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Incorporating prey refuge in a predator-prey system with imprecise parameter estimates

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Abstract: This article concerns with the optimal harvesting of a predator-prey model with prey refuge and imprecise biological parameters. We consider the above model under impreciseness and introduce parametric functional form of an interval which are different from that of models with precise biological parameters. We study the existence and stability of the equilibria. The bionomic equilibria of the model are discussed. Also, the optimal harvesting policy is derived by using Pontryagin's maximal principle. Numerical simulations are presented to verify the feasibility of our analytical results.

Keywords: Predator-prey system; refuge; interval number; equilibrium; stability; optimal harvesting policy.

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1 Introduction

It is well known that a few recent studies show that bio-economic modelling of the exploitation of biological resources has gained importance. In fact, the techniques and issues associated with the bio-economic exploitation of these resources have been discussed in detailed by Clark [1, 2]. Let $x(t)$ and $y(t)$ denote, respectively, the densities of prey and predator, and a general predator-prey system with harvesting can be written as

$$\begin{cases} \dfrac{dx(t)}{dt} = rx\Big(1 - \dfrac{x}{K}\Big) - \varphi(x)y - H_1(x), \\ \dfrac{dy(t)}{dt} = -dy - sy^2 + f\varphi(x)y - H_2(y), \end{cases} \tag{1.1}$$

where r is the biotic potential (or the intrinsic growth rate) of the prey in the absence of the predator, K is the prey's carrying capacity, d is the mortality rate and s is intra-specific competitive rate of the predator, f stands for the efficiency rate with which captured prey are converted to new predators, $\varphi(x)$ represents the functional response of the predator and satisfies the following assumption

$$\varphi(0) = 0, \quad \varphi'(x) > 0 \ (x > 0),$$

and the functions $H_1(x)$, $H_2(y)$ are non-negative and represent the effects of harvesting on the prey and predator, respectively. If $H_1(x) \equiv 0, H_2(y) \equiv 0$, the terms represent no harvesting. If $H_1(x)$ and $H_2(y)$ are positive constants, the terms represent constant time rates at which prey and predator are harvested from system (1.1), respectively. And if $H_1(x) = \bar{H}_1 x, H_2(y) = \bar{H}_2 y$, the terms stand for that harvesting is proportional to the densities of prey and predator, respectively.

Note that many prey spend much of their lives near or in refuges to avoid predators, which include holes, crevices, thick vegetation, shells or tubes and so on. Since Gause et al.[3] and Maynard Smith [4] introduced a quantity x_R of the prey that involves refuges, the concept of prey refuges has attracted the attention of ecologists and mathematicians because it exists extensively in predator-prey communities. A lot of literatures showed that prey refuges have significant effects on the population dynamic (see [5, 6, 7, 8, 9, 10, 11, 12] in detail). Incorporating prey refuges in the above system, system (1.1) turns into

$$\begin{cases} \dfrac{dx(t)}{dt} = rx\Big(1 - \dfrac{x}{K}\Big) - \varphi(x - x_R)y - H_1(x), \\ \dfrac{dy(t)}{dt} = -dy - sy^2 + f\varphi(x - x_R)y - H_2(y). \end{cases} \tag{1.2}$$

In view of [5], the quantity x_R can be considered from two points: the quantity of prey using refuges is proportional to the density of prey, that is $x_R = mx$ $(0 < m < 1)$; the quantity of prey using refuges is a constant number, that is $x_R = R$.

In this paper, with $x_R = mx$ and proportional harvesting, we consider the following predator-prey system

$$\begin{cases} \dfrac{dx(t)}{dt} = rx\Big(1 - \dfrac{x}{K}\Big) - c(1-m)xy - q_1E_1x, \\ \dfrac{dy(t)}{dt} = -dy - d_1y^2 + e(1-m)xy - q_2E_2y, \end{cases} \tag{1.3}$$

where c means the decreasing rate of prey due to presence of predator and $e = cf$. E_1, E_2 and q_1, q_2 stand for the harvesting efforts and the catchability coefficients of prey and predator, respectively, and the catch-rate functions $q_1E_1x_1$ and $q_2E_2x_2$ are satisfied CPUE (catch-per-unit-effort) hypothesis [1].

As we know, harvesting has an important influence on the dynamic evolution of a population. Researchers treat the problem of harvesting policy in managing natural resources as a dominant theme in ecology owing to its significance. Optimal harvesting problem have been studied in environmental and renewable resource economics. A lot of good work on this topic is emerged in recent years (see [13, 14, 15, 16, 17, 18, 19] in detail). Song and Chen [15] researched a competitive population model of two species with harvesting. The study of one prey one predator harvesting model with imprecise biological parameters was presented by Pal et al. in [17]. Liu and Bai [19] gained the optimal harvesting policy for a stochastic predator-prey model. A problem on optimal harvesting policy is also discussed in this article.

On the other hand, the parameters in system (1.3) are always accurate, however, this assumption is impossible due to the lack of information, lack of data, mistakes in the measurement process and determining the initial conditions. Therefore, the model with imprecise parameters are more realistic and significant in nature. Stochastic approach, fuzzy approach, fuzzy-stochastic approach, etc. are useful approaches in managing the models with imprecise parameters, see [17, 18, 19] for example. In this contribution, we prepare to discuss imprecise parameters using fuzzy approach. To this end, we firstly give the following two definitions.

Definition 1.1 (see [17]). (Interval number) An interval number A is represented by closed interval $[a^l, a^u]$ and defined by $A = [a^l, a^u] = \{x | a^l \le x \le a^u, x \in \mathbb{R}\}$, where $\mathbb{R}$ is the set of real numbers and a^l, a^u are the left and right limit of the interval number respectively. Also every real number can be represented by the interval number $[a, a]$, for all $a \in \mathbb{R}$.

Definition 1.2 (see [17]). (Interval-valued function) Let $a > 0, b > 0$ and consider the interval $[a, b]$. From a mathematical point of view, any real number can be represented on a line. Similarly, we can represent an interval by a function. If the interval is of the form $[a, b]$, the interval-valued function is taken as $h(k) = a^{(1-k)}b^k$ for $k \in [0, 1]$.

For any two interval numbers $A = [a^l, a^u]$ and $B = [b^l, b^u]$, we define the following arithmetic operations on interval valued functions:

Addition: $A + B = [a^l, a^u] + [b^l, b^u] = [a^l + b^l, a^u + b^u]$ if $a^l + b^l > 0$. The interval-valued function for the interval $A + B$ is defined as $h(k) = (a^L)^{(1-k)}(a^U)^k$ where $a^L = a^l + b^l$ and

$a^U = a^u + b^u$.

Subtraction: $A - B = [a^l, a^u] - [b^l, b^u] = [a^l - b^u, a^u - b^l]$ if $a^l - b^u > 0$. The interval-valued function for the interval $A - B$ is defined $h(k) = (b^L)^{(1-k)}(b^U)^k$ where $b^L = a^l - b^u$ and $b^U = a^u - b^l$.

Scalar multiplication:

$$\alpha A = \alpha[a^l, a^u] = \begin{cases} [\alpha a^l, \alpha a^u] & \text{if } \alpha \geq 0 \\ [\alpha a^u, \alpha a^l] & \text{if } \alpha < 0 \end{cases} \quad \text{if } a^l > 0.$$

The interval-valued function interval αA is defined as

$$h(k) = (v^L)^{(1-k)}(v^U)^k \text{ if } \alpha \geq 0 \text{ and } h(k) = -(w^U)^{(1-k)}(w^L)^k \text{ if } \alpha < 0,$$

where $v^L = \alpha a^l, v^U = \alpha a^u, w^U = |\alpha| a^u$ and $w^L = |\alpha| a^l$.

Considering impreciseness of the parameters in system (1.3), we denote $\bar{r}, \bar{c}, \bar{d}, \bar{s}$ and $\bar{e}$ by the interval numbers of r, c, d, s and e, respectively. Then, system (1.3) can be expressed as

$$\begin{cases} \dfrac{dx(t)}{dt} = \bar{r}x\Big(1 - \dfrac{x}{K}\Big) - \bar{c}(1-m)xy - q_1 E_1 x, \\ \dfrac{dy(t)}{dt} = -\bar{d}y - \bar{s}y^2 + \bar{e}(1-m)xy - q_2 E_2 y, \end{cases} \tag{1.4}$$

where $\bar{r} \in [r^l, r^u], \bar{c} \in [c^l, c^u], \bar{d} \in [d^l, d^u], \bar{s} \in [s^l, s^u], \bar{e} \in [e^l, e^u]$ and $r^l > 0, c^l > 0, d^l > 0, s^l > 0, e^l > 0$.

Motivated by Theorem 1 in [17], we can easily proof that system (1.4) is equivalent to the following system

$$\begin{cases} \dfrac{dx(t;k)}{dt} = (r^l)^{1-k}(r^u)^k x - (r^u)^{1-k}(r^l)^k \dfrac{x^2}{K} - (c^u)^{1-k}(c^l)^k(1-m)xy - q_1 E_1 x, \\ \dfrac{dy(t;k)}{dt} = -(d^u)^{1-k}(d^l)^k y - (s^u)^{1-k}(s^l)^k y^2 + (e^l)^{1-k}(e^u)^k(1-m)xy - q_2 E_2 y. \end{cases} \tag{1.5}$$

If we neglect the roles of the intra-specific competitive rates of the prey and predator, and the prey refuges, then system (1.5) can be reduced to equations (7) and (8) in [17]. In this paper, some results which are different from that in [17] are obtained.

The rest of this paper is organized as follows. In the next section, the existence and stability of the equilibria of system (1.5) are analyzed in detail. Also, the existence of bionomic equilibria of system (1.5) is discussed in Section 3. Furthermore, we have studied the optimal harvesting policy for system (1.5) in Section 4. Finally, in Section 5, we give three numerical examples and two tables to substantiate our analytical results.

2 Existence and stability of equilibria

In this section, the existence and stability of equilibria of system (1.5) are investigated.

After an algebraic calculation, we derive the equilibria of system (1.5) as follows

(i) Trivial equilibrium: $A_0 = (0, 0)$.

(ii) Axial equilibrium: $A_1 = \left(\frac{K[(r^l)^{1-k}(r^u)^k - q_1E_1]}{(r^u)^{1-k}(r^l)^k}, 0\right)$ exists if $(r^l)^{1-k}(r^u)^k - q_1E_1 > 0$.

(iii) Interior Equilibrium: $A^* = (x^*, y^*)$, where

$$\begin{aligned} x^* &= \frac{K(s^u)^{1-k}(s^l)^k[(r^l)^{1-k}(r^u)^k - q_1E_1] + K(c^u)^{1-k}(c^l)^k(1-m)[(d^u)^{1-k}(d^l)^k + q_2E_2]}{(r^u)^{1-k}(r^l)^k(s^u)^{1-k}(s^l)^k + K(c^u)^{1-k}(c^l)^k(e^l)^{1-k}(e^u)^k(1-m)^2}, \\ y^* &= \frac{(e^l)^{1-k}(e^u)^k(1-m)x^* - [(d^u)^{1-k}(d^l)^k + q_2E_2]}{(s^u)^{1-k}(s^l)^k}, \end{aligned} \tag{2.1}$$

exists if

$$(e^l)^{1-k}(e^u)^k(1-m)x^* > (d^u)^{1-k}(d^l)^k + q_2E_2. \tag{2.2}$$

Now we begin to study the local stability of equilibria A_0, A_1 and A^*, respectively.

Theorem 2.1. The following conclusions are satisfied:

(i) The trivial equilibrium A_0 is locally asymptotically stable if $(r^l)^{1-k}(r^u)^k < q_1E_1$;

(ii) The axial equilibrium A_1 exists and is locally asymptotically stable if

$$(r^l)^{1-k}(r^u)^k > q_1E_1$$

and

$$K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1] < (r^u)^{1-k}(r^l)^k[(d^u)^{1-k}(d^l)^k + q_2E_2],$$

while the trivial equilibrium A_0 becomes unstable;

(iii) The interior equilibrium A^* exists and is locally asymptotically stable if

$$(e^l)^{1-k}(e^u)^k(1-m)x^* > (d^u)^{1-k}(d^l)^k + q_2E_2,$$

where x^* is defined in (2.1).

Proof. The Jacobian matrix of system (1.5) is given by

$$M = \begin{pmatrix} [(r^l)^{1-k}(r^u)^k - q_1E_1] - 2(r^u)^{1-k}(r^l)^k\frac{x}{K} - (c^u)^{1-k}(c^l)^k(1-m)y & -(c^u)^{1-k}(c^l)^k(1-m)x \\ (e^l)^{1-k}(e^u)^k(1-m)y & -[(d^u)^{1-k}(d^l)^k + q_2E_2] - 2(s^u)^{1-k}(s^l)^k y + (e^l)^{1-k}(e^u)^k(1-m)x \end{pmatrix}. \tag{2.3}$$

The Jacobian matrix $M_0 = M(0, 0)$ of the system (1.5) at A_0 is

$$M_0 = \begin{pmatrix} (r^l)^{1-k}(r^u)^k - q_1E_1 & 0 \\ 0 & -(d^u)^{1-k}(d^l)^k - q_2E_2 \end{pmatrix}. \tag{2.4}$$

The characteristic equation of the above matrix can be expressed as $\det(M_0 - \lambda I) = 0$ (I represents an identity matrix), then

$$\lambda_1 = (r^l)^{1-k}(r^u)^k - q_1E_1, \quad \lambda_2 = -(d^u)^{1-k}(d^l)^k - q_2E_2. \tag{2.5}$$

Hence, A_0 is a stable node when $(r^l)^{1-k}(r^u)^k < q_1E_1$; A_0 is a saddle point when $(r^l)^{1-k}(r^u)^k > q_1E_1$.

The Jacobian matrix $M_1 = M(\frac{K[(r^l)^{1-k}(r^u)^k - q_1E_1]}{(r^u)^{1-k}(r^l)^k}, 0)$ of the system (1.5) at A_1 is

$$M_1 = \begin{pmatrix} -[(r^l)^{1-k}(r^u)^k - q_1E_1] & -\frac{K(c^u)^{1-k}(c^l)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1]}{(r^u)^{1-k}(r^l)^k} \\ 0 & -[(d^u)^{1-k}(d^l)^k + q_2E_2] + \frac{K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1]}{(r^u)^{1-k}(r^l)^k} \end{pmatrix}. \tag{2.6}$$

The characteristic equation of the above matrix is written as $\det(M_1 - \lambda I) = 0$, then

$$\begin{aligned} \lambda_1 &= -[(r^l)^{1-k}(r^u)^k - q_1E_1], \\ \lambda_2 &= -[(d^u)^{1-k}(d^l)^k + q_2E_2] + \frac{K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1]}{(r^u)^{1-k}(r^l)^k}. \end{aligned} \tag{2.7}$$

We know that if $(r^l)^{1-k}(r^u)^k > q_1E_1$, then A_1 exists. So A_1 is a stable node when

$$K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1] < (r^u)^{1-k}(r^l)^k[(d^u)^{1-k}(d^l)^k + q_2E_2], \tag{2.8}$$

and A_1 is a saddle when

$$K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1] > (r^u)^{1-k}(r^l)^k[(d^u)^{1-k}(d^l)^k + q_2E_2]. \tag{2.9}$$

The Jacobian matrix M^* of system (1.5) at A^* is

$$M^* = \begin{pmatrix} -(r^u)^{1-k}(r^l)^k\frac{x^*}{K} & -(c^u)^{1-k}(c^l)^k(1-m)x^* \\ (e^l)^{1-k}(e^u)^k(1-m)y^* & -(s^u)^{1-k}(s^l)^k y^* \end{pmatrix}. \tag{2.10}$$

It is easy to see that

$$\det(M^*) = \Big[\frac{(r^u)^{1-k}(r^l)^k(s^u)^{1-k}(s^l)^k}{K} + (c^u)^{1-k}(c^l)^k(e^l)^{1-k}(e^u)^k(1-m)^2\Big]x^*y^* > 0, \tag{2.11}$$

and

$$\text{trace}(M^*) = -(r^u)^{1-k}(r^l)^k\frac{x^*}{K} - (s^u)^{1-k}(s^l)^k y^* < 0. \tag{2.12}$$

So if the condition in (2.2) holds, then the interior equilibrium A^* of system (1.5) is locally asymptotically stable.

3 Bionomic equilibria

The term bionomic equilibrium is a combination of the concepts of biological equilibrium and economic equilibrium. It is known that the biological equilibrium is achieved by $\frac{dx}{dt} =$

$\frac{dy}{dt} = 0$, which is discussed in Section 2. And the economic equilibrium is obtained if TR (the total revenue obtained by selling the harvested biomass) equals TC (the total cost for the effort devoted to harvesting). In this section, the bionomic equilibria of system (1.5) will be discussed in detail. First of all, we denote c_1, c_2 the fishing cost per unit effort for the prey x and the predator y, respectively. p_1, p_2 measure the price per unit biomass of the prey x and the predator y, respectively.

The net economic rent or net revenue (N) is expressed as

$$N = (p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 = N_1 + N_2, \tag{3.1}$$

where

$$N_1 = (p_1q_1x - c_1)E_1, \quad N_2 = (p_2q_2y - c_2)E_2. \tag{3.2}$$

Here, N_1 and N_2 represent the net revenues for the prey x and the predator y, respectively. Then we obtain the bionomic equilibria $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ of system (1.5) by the following equations

$$\begin{aligned}
&(r^l)^{1-k}(r^u)^k x - (r^u)^{1-k}(r^l)^k \frac{x^2}{K} - (c^u)^{1-k}(c^l)^k(1-m)xy - q_1E_1x = 0,\\
&-(d^u)^{1-k}(d^l)^k y - (s^u)^{1-k}(s^l)^k y^2 + (e^l)^{1-k}(e^u)^k(1-m)xy - q_2E_2y = 0,\\
&(p_1q_1x - c_1)E_1 + (p_2q_2y - c_2)E_2 = 0.
\end{aligned} \tag{3.3}$$

From the following four cases, we can obtain the bionomic equilibria of system (1.5).

Case I. If $c_1 > p_1q_1x$, that is, the cost is more than the revenue for the prey x, so fishing of the prey x is not suitable and should be stopped. Then only the predator y fishing remains possible. Hence $E_1 = 0$ and $c_2 < p_2q_2y$, we calculate that $y_\infty = \frac{c_2}{p_2q_2}$ and $(x_\infty, E_{2\infty})$ will be any point on the line

$$(e^l)^{1-k}(e^u)^k(1-m)x - q_2E_2 = (d^u)^{1-k}(d^l)^k + (s^u)^{1-k}(s^l)^k \frac{c_2}{p_2q_2}$$

in the first quadrant of the xE_2-plane.

Case II. If $c_2 > p_2q_2y$, that is, the cost is more than the revenue for the predator y, thus fishing of the predator y is not practicable and should be stopped. Then only the prey x fishing remains operational. Therefore $E_2 = 0$ and $c_1 < p_1q_1x$, it is easy to see that $x_\infty = \frac{c_1}{p_1q_1}$ and $(y_\infty, E_{1\infty})$ will be any point on the line

$$(c^u)^{1-k}(c^l)^k(1-m)y + q_1E_1 = (r^l)^{1-k}(r^u)^k - \frac{c_1(r^u)^{1-k}(r^l)^k}{Kp_1q_1}$$

in the first quadrant of the yE_1-plane if $(r^l)^{1-k}(r^u)^k > \frac{c_1(r^u)^{1-k}(r^l)^k}{Kp_1q_1}$.

Case III. If $c_1 > p_1q_1x$ and $c_2 > p_2q_2y$, that is to say, the costs of the prey x and the predator y are more than the revenue, so we should stop harvesting both the prey x and the predator y i.e., the whole system will be closed.

Case IV. If $c_1 < p_1q_1x$ and $c_2 < p_2q_2y$, then fishing of both the prey x and the predator y will be in operation. One yields that $x_\infty = \frac{c_1}{p_1q_1}$ and $y_\infty = \frac{c_2}{p_2q_2}$. Substituting the values of x_∞ and y_∞ into the first and second equations of (3.3) we have

$$E_{1\infty} = \frac{Kp_1p_2q_1q_2(r^l)^{1-k}(r^u)^k - c_1p_2q_2(r^u)^{1-k}(r^l)^k - Kc_2p_1q_1(c^u)^{1-k}(c^l)^k(1-m)}{Kp_1p_2q_1^2q_2} \quad (3.4)$$

and

$$E_{2\infty} = \frac{c_1p_2q_2(e^l)^{1-k}(e^u)^k(1-m) - p_1p_2q_1q_2(d^u)^{1-k}(d^l)^k - c_2p_1q_1(s^u)^{1-k}(s^l)^k}{p_1p_2q_1q_2^2}. \quad (3.5)$$

We easily know that $E_{1\infty} > 0$ and $E_{2\infty} > 0$ provided

$$Kp_1p_2q_1q_2(r^l)^{1-k}(r^u)^k > c_1p_2q_2(r^u)^{1-k}(r^l)^k + Kc_2p_1q_1(c^u)^{1-k}(c^l)^k(1-m) \quad (3.6)$$

and

$$c_1p_2q_2(e^l)^{1-k}(e^u)^k(1-m) > p_1p_2q_1q_2(d^u)^{1-k}(d^l)^k + c_2p_1q_1(s^u)^{1-k}(s^l)^k. \quad (3.7)$$

Therefore, from conditions (3.6) and (3.7), there exists the nontrivial bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$.

Based on the above discussion we have the following Theorem 3.1.

Theorem 3.1. The following conclusions are satisfied:

(i) The trivial bionomic equilibrium $(x_\infty, y_\infty, 0, E_{2\infty})$ exists, in which $y_\infty = \frac{c_2}{p_2q_2}$ and $(x_\infty, E_{2\infty})$ will be any point on the line

$$(e^l)^{1-k}(e^u)^k(1-m)x - q_2E_2 = (d^u)^{1-k}(d^l)^k + (s^u)^{1-k}(s^l)^k\frac{c_2}{p_2q_2}$$

in the first quadrant of the xE_2-plane;

(ii) The trivial bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, 0)$ exists if $(r^l)^{1-k}(r^u)^k > \frac{c_1(r^u)^{1-k}(r^l)^k}{Kp_1q_1}$, in which $x_\infty = \frac{c_1}{p_1q_1}$ and $(y_\infty, E_{1\infty})$ will be any point on the line

$$(c^u)^{1-k}(c^l)^k(1-m)y + q_1E_1 = (r^l)^{1-k}(r^u)^k - \frac{c_1(r^u)^{1-k}(r^l)^k}{Kp_1q_1}$$

in the first quadrant of the yE_1-plane;

(iii) The nontrivial bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$ exists if (3.6) and (3.7) hold, in which $x_\infty = \frac{c_1}{p_1q_1}$, $y_\infty = \frac{c_2}{p_2q_2}$, and $E_{1\infty}$ and $E_{2\infty}$ are defined in (3.4) and (3.5), respectively.

4 Optimal harvesting policy

In this section, to achieve the optimal harvesting policy of system (1.5), that is, to maximize the following objective function J of system (1.5), optimal control theory provides the correct approach. The form of J is expressed as follows

$$J(E_1, E_2) = \int_0^\infty e^{-\delta t}[(p_1q_1x - c_1)E_1(t) + (p_2q_2y - c_2)E_2(t)]dt, \quad (4.1)$$

which subject to the state equation (1.5) by invoking Pontryagin's maximal principle [20] and the control variables $E_i(t)$ are subjected to $0 \leq E_i(t) \leq E_i^{\max}, i = 1, 2$, and δ represents the instantaneous annual rate of discount.

We firstly construct the Hamiltonian as follows

$$\begin{aligned} H = {} & e^{-\delta t}[(p_1 q_1 x - c_1)E_1 + (p_2 q_2 y - c_2)E_2] + \lambda_1[(r^l)^{1-k}(r^u)^k x - (r^u)^{1-k}(r^l)^k \frac{x^2}{K} \\ & -(c^u)^{1-k}(c^l)^k(1-m)xy - q_1E_1x] + \lambda_2[-(d^u)^{1-k}(d^l)^k y - (s^u)^{1-k}(s^l)^k y^2 \\ & +(e^l)^{1-k}(e^u)^k(1-m)xy - q_2E_2y], \end{aligned} \tag{4.2}$$

where λ_1 and λ_2 represent the adjoint variables. A simple computation shows that

$$\frac{\partial H}{\partial E_1} = e^{-\delta t}(p_1q_1x - c_1) - \lambda_1 q_1 x = \mu_1(t), \quad \frac{\partial H}{\partial E_2} = e^{-\delta t}(p_2q_2y - c_2) - \lambda_2 q_2 y = \mu_2(t). \tag{4.3}$$

Obviously, the optimal control $E_i(t)(i = 1, 2)$ must satisfy the following conditions

$$E_i(t) = \begin{cases} E_i^{\max} & \text{if } \mu_i(t) > 0, \\ 0 & \text{if } \mu_i(t) < 0. \end{cases} \tag{4.4}$$

The functions $\mu_i(t)(i = 1, 2)$ are called switching functions as a result of that $\mu_i(t)$ lead $E_i(t)$ to switch between level 0 and $E_i^{\max}$. It follows from the sign of the switching functions $\mu_i(t)$ that the optimal control $E_i(t)$ are bang-bang switchings from one extreme point to other one. But if $\mu_i(t) = 0$, the Hamiltonian function H will be independent of the control variable $E_i(t)$ and the optimal control can not be determined by the above procedure. Then they become singular controls $E_i^*(t), 0 < E_i^*(t) < E_i^{\max}, i = 1, 2$. Hence the corresponding optimal harvesting policy should be

$$E_i(t) = \begin{cases} E_i^{\max} & \text{if } \mu_i(t) > 0, \\ 0 & \text{if } \mu_i(t) < 0, \\ E_i^* & \text{if } \mu_i(t) = 0, \end{cases} \quad \text{for } i = 1, 2. \tag{4.5}$$

When $\mu_i(t) = 0$ $(i = 1, 2)$, from (4.3) we derive that

$$\lambda_1 = e^{-\delta t}\Big(p_1 - \frac{c_1}{q_1 x}\Big), \quad \lambda_2 = e^{-\delta t}\Big(p_2 - \frac{c_2}{q_2 y}\Big). \tag{4.6}$$

By Pontryagin's maximum principle [20], the adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}. \tag{4.7}$$

It follows from the first equation of (4.7) and (4.2) that

$$\begin{aligned} \frac{d\lambda_1}{dt} = {} & -e^{-\delta t}p_1q_1E_1 - \lambda_1[(r^l)^{1-k}(r^u)^k - 2(r^u)^{1-k}(r^l)^k \frac{x}{K} \\ & - (c^u)^{1-k}(c^l)^k(1-m)y - q_1E_1] - \lambda_2(e^l)^{1-k}(e^u)^k(1-m)y, \end{aligned} \tag{4.8}$$

which, by equilibrium conditions, becomes

$$\frac{d\lambda_1}{dt} = -e^{-\delta t} p_1 q_1 E_1 + \lambda_1 (r^u)^{1-k} (r^l)^k \frac{x}{K} - \lambda_2 (e^l)^{1-k} (e^u)^k (1-m) y. \tag{4.9}$$

Substituting (4.6) into (4.9), one has

$$\begin{aligned} \frac{d\lambda_1}{dt} = & -e^{-\delta t} p_1 q_1 E_1 + e^{-\delta t} (r^u)^{1-k} (r^l)^k \frac{x}{K} \Big(p_1 - \frac{c_1}{q_1 x} \Big) \\ & - e^{-\delta t} (e^l)^{1-k} (e^u)^k (1-m) y \Big(p_2 - \frac{c_2}{q_2 y} \Big). \end{aligned} \tag{4.10}$$

On integration of (4.10) we have

$$\lambda_1 = \frac{1}{\delta} e^{-\delta t} \Big[p_1 q_1 E_1 - (r^u)^{1-k} (r^l)^k \frac{x}{K} \Big(p_1 - \frac{c_1}{q_1 x} \Big) + (e^l)^{1-k} (e^u)^k (1-m) y \Big(p_2 - \frac{c_2}{q_2 y} \Big) \Big], \tag{4.11}$$

in which, we neglect the constant of integration in order to guarantee shadow price $\lambda_1 e^{\delta t}$ of the prey x is bounded. Similarly, we obtain that

$$\lambda_2 = \frac{1}{\delta} e^{-\delta t} \Big[p_2 q_2 E_2 - (c^u)^{1-k} (c^l)^k (1-m) x \Big(p_1 - \frac{c_1}{q_1 x} \Big) - (s^u)^{1-k} (s^l)^k y \Big(p_2 - \frac{c_2}{q_2 y} \Big) \Big]. \tag{4.12}$$

According to the first equation of (4.6) and (4.11), one yields that

$$\begin{aligned} e^{-\delta t} \Big(p_1 - \frac{c_1}{q_1 x} \Big) = & \frac{1}{\delta} e^{-\delta t} \Big[p_1 q_1 E_1 - (r^u)^{1-k} (r^l)^k \frac{x}{K} \Big(p_1 - \frac{c_1}{q_1 x} \Big) \\ & + (e^l)^{1-k} (e^u)^k (1-m) y \Big(p_2 - \frac{c_2}{q_2 y} \Big) \Big]. \end{aligned} \tag{4.13}$$

Analogously, from the second equation of (4.6) and (4.12) that

$$\begin{aligned} e^{-\delta t} \Big(p_2 - \frac{c_2}{q_2 y} \Big) = & \frac{1}{\delta} e^{-\delta t} \Big[p_2 q_2 E_2 - (c^u)^{1-k} (c^l)^k (1-m) x \Big(p_1 - \frac{c_1}{q_1 x} \Big) \\ & - (s^u)^{1-k} (s^l)^k y \Big(p_2 - \frac{c_2}{q_2 y} \Big) \Big]. \end{aligned} \tag{4.14}$$

Therefore, we achieve the optimal harvesting efforts E_1 and E_2 as follows

$$E_1 = \frac{\Big[\delta + (r^u)^{1-k} (r^l)^k \frac{x}{K} \Big] \Big(p_1 - \frac{c_1}{q_1 x} \Big) - (e^l)^{1-k} (e^u)^k (1-m) y \Big(p_2 - \frac{c_2}{q_2 y} \Big)}{p_1 q_1} \tag{4.15}$$

and

$$E_2 = \frac{\Big[\delta + (s^u)^{1-k} (s^l)^k y \Big] \Big(p_2 - \frac{c_2}{q_2 y} \Big) + (c^u)^{1-k} (c^l)^k (1-m) x \Big(p_1 - \frac{c_1}{q_1 x} \Big)}{p_2 q_2}. \tag{4.16}$$

Together with (4.15) and (4.16), solving steady state equations we gain the optimal equilibrium (x_δ, y_δ) and optimal harvesting effort $(E_{1\delta}, E_{2\delta})$.

5 Numerical simulations and discussions

In this section, we give three numerical examples and two tables to illustrate the feasibility of our analytical results.

Example 5.1. Consider the following system with imprecise parameters:

$$\begin{cases} \dfrac{dx}{dt} = (2.0)^{1-k}(2.4)^k x - (2.4)^{1-k}(2.0)^k \dfrac{x^2}{5} - (1.5)^{1-k}(1.2)^k(1-0.1)xy - q_1E_1x, \\ \dfrac{dy}{dt} = -(0.5)^{1-k}(0.3)^k y - (0.08)^{1-k}(0.06)^k y^2 + (0.6)^{1-k}(0.8)^k(1-0.1)xy - q_2E_2y. \end{cases} \tag{5.1}$$

We set $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 15$ and $E_2 = 10$, it is easy to verify that

$$(r^l)^{1-k}(r^u)^k - q_1E_1 \approx -1 < 0. \tag{5.2}$$

Then consider (i) in Theorem 2.1, the trivial equilibrium $A_0 = (0, 0)$ is locally asymptotically stable (see Figure 1). These figures show that both the prey x and the predator y decrease to zero, that is, system (5.1) approaches to the trivial equilibrium A_0.

Assign $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 15$, a simple computation shows that

$$\begin{aligned} &(r^l)^{1-k}(r^u)^k - q_1E_1 \approx 1 > 0, \\ &K(e^l)^{1-k}(e^u)^k(1-m)[(r^l)^{1-k}(r^u)^k - q_1E_1] - (r^u)^{1-k}(r^l)^k[(d^u)^{1-k}(d^l)^k + q_2E_2] \approx -5.7 < 0, \end{aligned} \tag{5.3}$$

which, together with (ii) in Theorem 2.1, means that the axial equilibrium $A_1 = (2.0833, 0)$ is locally asymptotically stable (see Figure 2). From the figures, the prey x exists, however, the predator y goes to extinct, which is equal to that system (5.1) approaches to the axial equilibrium A_1.

Considering $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 2$, it derives that

$$(e^l)^{1-k}(e^u)^k(1-m)x^* - [(d^u)^{1-k}(d^l)^k + q_2E_2] \approx 0.0113 > 0. \tag{5.4}$$

According to (iii) in Theorem 2.1, we easily know that the axial equilibrium $A_2 = (1.6875, 0.14 07)$ is locally asymptotically stable (see Figure 3). In these figures, both the prey x and the predator y exist, i.e. system (5.1) approaches to the interior equilibrium A^*.

On the other hand, from Figures 4-6, we consider the effect of imprecise parameters. In Figure 4, assign $q_1 = 0.2, q_2 = 0.2, E_1 = 15$ and $E_2 = 10$, it implies that the trivial equilibrium A_0 of system (5.1) always exists for different values of k ($k \in [0, 1]$), and the values of the prey x and the predator y are invariant in zero with increasing k. In Figure 5, set $q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 15$, we can see that the axial equilibrium A_1 of system (5.1) exists for all $k \in [0, 1]$, and the values of the prey x is increasing and the predator y is invariant in zero with increasing k. For $q_1 = 0.2, q_2 = 0.2, E_1 = 5$ and $E_2 = 2$, Figure 6 shows that the interior equilibrium A^* of system (5.1) always exists for different values of

k ($k \in [0,1]$), and the values of the prey x is decreasing but the predator y is increasing with increasing k.

The following example is used to illustrate the existence of the nontrivial bionomic equilibrium.

Example 5.2. Consider the following system with imprecise parameters:

$$\begin{cases} \dfrac{dx}{dt} = (1.5)^{1-k}(1.6)^k x - (1.6)^{1-k}(1.5)^k \dfrac{x^2}{10} - (0.3)^{1-k}(0.25)^k(1-0.1)xy - q_1 E_1 x, \\ \dfrac{dy}{dt} = -(0.5)^{1-k}(0.45)^k y - (0.2)^{1-k}(0.15)^k y^2 + (1.3)^{1-k}(1.35)^k(1-0.1)xy - q_2 E_2 y, \end{cases} \tag{5.5}$$

with $q_1 = 0.92, q_2 = 0.95, p_1 = 20, p_2 = 25, c_1 = 30, c_2 = 15$ and $k \in [0,1]$. A simple computation shows that

$$Kp_1p_2q_1q_2(r^l)^{1-k}(r^u)^k - c_1p_2q_2(r^u)^{1-k}(r^l)^k - Kc_2p_1q_1(c^u)^{1-k}(c^l)^k(1-m) \geq 4669.8000 > 0 \tag{5.6}$$

and

$$c_1p_2q_2(e^l)^{1-k}(e^u)^k(1-m) - p_1p_2q_1q_2(d^u)^{1-k}(d^l)^k - c_2p_1q_1(s^u)^{1-k}(s^l)^k \geq 559.9250 > 0. \tag{5.7}$$

According to (iii) in Theorem 3.1, system (5.5) exists the nontrivial bionomic equilibria for different values of k. In Table 5.1, we show the nontrivial bionomic equilibria ($x_\infty, y_\infty, E_{1\infty}$, $E_{2\infty}$).

Table 5.1. Nontrivial bionomic equilibria for different k.

k	Nontrivial bionomic equilibrium $(x_\infty, y_\infty, E_{1\infty}, E_{2\infty})$
0	$(1.6304, 0.6316, 1.1615, 1.3487)$
0.2	$(1.6304, 0.6316, 1.1930, 1.3824)$
0.5	$(1.6304, 0.6316, 1.2402, 1.4318)$
0.8	$(1.6304, 0.6316, 1.2873, 1.4802)$
1	$(1.6304, 0.6316, 1.3188, 1.5118)$

From Table 5.1, we can see that x_∞ and y_∞ are invariable with increasing k, and $E_{1\infty}$ and $E_{2\infty}$ are increasing as k increases.

In order to find the optimal equilibrium and optimal harvesting effort, we consider the following example.

Example 5.3. Consider the following system with imprecise parameters:

$$\begin{cases} \dfrac{dx}{dt} = (1.8)^{1-k}(1.85)^k x - (1.85)^{1-k}(1.8)^k \dfrac{x^2}{10} - (2.5)^{1-k}(2.45)^k(1-0.1)xy - q_1 E_1 x, \\ \dfrac{dy}{dt} = -(0.015)^{1-k}(0.012)^k y - (0.01)^{1-k}(0.008)^k y^2 + (0.2)^{1-k}(0.21)^k(1-0.1)xy - q_2 E_2 y, \end{cases} \tag{5.8}$$

with $q_1 = 0.95, q_2 = 0.85, p_1 = 30, p_2 = 25, c_1 = 25, c_2 = 15, \delta = 0.001$ and $k \in [0, 1]$. So, for different values of k, the optimal equilibria (x_δ, y_δ) and optimal harvesting efforts $(E_{1\delta}, E_{2\delta})$ are displayed in Table 5.2.

Table 5.2. Optimal equilibria and optimal harvesting efforts for different k.

k	Optimal equilibrium (x_δ, y_δ)	Optimal harvesting effort $(E_{1\delta}, E_{2\delta})$
0	$(0.9310, 0.7199)$	$(0.0083, 0.1711)$
0.2	$(0.9322, 0.7281)$	$(0.0072, 0.1743)$
0.5	$(0.9340, 0.7405)$	$(0.0054, 0.1791)$
0.8	$(0.9359, 0.7530)$	$(0.0035, 0.1840)$
1	$(0.9371, 0.7616)$	$(0.0022, 0.1871)$

According to Table 5.2, it is easy to see that the optimal equilibria are increasing with increasing k. Also, the optimal harvesting efforts of the predator y are increasing as k increases, however, the optimal harvesting efforts of the prey x are decreasing as k increases.

6 Conclusions

In this paper, we study a predator-prey model with a prey refuge under harvesting. As far as we know, most ecological models with precise biological parameters are investigated, however, the accurate estimate in our real world can not come true easily. So the method of interval-valued function is applied in our system for solving the problem about imprecise parameters. And then we analyze the sufficient conditions for the existence and stability of equilibria of our imprecise harvesting system. All possible bionomic equilibria of the system are obtained in detail. We also discuss the optimal harvesting policy by applying Pontryagin's maximal principle, and the optimal equilibrium and optimal harvesting effort can be derived. In our opinion, the factor on the impreciseness of parameters for many bioeconomic models can not be ignored, and the fuzzy approach is good for handling such type of model in practice, so many existing models of biomathematics can be considered under impreciseness by the above approach. On the other hand, for the above model with a constant prey refuge, or with other types of functional response, we leave it for later discussion.

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Figure legends

Figure 1. (a) Time-series of the prey x and the predator y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 15, E_2 = 10$ and initial values $x(0) = 0.25$ and $y(0) = 0.15$ for $t \in [0, 50]$. (b) Phase portrait of x and y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 15, E_2 = 10$ and different initial values for $t \in [0, 50]$.

Figure 2. (a) Time-series of the prey x and the predator y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 15$ and initial values $x(0) = 0.25$ and $y(0) = 0.15$ for $t \in [0, 50]$. (b) Phase portrait of x and y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 15$ and different initial values for $t \in [0, 50]$.

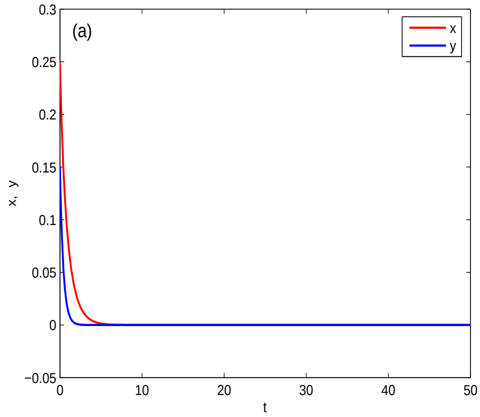
Figure 3. (a) Time-series of the prey x and the predator y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ and initial values $x(0) = 0.25$ and $y(0) = 0.15$ for $t \in [0, 50]$. (b) Phase portrait of x and y with $k = 0, q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ and different initial values for $t \in [0, 50]$.

Figure 4. (a)-(e) Time-series of the prey x and the predator y with $q_1 = 0.2, q_2 = 0.2, E_1 = 15, E_2 = 10$ and initial values $(0.25, 0.15)$ for $k = 0, k = 0.2, k = 0.5, k = 0.8$ and $k = 1$, respectively, $t \in [0, 50]$. (f) Dynamical behavior of the prey x and the predator y with respect to k and the values of other parameters are the same to the above values.

Figure 5. (a)-(e) Time-series of the prey x and the predator y with $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 15$ and initial values $(0.25, 0.15)$ for $k = 0, k = 0.2, k = 0.5, k = 0.8$ and $k = 1$, respectively, $t \in [0, 50]$. (f) Dynamical behavior of the prey x and the predator y with respect to k and the values of other parameters are the same to the above values.

Figure 6. (a)-(e) Time-series of the prey x and the predator y with $q_1 = 0.2, q_2 = 0.2, E_1 = 5, E_2 = 2$ and initial values $(0.25, 0.15)$ for $k = 0, k = 0.2, k = 0.5, k = 0.8$ and $k = 1$, respectively, $t \in [0, 50]$. (f) Dynamical behavior of the prey x and the predator y with respect to k and the values of other parameters are the same to the above values.

Figure 1



(a)
x
y
x, y
t
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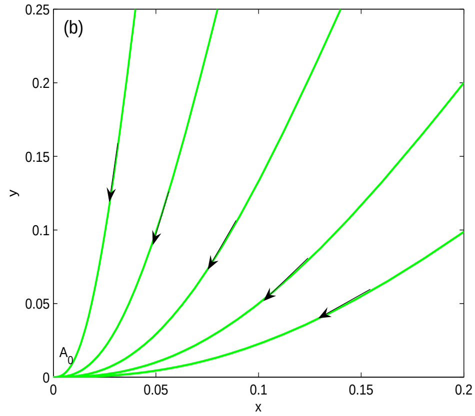
0.25
0.2
0.15
0.1
0.05
0
−0.05
0
10
20
30
40
50



(b)
y
x
A_0
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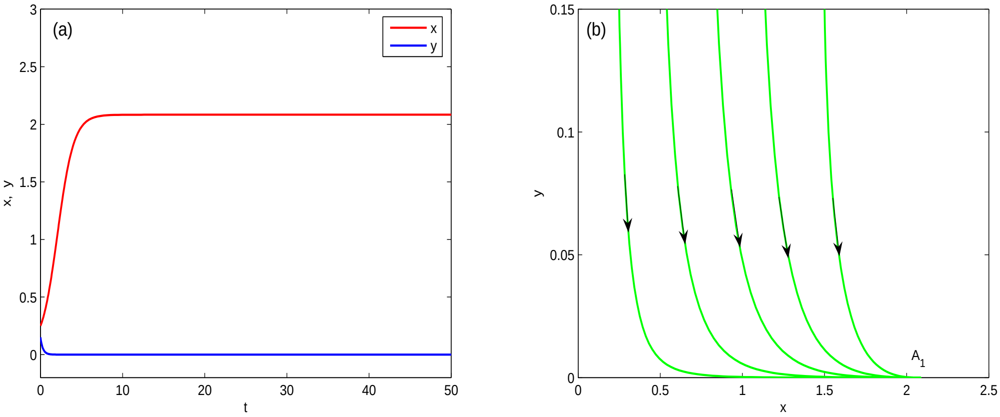
0.05
0
0
0.05
0.1
0.15
0.2

Figure 2



(a)
x
y
x, y
t
0
10
20
30
40
50
0
0.5
1
1.5
2
2.5
3
(b)
A1
y
x
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0.5
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0.05
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Figure 3

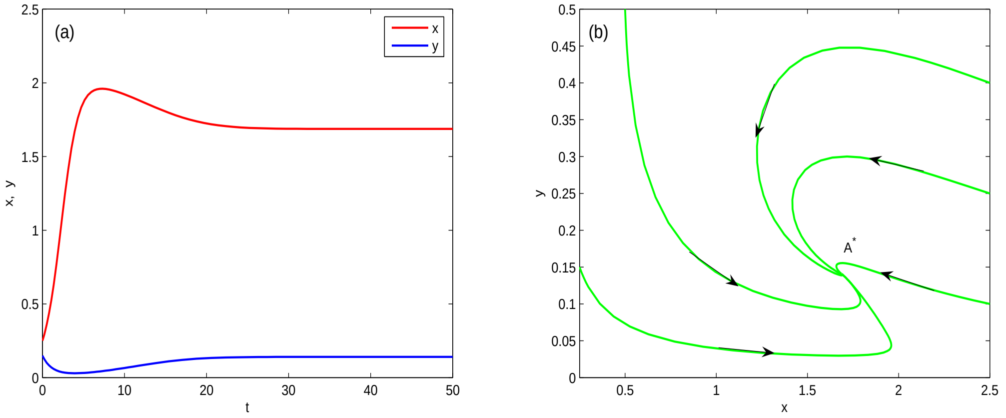

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Figure 4

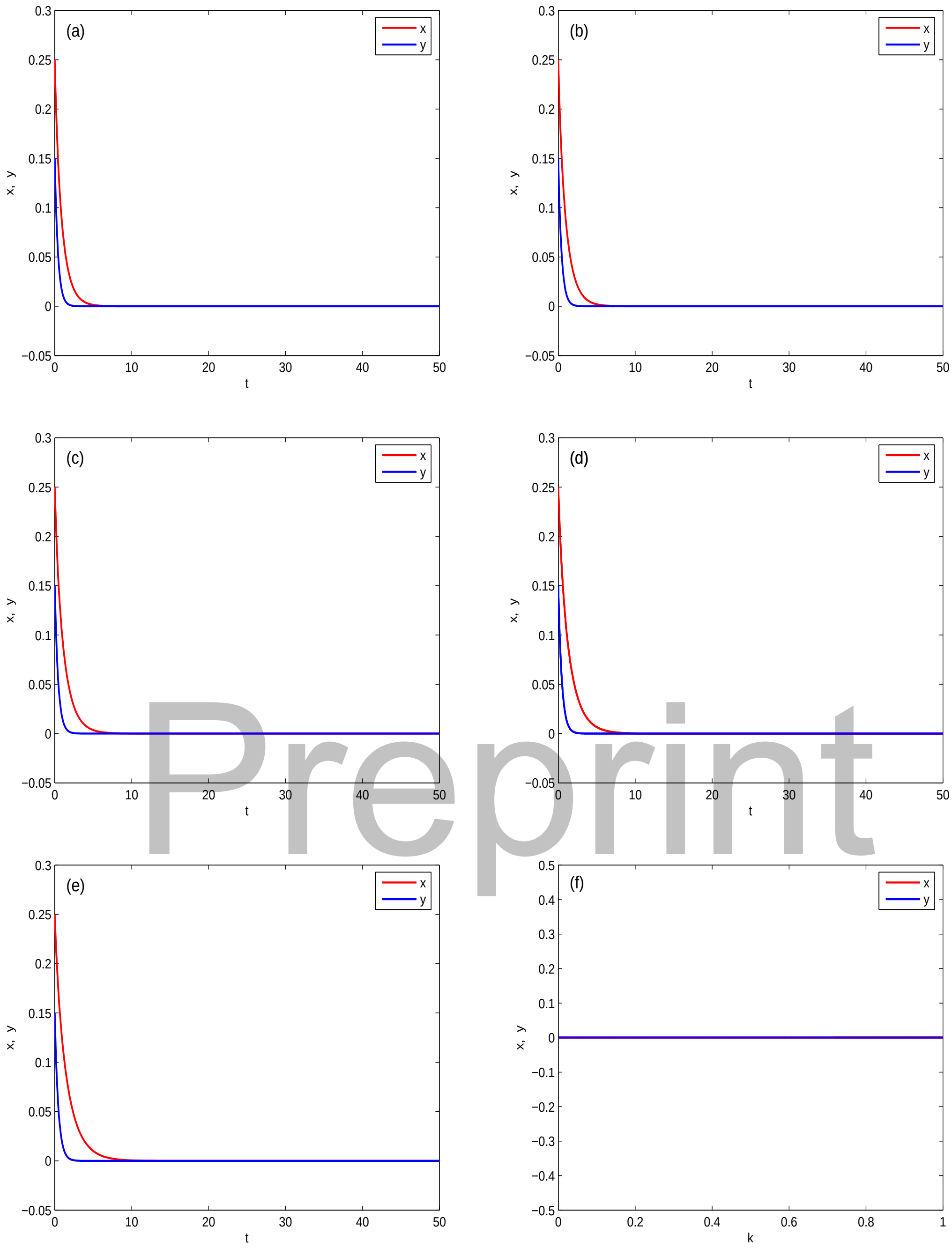

Figure 5

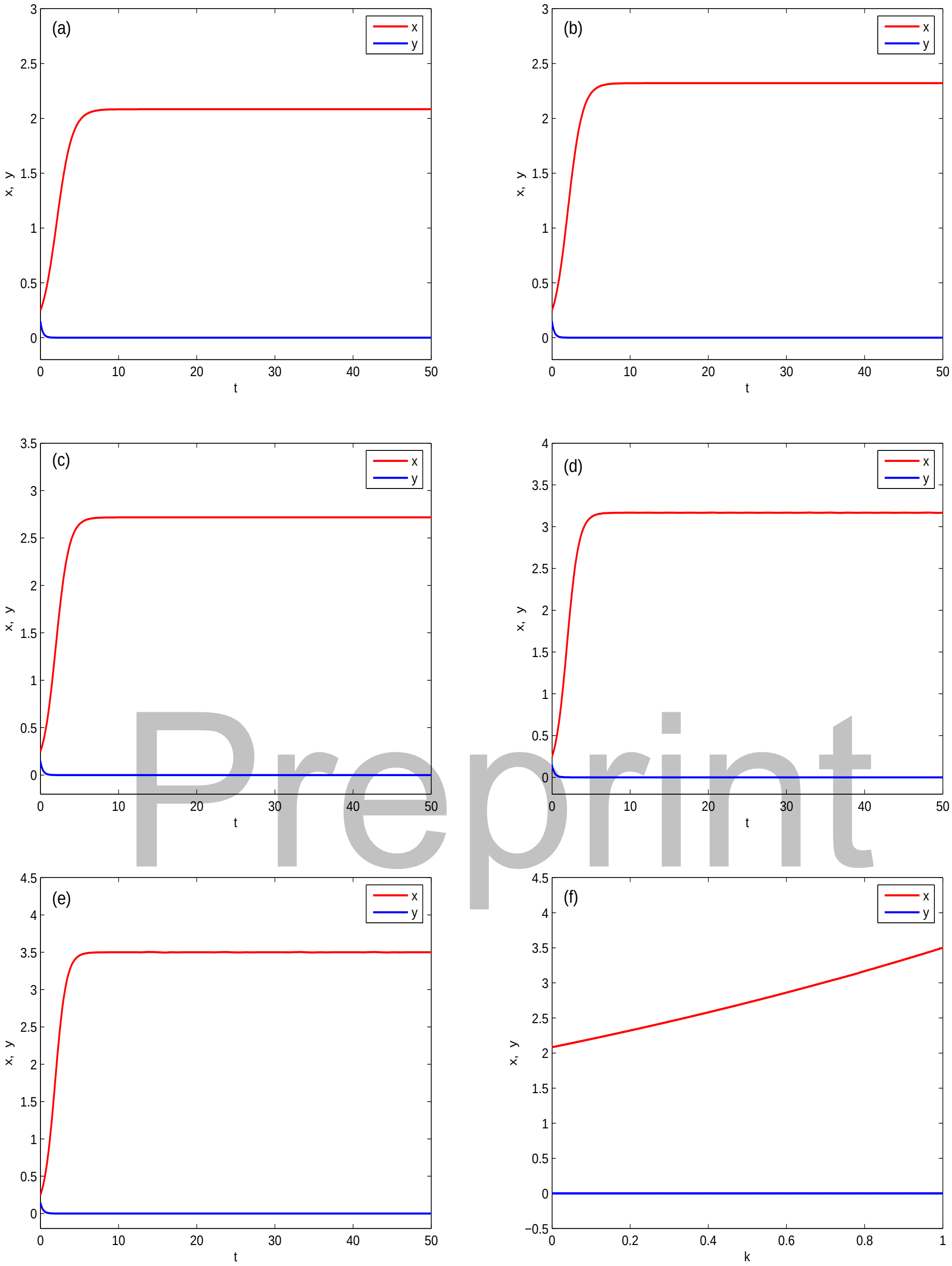

Figure 6