N \otimes A$ and $B_a = I_N \otimes B$, with A and B defined as in (1).

The LQR problem for the system (2) is described through the cost function of the form

$$\begin{aligned}J\big(\boldsymbol{u}(t), \boldsymbol{x_0}\big) = \int_0^\infty \Big(&\sum_{i=1}^N \big(\boldsymbol{x_i}(t)^T Q_{ii} \boldsymbol{x_i}(t) + \boldsymbol{u_i}(t)^T R_{ii} \boldsymbol{u_i}(t)\big) + \\ &\sum_{i=1}^N \sum_{\substack{j=1 \\ j>i}}^N \Big(\big(\boldsymbol{x_i}(t) - \boldsymbol{x_j}(t)\big)^T Q_{ij} \big(\boldsymbol{x_i}(t) - \boldsymbol{x_j}(t)\big) \Big) \Big) \,\mathrm{d}t.\end{aligned}$$

In the more compact notation the LQR cost is defined as:

$$J\big(\boldsymbol{u}(t), \boldsymbol{x_0}\big) = \int_0^\infty \big(\boldsymbol{x}(t)^T Q_a \boldsymbol{x}(t) + \boldsymbol{u}(t)^T R_a \boldsymbol{u}(t)\big)\,\mathrm{d}t \tag{3}$$

where the matrices Q_a and R_a have the following structure:

$$Q_a = \begin{pmatrix} Q_{a_{11}} & Q_{a_{12}} & \ldots & Q_{a_{1N}} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{a_{N1}} & Q_{a_{N2}} & \ldots & Q_{a_{NN}} \end{pmatrix}, R_a = I_N \otimes R, \tag{4}$$

with $Q_{a_{ii}} = \sum_{k=1}^N Q_{ik}$ for $i = 1, \ldots, N$; $Q_{a_{ij}} = -Q_{ij}$ for $i,j = 1, \ldots, N, \;\; i \neq j$; $Q_{ii} = Q_{ii}^T \geq 0$ and $R_{ii} = R_{ii}^T > 0$ for $\forall i$, $Q_{ij} = Q_{ij}^T = Q_{ji} \geq 0$ for $\forall i \neq j$.

We are assuming that the pairs (A, B), (A_a, B_a) are stabilizable and the pairs (A, C), (A_a, C_a) are observable for any $Q = Q^T \geq 0$ and Q_a as in (4) (where $C^T C = Q$,

$C_a^T C_a = Q_a$). Then, the optimal control law for the quadratic cost function (3) is given by

$$\boldsymbol{u} = -R_a^{-1} B_a^T P_a \boldsymbol{x} \tag{5}$$

where P_a is the symmetric positive definite stabilizing solution of the following (large-scale) Algebraic Riccati Equation (ARE):

$$A_a^T P_a + P_a A_a - P_a B_a R_a^{-1} B_a^T P_a + Q_a = 0. \tag{6}$$

If the weighting matrices in (4) are chosen as $Q_{a_{ii}} = Q_1$ $\forall i = 1, \ldots, N$, and $Q_{a_{ij}} = Q_2$ $\forall i = 1, \ldots, N$, $i \neq j$, P_a is of the form:

$$P_a = \begin{pmatrix} P_{a_{11}} & P_{a_{12}} & \ldots & P_{a_{12}} \\ P_{a_{12}} & P_{a_{11}} & \ldots & P_{a_{12}} \\ \vdots & \vdots & \ddots & \vdots \\ P_{a_{12}} & \ldots & \ldots & P_{a_{11}} \end{pmatrix} \tag{7}$$

where individual blocks of P_a, $P_a[(i-1)n+1 : in, (j-1)n+1 : jn]$ for $i, j = 1, \ldots, N$, have the following properties:

(1) $\sum_{j=1}^{N} P_{a_{ij}} = P$ for all $i = 1, \ldots, N$, where $P \in \mathbb{R}^{n \times n}$ is the symmetric positive definite solution of the ARE:
$$A^T P + PA - PBR^{-1}B^T P + Q_1 = 0. \tag{8}$$
Therefore, $P_{a_{11}} = P - (N-1)P_{a_{12}}$.

(2) All off-diagonal blocks of P_a, namely $P_{a_{ij}}$ for $i \neq j$, are equal matrices, denoted as $P_{a_{12}}$. Furthermore, $P_{a_{12}}$ is the negative semi-definite solution of the ARE
$$A_{cl}^T P_{a_{12}} + P_{a_{12}} A_{cl} + N P_{a_{12}} X P_{a_{12}} - Q_2 = 0 \tag{9}$$
where $A_{cl} = A - BR^{-1}B^T P$ and $X = BR^{-1}B^T$.

Similarly, this structure of diagonal and off-diagonal blocks will be preserved in the gain matrix K_a, given as

$$K_a = R_a^{-1} B^T P_a. \tag{10}$$

For more details and proofs see Borrelli and Keviczky (2008).

3.2 Distributed LQR Control of Multi-Agent Systems

The *distributed optimal control problem* can be described as (Borrelli and Keviczky (2008)):

$$\begin{aligned} \min_{\tilde{K}} \tilde{J}\big(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{x}}_0\big) &= \int_0^\infty \big(\tilde{\boldsymbol{x}}(t)^T \tilde{Q} \tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{u}}(t)^T \tilde{R} \tilde{\boldsymbol{u}}(t)\big)\, \mathrm{d}t \\ \text{subj. to } & \dot{\tilde{\boldsymbol{x}}}(t) = \tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}\tilde{\boldsymbol{u}}, \;\; \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0 \\ & \tilde{K} \in \mathcal{K}_{n,m}^{N_d}(\mathcal{G}) \\ & \tilde{Q} \in \mathcal{K}_{n,n}^{N_d}(\mathcal{G}), \;\; \tilde{R} = I_{N_d} \otimes R \end{aligned} \tag{11}$$

where N_d is a number of identical decoupled systems; $\tilde{A} = I_{N_d} \otimes A$ and $\tilde{B} = I_{N_d} \otimes B$ with A and B defined as in (1); $\tilde{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{u}}(t)$ are the vectors which collect the states and inputs of the N_d systems; $\mathcal{K}_{n,m}^{N_d}(\mathcal{G})$ is defined as in Definition 3; $\tilde{Q} = \tilde{Q}^T \geq 0$ and $\tilde{R} = \tilde{R}^T > 0$.

In general, computing the solution of (11) is an NP-hard problem. Therefore, the procedure for designing a suboptimal distributed controller is given next.

Theorem 1. (Borrelli and Keviczky (2008)). Consider the large-scale system composed of N_d identical decoupled subsystems whose collective dynamics is described as:

$$\dot{\tilde{\boldsymbol{x}}}(t) = \tilde{A}\tilde{\boldsymbol{x}} + \tilde{B}\tilde{\boldsymbol{u}}, \;\; \tilde{\boldsymbol{x}}(0) = \tilde{\boldsymbol{x}}_0. \tag{12}$$

LQR problem for the system (12) is defined through the cost function:

$$J\big(\tilde{\boldsymbol{u}}(t), \tilde{\boldsymbol{x}}_0\big) = \int_0^\infty \big(\tilde{\boldsymbol{x}}(t)^T \tilde{Q} \tilde{\boldsymbol{x}}(t) + \tilde{\boldsymbol{u}}(t)^T \tilde{R} \tilde{\boldsymbol{u}}(t)\big)\, \mathrm{d}t \tag{13}$$

where $\tilde{Q}$ and $\tilde{R}$ are structured as in (4) with $\tilde{Q}_{ii} = \tilde{Q}_1$ for all $i = 1, \ldots, N$ and $\tilde{Q}_{ij} = \tilde{Q}_2$ for all $j = 1, \ldots, N$, $i \neq j$. Then, the distributed controller can be constructed as:

$$\tilde{K} = I_{N_d} \otimes R^{-1} B^T P - M \otimes R^{-1} B^T P_{12} \tag{14}$$

where P is the symmetric positive definite solution of (8) and P_{12} represents the off-diagonal block of P_a in (7). The size of the problem in (6) is a function of the maximum vertex degree in network, such that $N_{min} = d_{max}(\mathcal{G}) + 1$ agents. The structure of the graph $\mathcal{G}$ is reflected through matrix M given by

$$M = aI_{N_d} - b\mathbf{A}(\mathcal{G}), \quad b \geq 0 \tag{15}$$

where $\mathbf{A}(\mathcal{G})$ is the adjacency matrix. Coefficients a and b are chosen to satisfy $a - bd_{max} \geq 0$, which follows from the gain margin properties of the proposed controller. Then, the closed loop system:

$$\tilde{A_{cl}} = \tilde{A} - \tilde{B}\tilde{K} = I_{N_d} \otimes A + (I_{N_d} \otimes B)\tilde{K} \tag{16}$$

will be asymptotically stable and $\tilde{P}$ is the unique solution of the following Lyapunov equation:

$$\tilde{A}_{cl}^T \tilde{P} + \tilde{P}\tilde{A}_{cl} + \tilde{Q} + \tilde{K}^T \tilde{R} \tilde{K} = 0. \tag{17}$$

Proof. See Borrelli and Keviczky (2008).

4. MAIN RESULTS

In this section the method for identifying optimal and near-optimal distributed control schemes is proposed. Two different control designs given in Section 3 are compared with regards to their performance cost. Note that in the case of complete graph distributed gain matrix ($\tilde{K}$) in (14) is equivalent to the centralized (large-scale) gain matrix (K_a) in (10) and optimality is preserved. Next results are valid for multi-agent systems consisting of minimum 4 agents.

Consider the distributed multi-agent network that differs from centralized only in a single link (i.e. one agent is not connected to the remaining $(N-1)$ agents and has degree $d = N - 2$). Then, two gain matrices will differ for ΔK which in the case of $(1,2) \notin \mathcal{E}$ is given by

$$\Delta K = \begin{pmatrix} 0 & -R^{-1}B^T P_{a_{12}} & \ldots & 0 \\ -R^{-1}B^T P_{a_{12}} & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 \end{pmatrix} \tag{18}$$

and the following result can be established.

Theorem 2. Suppose that Theorem 1 holds and let $E = \tilde{P} - P_a$, where P_a is solution of (6) and $\tilde{P}$ is the solution of the Lyapunov equation:

$$(A_a - B_a\tilde{K})^T \tilde{P} + \tilde{P}(A_a - B_a\tilde{K}) + \tilde{K}^T R_a \tilde{K} + Q_a = 0. \tag{19}$$

Then, $E = E^T$ is the unique positive semi-definite solution of the following Lyapunov equation:

$$\tilde{A}_{cl}^T E + E\tilde{A}_{cl} + (\Delta K)^T R_a \Delta K = 0 \tag{20}$$

in which $\tilde{A}_{cl} = A_a - B_a R_a^{-1} B_a^T P_a + B_a \Delta K$ is Hurwitz. In particular, $E = E^T > 0$ if and only if the pair $(A_a - B_a R_a^{-1} B_a^T P_a, \Delta K)$ is observable.

Proof. Asymptotic stability of $(A_a - B_a K_a)$ and $(A_a - B_a\tilde{K})$ implies that P_a and $\tilde{P}$ are positive semidefinite solutions. Subtracting equation (6) from equation (19) shows (after some algebra) that E is the solution of (20).

Theorem 1 implies that $\tilde{A}_{cl}$ is Hurwitz which in turn implies that $E = E^T \geq 0$. From the standard theory of Lyapunov equations E is positive definite if and only if the pair $(\tilde{A}_{cl}, \Delta K)$ is observable, which is equivalent to the observability of the pair $(A_a - B_a R_a^{-1} B_a^T P_a, \Delta K)$.

Furthermore, E will be always singular when a single link is removed from a complete graph. The result is established in Theorem 4, but before stating and proving this theorem we need to state some preliminary results on the spectrum of the centralized closed-loop matrix.

Theorem 3. Consider LQR problem in (1)-(3) with state and control weighting matrices Q_1 and R, respectively. Let $A_{cl} = A - BR^{-1}B^T P$ be the closed-loop matrix of the problem defined where P is the symmetric positive definite solution of (8). Similarly, let $A_{cl_a} = A_a - B_a R_a^{-1} B_a^T P_a$ be the closed-loop matrix of the (large-scale) centralized LQR problem in (6) with state and control weighting matrices Q_a and R_a respectively. P_a is the symmetric positive definite solution of (6) decomposed into N^2 blocks of dimension $n \times n$ as in (7). Then, the spectrum of A_{cl_a}, i.e. $\mathcal{S}(A_{cl_a})$ is given by:

$$\mathcal{S}(A_{cl_a}) = \mathcal{S}(A_{cl}) \ \cup \ \underbrace{\mathcal{S}(A_{cl_{1-2}}) \ \cup \ \dots \ \cup \ \mathcal{S}(A_{cl_{1-2}})}_{\text{(N-1) times}} \tag{21}$$

where $A_{cl_{1-2}} = A - BR^{-1}B^T(P_{a_{11}} - P_{a_{12}})$, in which $P_{a_{11}}$ and $P_{a_{12}}$ are $n \times n$ blocks of P_a in (7).

Proof. The proof is straightforward, so full details are omitted due to space restrictions. The proof is based on the transformation of closed-loop matrix $A_{cl_a} = A_a - B_a R_a^{-1} B_a^T P_a$ into a block lower-triangular matrix by using the similarity transformations. The transformation matrix is given by

$$T = \begin{pmatrix} I & -I & 0 & \dots & 0 \\ 0 & I & -I & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \ddots & -I \\ 0 & 0 & \dots & \dots & I \end{pmatrix}. \tag{22}$$

Since eigenvalues of closed-loop matrix are preserved under similarity transformations equation (21) follows.

Theorem 4. Consider the fully connected multi-agent network consisting of at least four agents. Then, if a single link is removed between any two agents, E will be singular.

Proof. Using results of Theorem 1 and Theorem 2, $\tilde{A}_{cl}$ is Hurwitz and $E = E^T \geq 0$. To show that E is singular it suffices to show that the pair $(A_a - B_a R_a^{-1} B_a^T P_a, \Delta K)$ is unobservable which is equivalent to the existence of $\lambda \in \mathbb{C}$ such that the matrix $\begin{pmatrix} A_{cl_a} - \lambda I \\ \hdashline \Delta K \end{pmatrix}$ is rank deficient. For the network of N agents and link removal between agents 1 and 2 this matrix can be written as:

$$\begin{pmatrix} A_{11} - \lambda I & -XP_{a_{12}} & \dots & -XP_{a_{12}} \\ -XP_{a_{12}} & A_{11} - \lambda I & \dots & -XP_{a_{12}} \\ \vdots & \vdots & \ddots & \vdots \\ -XP_{a_{12}} & -XP_{a_{12}} & \dots & A_{11} - \lambda I \\ \hdashline 0 & -R^{-1}B^T P_{a_{12}} & \dots & 0 \\ -R^{-1}B^T P_{a_{12}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{23}$$

where $A_{11} = A - XP_{a_{11}}$. Using the state-space transformation $\begin{pmatrix} T(A_{cl_a} - \lambda I)T^{-1} \\ \hdashline \Delta K T^{-1} \end{pmatrix}$, where the large-scale transformation matrix is given by

$$T = \left(\begin{array}{cccc:cccc} I & I & I & I & I & I & \dots & I \\ 0 & I & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -I & I & 0 & 0 & 0 & \dots & 0 \\ 0 & I & I & I & I & I & \dots & I \\ \hdashline 0 & 0 & 0 & 0 & I & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & I \end{array}\right), \tag{24}$$

matrix in (23) will lose the rank along the third column if λ is chosen as an eigenvalue of the matrix $A - X(P_{a_{11}} - P_{a_{12}})$ (using Theorem 3). Since the rank of a matrix remains invariant under similarity transformations the system $(\tilde{A}_{cl}, \Delta K)$ is unobservable in this case and hence E is singular.

Remark 1. Theorem 4 is proved under assumption that link between agents 1 and 2 is removed from the complete graph of any size. Different choice of link to be removed will change the structure of ΔK, but eigenvalue distribution in $\tilde{P}$ and $\tilde{A}_{cl}$ will be unchanged (as all similar matrices have the same spectrum). Also, this can be related to the automorphism group of a graph $\mathcal{G}$ that arises in the enumeration of nodes known as labeling (for more details see Cameron (2001)). Therefore, E is singular for any of $\frac{N(N-1)}{2}$ configurations corresponding to complete graph of N agents with a single link removed.

The analysis can be extended to the case when more than one link is removed from a complete network. Then, by using appropriate transformation matrices we can always choose the direction where E is singular. However, the complexity of the analysis will increase with the number of agents and number of links to be removed. Proofs are omitted due to space restrictions, but an numerical example is given in Section 5.

Next, we present the results obtained by comparing the costs of the centralized LQR controller and distributed LQR controller.

Proposition 5. The cost of using distributed LQR design will be always equal or higher that the optimal cost imposed by centralized LQR design.

Proof. We assume that the distributed controller is stabilizing. Therefore, cost is finite. Using Theorem 2, $\tilde{P} = P_a + E$ with all matrices being symmetric positive semi-definite. Applying Weyl's inequality (see Horn and Johnson (1994)) we get:

$$\lambda_k(P_a) + \lambda_n(E) \leq \lambda_k(\tilde{P}) \ \text{ for any } \ 1 \leq k \leq n$$

where the eigenvalues are indexed in decreasing order. The required result follows from Theorem 2 since $\lambda_n(E) \geq 0$.

The case when two costs are equal is summarised in the next theorem.

Theorem 6. The cost of a stabilizing distributed controller defined in Theorem 1 is equal to the cost of the centralized optimal LQR controller if and only if the pair $(A_a - B_a R_a^{-1} B_a^T P_a, \Delta K)$ is unobservable and $\tilde{\boldsymbol{x}}_0 \in \text{Ker}(E)$ where $E = \tilde{P} - P_a$.

Proof. The cost of a stabilizing distributed controller is:

$$J(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{x}}_0)=\tilde{\boldsymbol{x}}_0^T\tilde{P}\tilde{\boldsymbol{x}}_0=\tilde{\boldsymbol{x}}_0^T P_a\tilde{\boldsymbol{x}}_0+\tilde{\boldsymbol{x}}_0^T E\tilde{\boldsymbol{x}}_0. \tag{25}$$

The term $\tilde{\boldsymbol{x}}_0^T P_a\tilde{\boldsymbol{x}}_0$ represents the optimal LQR cost of the centralized controller. Since $E=E^T\geq 0$, the term $\tilde{\boldsymbol{x}}_0^T E\tilde{\boldsymbol{x}}_0$ is zero if and only if E is singular and $\tilde{\boldsymbol{x}}_0\in \mathrm{Ker}(E)$. Using Theorem 2 this will be satisfied only if the pair $(A_a -\ B_aR_a^{-1}B_a^TP_a,\Delta K)$ is unobservable.

Remark 2. Theorem 4 and Theorem 6 can be extended along various directions. Consider first the case of near-optimal distributed configurations for which the cost increase along specific directions $\boldsymbol{\xi}$ is small relative to the optimal LQR cost. For these directions the pair $(A_a -\ B_aR_a^{-1}B_a^TP_a,\Delta K)$ is close to unobservability, in the sense that for certain $\lambda_0\in\mathbb{C}$ and $\|\boldsymbol{\xi}\|=1$ the norm of the vector

$$\begin{pmatrix}\lambda_0 I -\ A_a +\ B_aR_a^{-1}B_a^TP_a\\ R_a^{1/2}K_a\end{pmatrix}\boldsymbol{\xi} \tag{26}$$

is small. Specifically, let $E=E^T>0$ be a solution of (20) with $\lambda_{min}(E)=\epsilon>0$. Then, the cost of the corresponding distributed controller (guaranteed to be stabilizing under the previous assumptions) is:

$$J(\tilde{\boldsymbol{u}},\tilde{\boldsymbol{x}}_0)=\tilde{\boldsymbol{x}}_0^T\tilde{P}\tilde{\boldsymbol{x}}_0=\tilde{\boldsymbol{x}}_0^T P_a\tilde{\boldsymbol{x}}_0+\tilde{\boldsymbol{x}}_0^T E\tilde{\boldsymbol{x}}_0 \tag{27}$$

where the term $\tilde{\boldsymbol{x}}_0^T E\tilde{\boldsymbol{x}}_0\geq\epsilon\|\tilde{\boldsymbol{x}}_0\|^2$. In particular, if $\tilde{\boldsymbol{x}}_0$ is chosen to lie in the eigenspace of E corresponding to its minimum eigenvalue, $\tilde{\boldsymbol{x}}_0^T E\tilde{\boldsymbol{x}}_0=\epsilon\|\tilde{\boldsymbol{x}}_0\|^2$. Therefore, for small values of ϵ and along these directions the cost increase from the optimal level will be minimal. Next consider the case that $E=E^T\geq 0$ and singular. Let $\lambda_1\geq\ldots\geq\lambda_m>\lambda_{m+1}=\ldots=\lambda_{m+r}>\lambda_{m+r+1}=\ldots=\lambda_n=0$. In this case there is no cost increase along all directions in the null-space of E. If $\tilde{\boldsymbol{x}}_0$ lies in the r-dimensional eigenspace corresponding to $\lambda_{m+1}(E)$, then the cost increase is exactly $\lambda_{m+1}(E)\|\tilde{\boldsymbol{x}}_0\|^2$. Thus the sequence of eigenvalues of E indicate the progressive deviation from optimality if the initial state lies in the corresponding eigenspace. A final measure of deviation from optimality for each decentralized control scheme is average cost. Assuming that $\tilde{\boldsymbol{x}}_0$ is uniformly distributed on the surface of an n-dimensional hyper-sphere we define the average cost as the expected value:

$$\mu(E):=\frac{\int_{\|\xi\|=1}\boldsymbol{\xi}^T E\boldsymbol{\xi}\ dS}{\int_{\|\boldsymbol{\xi}\|=1}dS}=\frac{\mathrm{trace}(E)}{nN_d} \tag{28}$$

which may be considered as a measure of average cost increase due to decentralization over all initial state directions.

5. NUMERICAL EXAMPLE

Consider a network of $N=5$ identical, dynamically decoupled agents described by double-integrator dynamics in both spatial dimensions:

$$\ddot{\boldsymbol{x}}_i=\boldsymbol{u}_{x,i},\quad \ddot{\boldsymbol{y}}_i=\boldsymbol{u}_{y,i},\quad i=1,\ldots,5. \tag{29}$$

The interconnection structure is depicted in Fig. 1.

The collective dynamics in a state-space formulation is given by

$$\dot{\tilde{\boldsymbol{x}}}(t)=\tilde{A}\tilde{\boldsymbol{x}}+\tilde{B}\tilde{\boldsymbol{u}},\ \ \tilde{\boldsymbol{x}}(0)=\tilde{\boldsymbol{x}}_0. \tag{30}$$

where $\tilde{A}=I_5\otimes A$ and $\tilde{B}=I_5\otimes B$ with A and B defined as

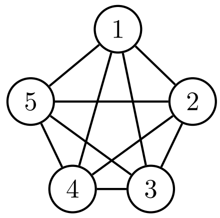


Fig. 1. The complete graph with $N=5$ agents

$$A=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\quad B=\begin{bmatrix}0&0\\1&0\\0&0\\0&1\end{bmatrix}. \tag{31}$$

The augmented system (30) is altered by adding dumping elements to the diagonal of matrix A (i.e. $diag(A)=(\,0.1\ -0.1\ 0.1\ -0.1\,)$) such that local and global stabilizability assumptions in Section 3.1 are satisfied. Then, the LQR problem for a formation in Fig. 1 is defined as

$$\min_{\tilde{K}}\tilde{J}\big(\tilde{\boldsymbol{u}}(t),\tilde{\boldsymbol{x}}_0\big)\quad \text{subj. to } \dot{\tilde{\boldsymbol{x}}}=\tilde{A}\tilde{\boldsymbol{x}}+\tilde{B}\tilde{\boldsymbol{u}},\ \tilde{\boldsymbol{x}}(0)=\tilde{\boldsymbol{x}}_0$$

where the cost function $\tilde{J}\big(\tilde{\boldsymbol{u}}(t),\tilde{\boldsymbol{x}}_0\big)$ is defined as in (13) with $\tilde{Q}$ (its diagonal and off-diagonal blocks) and $\tilde{R}$ structured as $\tilde{Q}_{ii}=\mathrm{diag}(5,\ 0,\ 5,\ 0)$, $\tilde{Q}_{ij}=\mathrm{diag}(-1,\ 0,\ -1,\ 0)$ and $\tilde{R}=I_5\otimes R$ in which $R=I_2$.

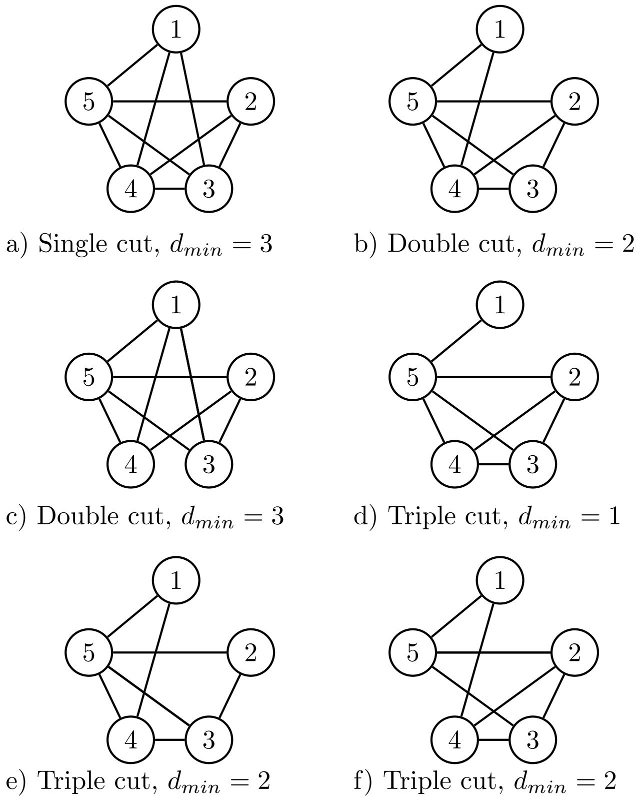
By using the distributed control method proposed in Section 3.2 each agent is stabilized and for the configuration given in Fig. 1 the cost is optimal due to the equivalence with centralized LQR problem. For a given initial state vector $\tilde{\boldsymbol{x}}_0$ such that $\|\tilde{\boldsymbol{x}}_0\|=1$ we get the optimal cost J_c. As we are interested in the minimum cost, average cost, and maximum cost imposed by centralized design, for a given example these are $J_{c_{min}}=0.705$, $J_{c_{avg}}=5.413$,and $J_{c_{max}}=10.381$, respectively.

Next, we consider a number of different distributed configurations obtained by removing one, two of three links from a complete graph in Fig. 1. Distributed configurations are depicted in Fig. 2. Please note that number of links removed could be larger than three and asymptotic stability will be still achieved, but these configurations are omitted due to space restrictions. However, same conclusions would apply.

Table 1 shows the cost change in distributed case for a different directions in space relative to the optimal design (i.e. centralized network). Minimum, average and maximum cost increase correspond to the $\lambda_{min}(E)$, $\frac{trace(E)}{\dim(E)}$ and $\lambda_{max}(E)$, respectively. Please note that number of possible configurations for up to four cuts is reduced to six based on graph properties in Remark 1.

Table 1. Cost increase for distributed designs

Configuration	a)	b)	c)	d)	e)	f)
Minimum cost increase	0	0	0	0	0	0
Average cost increase	0.010	0.021	0.019	0.033	0.031	0.030
Maximum cost increase	0.049	0.112	0.066	0.194	0.150	0.121



a) Single cut, $d_{min} = 3$ b) Double cut, $d_{min} = 2$

c) Double cut, $d_{min} = 3$ d) Triple cut, $d_{min} = 1$

e) Triple cut, $d_{min} = 2$ f) Triple cut, $d_{min} = 2$

Fig. 2. Different distributed configurations

Note that for each configuration there is always one direction for which optimality is preserved relative to the optimal centralized LQR controller. It can also be seen that for the same number of cuts the average and maximum cost decreases as the minimum degree of the network increases, which corresponds to a well-connected network. Future work will attempt to establish precise correlations between decentralized cost, the connectivity properties of the network and the corresponding structured controllability properties of each distributed scheme.

6. CONCLUSION

The paper has considered a distributed LQR design framework for a network of dynamically decoupled multi-agent systems with identical dynamics. Using the robustness properties of LQR optimal control, stability can still be guaranteed if the fully centralized solution is relaxed be removing up to a maximum number of network links. Necessary and sufficient conditions have been derived for which different distributed control configuration pattern do not entail loss of performance. Cost increase due to decentralization has also been quantified by introducing three cost measures corresponding to the worst-case, best-case and average directions in which the initial state of the system lies. The results of the paper have been illustrated with a numerical example. Future work will attempt to correlate the additional cost arising due to decentralization with the connectivity and structured controllability properties of the network and apply the results to the design of optimal or near-optimal controllers for networks of agents operating under communication constraints, external disturbances and model uncertainty.

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