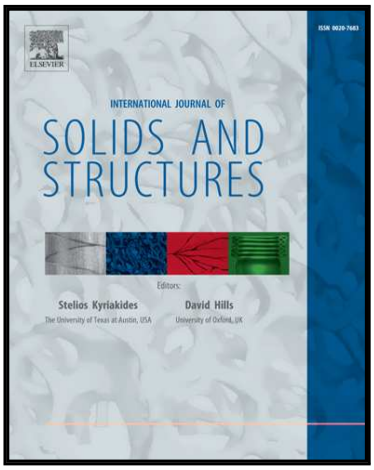

On a tensor cross product based formulation of large strain solid mechanics

Javier Bonet, Antonio J. Gil, Rogelio Ortigosa

PII: S0020-7683(15)00530-2
DOI: 10.1016/j.ijsolstr.2015.12.030
Reference: SAS 9011

To appear in: *International Journal of Solids and Structures*

Received date: 30 December 2014
Revised date: 21 October 2015
Accepted date: 27 December 2015

Please cite this article as: Javier Bonet, Antonio J. Gil, Rogelio Ortigosa, On a tensor cross product based formulation of large strain solid mechanics, *International Journal of Solids and Structures* (2016), doi: 10.1016/j.ijsolstr.2015.12.030

On a tensor cross product based formulation of large strain solid mechanics

Javier Bonet[1], Antonio J. Gil[2], Rogelio Ortigosa

Zienkiewicz Centre for Computational Engineering, College of Engineering Swansea University, Bay Campus, SA1 8EN, United Kingdom

Abstract

This paper describes in detail the formulation of large strain solid mechanics based on the tensor cross product, originally presented by de Boer [1], page 76, and recently re-introduced by Bonet et al. in [2] and [3]. The paper shows how the tensor cross product facilitates the algebra associated with the area and volume maps between reference and final configurations. These maps, together with the fibre map, make up the fundamental kinematic variables in polyconvex elasticity. The algebra proposed leads to novel expressions for the tangent elastic operator which neatly separates material from geometrical dependencies. The paper derives new formulas for the spatial and material stress and their corresponding elasticity tensors. These are applied to the simple case of a Mooney-Rivlin material model. The extension to transversely isotropic material models is also considered.

Keywords: Large strain elasticity, polyconvex elasticity, complementary energy, incompressible elasticity, tensor cross product, Generalised Gibbs energy function

1. Introduction

Large strain elastic and inelastic analysis by finite elements or other computational techniques is now well established for many engineering applications [4–16]. Often elasticity is described by means of a hyperelastic model

[1]Corresponding author: j.bonet@swansea.ac.uk

[2]Corresponding author: a.j.gil@swansea.ac.uk

defined in terms of a stored energy functional which depends on the deformation gradient of the mapping between initial and final configurations [4, 17–25]. It has also been shown that for the model to be well defined in a mathematical sense, this dependency with respect to the deformation gradient has to satisfy certain convexity criteria [4, 20, 21]. The most well-established of these criteria is the concept of polyconvexity [22–28] whereby the strain energy function must be expressed as a convex function of the components of the deformation gradient, its determinant and the components of its adjoint or co-factor. Numerous authors have previously incorporated this concept into computational models for both isotropic and non-isotropic materials for a variety of applications [29–34].

The classical approach consists of ensuring that the stored energy function satisfies the polyconvexity condition first but then proceed towards an evaluation of stresses and elasticity tensors by re-expressing the energy function in terms of the deformation gradient alone. This inevitably leads to the differentiation of inverse functions of the deformation gradient, its transpose or the inverse of the right Cauchy-Green tensor. These derivatives are readily obtained using standard algebra but can lead to lengthy expressions. An alternative approach has recently been proposed by Bonet et al. in [2] and [3] by recovering the concept of the tensor cross-product originally introduced by de Boer [1] but not previously used in continuum mechanics. This tensor cross product allows for simpler expressions to be obtained for the area and volume maps and their derivatives. The resulting formulas for the elasticity tensors provide useful physical insights by separating positive definite material components from geometrical components.

The paper explores the proposed formulation both in the reference setting, using Piola-Kirchhoff stress tensors and in the spatial setting using Kirchhoff and Cauchy stress tensors. Some formulas derived with the tensor cross product formulation are compared against their classical equivalent versions in order to demonstrate the advantages of the proposed methodology. Both isotropic and anisotropic cases are considered, in the latter case anisotropy is restricted to the simple transversely isotropic case. The paper illustrates the proposed concepts using the well established model of a Mooney-Rivlin material.

The paper is organised as follows. Section 2 introduces the novel tensor cross product notation in the context of large strain deformation. Whilst this product had already been proposed by de Boer in [1] (in German), it has not previously been described in the English literature or used in the context of solid mechanics, so most readers will be unfamiliar with it. This product is used to re-express the adjoint of the deformation gradient and its directional derivatives in a novel, simple and convenient manner. Section 3 reviews the definition of polyconvex elastic strain energy functions and defines a new set of stresses conjugate to the main kinematic variables. The relationships between these stresses and the standard first Piola-Kirchhoff stresses are provided. The section also derives complementary strain energy functions in terms of the new conjugate stresses. The algebra is greatly simplified via the tensor cross product. The fourth order elasticity tensors are derived in this section taking advantage of the tensor cross product operation leading to interesting insights into the consequences of convexity. Both compressible and nearly incompressible cases are discussed in the context of Mooney-Rivlin models, although the extension to more general strain energy functions is straight forward. Section 4 derives similar equations using entirely material tensors such as the right Cauchy-Green tensor and the second Piola-Kirchhoff tensor or spatial tensors such as the Kirchhoff or Cauchy stresses. Expressions for both material and spatial elasticity tensor are given in the context of the new proposed notation. Section 5 particularises the above expressions for the case of isotropic and transversely isotropic materials. A number of mixed and complementary energy variational principles are presented in Section 6. Several of these have been used in [2] for the purpose of constructing novel finite element approximations. Finally, Section 7 provides some concluding remarks and a summary of the key contributions of this paper.

2. Definitions and notation

2.1. Motion and deformation

Consider the three dimensional deformation of an elastic body from its initial configuration occupying a volume V, of boundary ∂V, into a final configuration at volume v, of boundary ∂v (see Figure 1). The standard nomenclature for the deformation gradient tensor $\boldsymbol{F}$ and the Jacobian J of the deformation are used

$$d\boldsymbol{x} = \boldsymbol{F} d\boldsymbol{X}; \qquad \boldsymbol{F} = \boldsymbol{\nabla}_0 \boldsymbol{x}; \tag{1a}$$

$$dv = J dV; \qquad J = \det\left(\boldsymbol{\nabla}_0 \boldsymbol{x}\right), \tag{1b}$$

where $\boldsymbol{x}$ represents the current position of a particle originally at $\boldsymbol{X}$ and $\boldsymbol{\nabla}_0 := \frac{\partial}{\partial \boldsymbol{X}}$ denotes the gradient with respect to material coordinates. Virtual or linear incremental variations of $\boldsymbol{x}$ will be denoted $\delta\boldsymbol{v}$ and $\boldsymbol{u}$, respectively. It will be assumed that $\boldsymbol{x}$ satisfy appropriate prescribed displacement based boundary conditions in $\partial_u V$, and that $\delta\boldsymbol{v}$ and $\boldsymbol{u}$ will satisfy the equivalent homogeneous conditions in this section of the boundary. Additionally, the body is under the action of certain body forces per unit undeformed volume $\boldsymbol{f}_0$ and traction per unit undeformed area $\boldsymbol{t}_0$ in $\partial_t V$, where $\partial_t V \cup \partial_u V = \partial V$ and $\partial_t V \cap \partial_u V = \emptyset$.

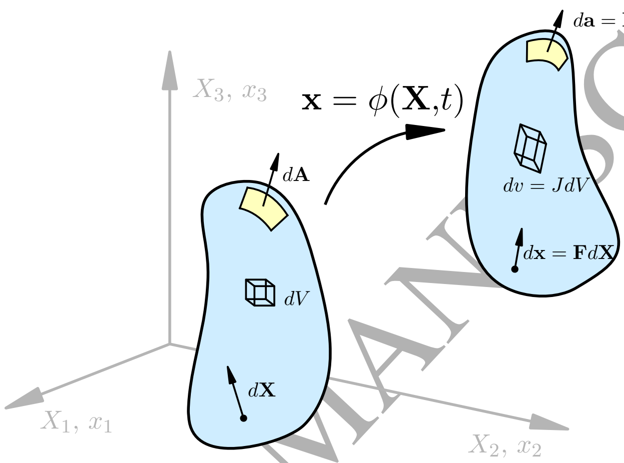

Figure 1: Deformation mapping of a continuum and associated kinematics magnitudes: $\boldsymbol{F}, \boldsymbol{H}, J$.

The element area vector is mapped from initial $d\boldsymbol{A}$ to final $d\boldsymbol{a}$ configuration by means of the two-point tensor $\boldsymbol{H}$, which is related to the deformation gradient via Nanson's rule [4]:

$$d\boldsymbol{a} = \boldsymbol{H} d\boldsymbol{A}; \qquad \boldsymbol{H} = \det(\boldsymbol{\nabla}_0 \boldsymbol{x})\,(\boldsymbol{\nabla}_0 \boldsymbol{x})^{-T}. \tag{2}$$

Clearly, the components of this tensor are the order 2 minors of the deformation gradient and it is often referred to as the co-factor or adjoint tensor, that is $\boldsymbol{H} = \mathrm{Cof}\,(\boldsymbol{\nabla}_0 \boldsymbol{x})$. This tensor and its derivatives will feature heavily in the formulation that follows as it is a key variable for polyconvex elastic models. Its evaluation and, more importantly, the evaluation of its derivatives using equation (2) is not ideal, and a more convenient formula can be derived for three dimensional applications. This relies on the use of a tensor cross product operation, presented from the first time in Reference [1], page 76, but included in 2.2 for completeness.

The relationships between $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and the geometry $\boldsymbol{x}$ via equations (1)-(2) represent three geometric compatibility conditions, which can be re-expressed in a more helpful manner via the tensor cross product defined below.

2.2. Tensor cross product

The key elements of the framework proposed is the extension of the standard vector cross product to define the cross product between second order tensors and between tensors and vectors. This rediscovers the work of de Boer [1] which, to the best knowledge of the authors, does not appear in any English language publication. The original nomenclature in [1] is "Das äuβere Tensorprodukt von Tensoren" which has been translated here as tensor cross product.

The left cross product of a vector $\boldsymbol{v}$ and a second order tensor $\boldsymbol{A}$ to give a second order tensor denoted $\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A}$ is defined so that when applied to a general vector $\boldsymbol{w}$ gives:

$$(\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A})\, \boldsymbol{w} = \boldsymbol{v} \times (\boldsymbol{A}\boldsymbol{w})\,; \quad (\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A})_{ij} = \mathcal{E}_{ikl} v_k A_{lj}, \tag{3}$$

where $\mathcal{E}_{ikl}$ denote the standard third order alternating tensor components, repeated indices indicate summation and $\times$ is the standard vector cross product. Note that the notation $\boldsymbol{\times}$ instead of $\times$ is used if the outcome of the operation is a second order tensor rather than a vector. The effect of the above operation is to replace the columns of $\boldsymbol{A}$ by the cross products between $\boldsymbol{v}$ and the original columns of $\boldsymbol{A}$. Similarly, the right cross product of a second order tensor $\boldsymbol{A}$ by a vector $\boldsymbol{v}$ to give a second order tensor denoted $\boldsymbol{A} \boldsymbol{\times} \boldsymbol{v}$ is defined so that for every vector $\boldsymbol{w}$ the following relationship applies:

$$(\boldsymbol{A} \boldsymbol{\times} \boldsymbol{v})\, \boldsymbol{w} = \boldsymbol{A}\, (\boldsymbol{v} \times \boldsymbol{w})\,; \quad (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{v})_{ij} = \mathcal{E}_{jkl} A_{ik} v_l. \tag{4}$$

The effect is now to replace the rows of $\boldsymbol{A}$ by the cross products of its original rows by $\boldsymbol{v}$.

Finally, the cross product of two second order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ to give a new second order tensor denoted $\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B}$ is defined so that for any arbitrary vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ gives:

$$\boldsymbol{v} \cdot (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B})\, \boldsymbol{w} = (\boldsymbol{v} \boldsymbol{\times} \boldsymbol{A}) : (\boldsymbol{B} \boldsymbol{\times} \boldsymbol{w})\,; \quad (\boldsymbol{A} \boldsymbol{\times} \boldsymbol{B})_{ij} = \mathcal{E}_{ikl}\mathcal{E}_{jmn} A_{km} B_{ln}. \tag{5}$$

In this paper, the tensor cross product will be mostly applied between two-point tensors. For this purpose, the above definition can be particularised to second order two-point tensors or material tensors as,

$$(\boldsymbol{A} \times \boldsymbol{B})_{iI} = \mathcal{E}_{ijk}\mathcal{E}_{IJK}A_{jJ}B_{kK}; \quad (\boldsymbol{A} \times \boldsymbol{B})_{IJ} = \mathcal{E}_{IKL}\mathcal{E}_{JMN}A_{KM}B_{LN}. \tag{6}$$

Box 1 shows the practical evaluation of these products.

Finally, the cross vector product of two two-point tensors to give a spatial vector is also defined by a cross product operation with respect to the first indices and a contraction with respect to the second set of indices, so that,

$$\boldsymbol{v}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{v}\cdot\boldsymbol{\mathcal{E}} : (\boldsymbol{A}\boldsymbol{B}^T) = \mathrm{tr}\left(\boldsymbol{v}\times(\boldsymbol{A}\boldsymbol{B}^T)\right); \qquad (\boldsymbol{A}\times\boldsymbol{B})_i = \mathcal{E}_{ijk}A_{jI}B_{kI}. \tag{7}$$

Remark 1: It is easy to show using simply algebraic manipulations based on the permutation properties of $\boldsymbol{\mathcal{E}}$ or the fact that $\mathcal{E}_{ijk}\mathcal{E}_{kln} = \delta_{il}\delta_{jn} - \delta_{in}\delta_{jl}$, that the above tensor cross products satisfy the following properties (note that $\boldsymbol{v}$, $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{w}$, $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ denote arbitrary vectors and $\boldsymbol{A}$, $\boldsymbol{A}_1$, $\boldsymbol{A}_2$, $\boldsymbol{B}$, $\boldsymbol{B}_1$, $\boldsymbol{B}_2$ and $\boldsymbol{C}$ are second order tensors):

$$\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{B}\times\boldsymbol{A} \tag{8}$$

$$(\boldsymbol{A}\times\boldsymbol{B})^T = \boldsymbol{A}^T\times\boldsymbol{B}^T \tag{9}$$

$$\boldsymbol{A}\times(\boldsymbol{B}_1+\boldsymbol{B}_2) = \boldsymbol{A}\times\boldsymbol{B}_1 + \boldsymbol{A}\times\boldsymbol{B}_2 \tag{10}$$

$$\alpha\,(\boldsymbol{A}\times\boldsymbol{B}) = (\alpha\boldsymbol{A})\times\boldsymbol{B} = \boldsymbol{A}\times(\alpha\boldsymbol{B}) \tag{11}$$

$$(\boldsymbol{A}\times\boldsymbol{B}) : \boldsymbol{C} = (\boldsymbol{B}\times\boldsymbol{C}) : \boldsymbol{A} = (\boldsymbol{A}\times\boldsymbol{C}) : \boldsymbol{B} \tag{12}$$

$$\boldsymbol{A}\times\boldsymbol{I} = (\mathrm{tr}\boldsymbol{A})\,\boldsymbol{I} - \boldsymbol{A}^T \tag{13}$$

$$\boldsymbol{I}\times\boldsymbol{I} = 2\boldsymbol{I} \tag{14}$$

$$(\boldsymbol{A}\times\boldsymbol{A}) : \boldsymbol{A} = 6\,\det\boldsymbol{A} \tag{15}$$

$$\mathrm{Cof}\boldsymbol{A} = \frac{1}{2}\boldsymbol{A}\times\boldsymbol{A} \tag{16}$$

$$(\boldsymbol{v}_1\otimes\boldsymbol{v}_2)\times(\boldsymbol{w}_1\otimes\boldsymbol{w}_2) = (\boldsymbol{v}_1\times\boldsymbol{w}_1)\otimes(\boldsymbol{v}_2\times\boldsymbol{w}_2) \tag{17}$$

$$\boldsymbol{v}\times(\boldsymbol{A}\times\boldsymbol{w}) = (\boldsymbol{v}\times\boldsymbol{A})\times\boldsymbol{w} = \boldsymbol{v}\times\boldsymbol{A}\times\boldsymbol{w} \tag{18}$$

$$\boldsymbol{A}\times(\boldsymbol{v}\otimes\boldsymbol{w}) = -\boldsymbol{v}\times\boldsymbol{A}\times\boldsymbol{w} \tag{19}$$

$$(\boldsymbol{A}\times\boldsymbol{B})\,(\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{A}\boldsymbol{v})\times(\boldsymbol{B}\boldsymbol{w}) + (\boldsymbol{B}\boldsymbol{v})\times(\boldsymbol{A}\boldsymbol{w}) \tag{20}$$

$$(\boldsymbol{A}_1 \times \boldsymbol{A}_2)(\boldsymbol{B}_1 \times \boldsymbol{B}_2) = (\boldsymbol{A}_1\boldsymbol{B}_1) \times (\boldsymbol{A}_2\boldsymbol{B}_2) + (\boldsymbol{A}_1\boldsymbol{B}_2) \times (\boldsymbol{A}_2\boldsymbol{B}_1) \tag{21}$$

$$(\boldsymbol{A}_1\boldsymbol{B}) \times (\boldsymbol{A}_2\boldsymbol{B}) = (\boldsymbol{A}_1 \times \boldsymbol{A}_2)\,\mathrm{Cof}\boldsymbol{B} \tag{22}$$

Box 1. Enumeration of tensor cross products:

$$[\boldsymbol{v} \times \boldsymbol{A}] = \begin{bmatrix} v_y A_{zx} - v_z A_{yx} & v_y A_{zy} - v_z A_{yy} & v_y A_{zz} - v_z A_{yz} \\ v_z A_{xx} - v_x A_{zx} & v_z A_{xy} - v_x A_{zy} & v_z A_{xz} - v_x A_{zz} \\ v_x A_{yx} - v_y A_{xx} & v_x A_{yy} - v_y A_{xy} & v_x A_{yz} - v_y A_{xz} \end{bmatrix}$$

$$[\boldsymbol{A} \times \boldsymbol{w}] = \begin{bmatrix} A_{xy} v_z - A_{xz} v_y & A_{xz} v_x - A_{xx} v_z & A_{xx} v_y - A_{xy} v_x \\ A_{yy} v_z - A_{yz} v_y & A_{yz} v_x - A_{yx} v_z & A_{yx} v_y - A_{yy} v_x \\ A_{zy} v_z - A_{zz} v_y & A_{zz} v_x - A_{zx} v_z & A_{zx} v_y - A_{zy} v_x \end{bmatrix}$$

$$[\boldsymbol{A} \times \boldsymbol{B}] = \begin{bmatrix} [\boldsymbol{A} \times \boldsymbol{B}]_{xx} & [\boldsymbol{A} \times \boldsymbol{B}]_{xy} & [\boldsymbol{A} \times \boldsymbol{B}]_{xz} \\ [\boldsymbol{A} \times \boldsymbol{B}]_{yx} & [\boldsymbol{A} \times \boldsymbol{B}]_{yy} & [\boldsymbol{A} \times \boldsymbol{B}]_{yz} \\ [\boldsymbol{A} \times \boldsymbol{B}]_{zx} & [\boldsymbol{A} \times \boldsymbol{B}]_{zy} & [\boldsymbol{A} \times \boldsymbol{B}]_{zz} \end{bmatrix}$$

$$\begin{aligned}
[\boldsymbol{A} \times \boldsymbol{B}]_{xx} &= A_{yy}B_{zz} - A_{yz}B_{zy} + A_{zz}B_{yy} - A_{zy}B_{yz} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{xy} &= A_{yz}B_{zx} - A_{yx}B_{zz} + A_{zx}B_{yz} - A_{zz}B_{yx} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{xz} &= A_{yx}B_{zy} - A_{yy}B_{zx} + A_{zy}B_{yx} - A_{zx}B_{yy} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{yx} &= A_{xz}B_{zy} - A_{xy}B_{zz} + A_{zy}B_{xz} - A_{zz}B_{xy} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{yy} &= A_{zz}B_{xx} - A_{zx}B_{xz} + A_{xx}B_{zz} - A_{xz}B_{zx} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{yz} &= A_{zx}B_{xy} - A_{zy}B_{xx} + A_{xy}B_{zx} - A_{xx}B_{zy} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{zx} &= A_{xy}B_{yz} - A_{xz}B_{yy} + A_{yz}B_{xy} - A_{yy}B_{xz} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{zy} &= A_{xz}B_{yx} - A_{xx}B_{yz} + A_{yx}B_{xz} - A_{yz}B_{xx} \\
[\boldsymbol{A} \times \boldsymbol{B}]_{zz} &= A_{xx}B_{yy} - A_{xy}B_{yx} + A_{yy}B_{xx} - A_{yx}B_{xy}
\end{aligned}$$

2.3. Alternative expressions for the geometric compatibility conditions

Using equation (16) it is possible to express the area map tensor $\boldsymbol{H}$ as

$$\boldsymbol{H} = \frac{1}{2}(\boldsymbol{\nabla}_0\boldsymbol{x}) \times (\boldsymbol{\nabla}_0\boldsymbol{x}). \tag{23}$$

Analogousy, equation (15) leads to an alternative expression for the volume map J as

$$J = \frac{1}{6}\left((\boldsymbol{\nabla}_0 \boldsymbol{x}) \times (\boldsymbol{\nabla}_0 \boldsymbol{x})\right) : \boldsymbol{\nabla}_0 \boldsymbol{x}. \tag{24}$$

In order to simplify the notation in what follows, we define ($:=$) the geometrically exact deformation terms $\{\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\}$ as

$$\boldsymbol{F}_{\boldsymbol{x}} := \boldsymbol{\nabla}_0 \boldsymbol{x}; \tag{25a}$$
$$\boldsymbol{H}_{\boldsymbol{x}} := \frac{1}{2}\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{F}_{\boldsymbol{x}}; \tag{25b}$$
$$J_{\boldsymbol{x}} := \frac{1}{3}\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{F}_{\boldsymbol{x}}, \tag{25c}$$

so that the geometric compatibility conditions (1)-(2) can be re-written as

$$\boldsymbol{F} = \boldsymbol{F}_{\boldsymbol{x}}; \tag{26a}$$
$$\boldsymbol{H} = \boldsymbol{H}_{\boldsymbol{x}}; \tag{26b}$$
$$J = J_{\boldsymbol{x}}. \tag{26c}$$

Remark 2: Note that in the exact continuum mechanics context, the geometric compatibility conditions (26) are satisfied strongly, namely $\boldsymbol{F} \equiv \boldsymbol{F}_{\boldsymbol{x}}$, $\boldsymbol{H} \equiv \boldsymbol{H}_{\boldsymbol{x}}$ and $J \equiv J_x$ at each material point. However, in the context of approximate solutions such as in computational mechanics, this is not necessarily true in general (i.e. $\boldsymbol{F} \not\equiv \boldsymbol{F}_{\boldsymbol{x}}$, $\boldsymbol{H} \not\equiv \boldsymbol{H}_{\boldsymbol{x}}$ and $J \not\equiv J_x$ in a point-wise manner). In the latter case, the geometric compatibility equations can be weakly enforced via, for instance, a mixed variational principle (refer to Section 6). In this case, the three deformation measures $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ are in effect independent from each other and only indirectly related through their relationship to the geometry $\boldsymbol{x}$ via the enforcement of the geometric compatibility conditions. Hence, direct relationships between J, $\boldsymbol{H}$ and $\boldsymbol{F}$, such as $J = \det \boldsymbol{F}$ or $\boldsymbol{H} = \frac{1}{2}\boldsymbol{F} \times \boldsymbol{F}$ will not be considered in this paper to be valid outside the continuum context. In contrast, the geometrically exact deformation maps $\boldsymbol{F}_{\boldsymbol{x}}$, $\boldsymbol{H}_{\boldsymbol{x}}$ and J_x do satisfy relationships like $\boldsymbol{H}_{\boldsymbol{x}} \equiv \frac{1}{2}\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{F}_{\boldsymbol{x}}$ and $J_{\boldsymbol{x}} \equiv \det \boldsymbol{F}_{\boldsymbol{x}}$, as these are simply a consequence of definitions (25).

It is also possible to derive alternative geometric compatibility equations for $\boldsymbol{H}$ and J. For instance, combining equations (25b) with (26b) and noting that the derivatives of $\boldsymbol{F}_{\boldsymbol{x}}$ are second derivatives of $\boldsymbol{x}$ and therefore symmetric, gives, after simple use of the product rule

$$\boldsymbol{H} = \boldsymbol{H}_{\boldsymbol{x}}; \qquad \boldsymbol{H}_{\boldsymbol{x}} \equiv \frac{1}{2}\mathrm{CURL}\left(\boldsymbol{x} \times \boldsymbol{F}_{\boldsymbol{x}}\right), \tag{27}$$

where the material CURL of a second order two point tensor is defined in the usual fashion by

$$(\mathrm{CURL}\boldsymbol{A})_{iI} = \mathcal{E}_{IJK}\frac{\partial A_{iK}}{\partial X_J}. \tag{28}$$

Similarly, combining equations (25c) with (26c), an alternative equation for the volume map J emerges as:

$$J = J_{\boldsymbol{x}}; \qquad J_{\boldsymbol{x}} \equiv \frac{1}{3}\mathrm{DIV}\left({\boldsymbol{H}_{\boldsymbol{x}}}^T\boldsymbol{x}\right), \tag{29}$$

where the material divergence is defined by the contraction

$$(\mathrm{DIV}\boldsymbol{A})_i = \frac{\partial A_{iI}}{\partial X_I}. \tag{30}$$

It is clear from equations (25a) and (27) that the following identities are fulfilled, namely $\mathrm{CURL}\left(\boldsymbol{F}_{\boldsymbol{x}}\right) \equiv 0$ and $\mathrm{DIV}\left(\boldsymbol{H}_{\boldsymbol{x}}\right) \equiv 0$. As a result of these identities, it is then possible to show that $\boldsymbol{F}$ and $\boldsymbol{H}$ should satisfy the following additional compatibility conditions, namely

$$\mathrm{CURL}\boldsymbol{F} = \boldsymbol{0}; \qquad \mathrm{DIV}\boldsymbol{H} = \boldsymbol{0}. \tag{31}$$

2.4. Differentiation of the deformation measures $\{\boldsymbol{F}, \boldsymbol{H}, J\}$

Combining equations (25a) and (26a), the first and second directional derivatives of $\boldsymbol{F}$ with respect to geometry changes are

$$D\boldsymbol{F}\left[\delta\boldsymbol{v}\right] = \boldsymbol{\nabla}_0\delta\boldsymbol{v}; \qquad D^2\boldsymbol{F}\left[\delta\boldsymbol{v}; \boldsymbol{u}\right] = \boldsymbol{0}. \tag{32}$$

Combining equations (25b) and (26b), the first and second directional derivatives of $\boldsymbol{H}$ with respect to geometry changes are easily evaluated as

$$\boxed{D\boldsymbol{H}\left[\delta\boldsymbol{v}\right] = \boldsymbol{F}_{\boldsymbol{x}} \times D\boldsymbol{F}_{\boldsymbol{x}}\left[\delta\boldsymbol{v}\right] = \boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0\delta\boldsymbol{v}} \tag{33}$$

$$\boxed{D^2\boldsymbol{H}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] = D\boldsymbol{F}_{\boldsymbol{x}}\left[\boldsymbol{u}\right] \times D\boldsymbol{F}_{\boldsymbol{x}}\left[\delta\boldsymbol{v}\right] = \boldsymbol{\nabla}_0\delta\boldsymbol{v} \times \boldsymbol{\nabla}_0\boldsymbol{u}} \tag{34}$$

Similarly, the derivatives of the volume ratio J are easily expressed with the help of equation (12), the geometric compatibility condition (26c) and the identities (25c) and (25b) as

$$\boxed{DJ\left[\delta\boldsymbol{v}\right] = \boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}} \tag{35}$$

$$\boxed{D^2J\left[\delta\boldsymbol{v};\boldsymbol{u}\right] = \boldsymbol{F}_{\boldsymbol{x}} : \left(\boldsymbol{\nabla}_0\delta\boldsymbol{v} \times \boldsymbol{\nabla}_0\boldsymbol{u}\right)} \tag{36}$$

The above formulas simplify greatly the manipulation of the derivatives of $\boldsymbol{H}$ and J by avoiding differentiating the inverse of the deformation gradient. They will be key to the development of the framework presented below.

Alternatively, the classical approach to compute the first directional derivative of $\boldsymbol{H}$ and J [4] is:

$$D\boldsymbol{H}\left[\delta\boldsymbol{v}\right] = J_{\boldsymbol{x}}G_{\delta\boldsymbol{v}}\boldsymbol{F}_{\boldsymbol{x}}{}^{-T} - J_{\boldsymbol{x}}\boldsymbol{G}_{\delta\boldsymbol{v}}\boldsymbol{F}_{\boldsymbol{x}}{}^{-T}; \tag{37a}$$

$$DJ\left[\delta\boldsymbol{v}\right] = J_{\boldsymbol{x}}G_{\delta\boldsymbol{v}}, \tag{37b}$$

where

$$G_{\delta\boldsymbol{v}} \equiv \boldsymbol{F}_{\boldsymbol{x}}{}^{-T} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; \qquad \boldsymbol{G}_{\delta\boldsymbol{v}} \equiv \boldsymbol{F}_{\boldsymbol{x}}{}^{-T}\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\right)^T. \tag{38}$$

The second directional derivatives of $\boldsymbol{H}$ and J [4] are:

$$\begin{aligned} D^2\boldsymbol{H}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] = {} & J_{\boldsymbol{x}}\boldsymbol{G}_{\delta v}\boldsymbol{G}_{\boldsymbol{u}}\boldsymbol{F}_{\boldsymbol{x}}{}^{-T} + J_{\boldsymbol{x}}\boldsymbol{G}_{\boldsymbol{u}}\boldsymbol{G}_{\delta v}\boldsymbol{F}_{\boldsymbol{x}}^{-T} \\ & - J_{\boldsymbol{x}}G_{\delta\boldsymbol{v}}\boldsymbol{G}_{\boldsymbol{u}}\boldsymbol{F}_{\boldsymbol{x}}^{-T} - J_{\boldsymbol{x}}G_{\boldsymbol{u}}\boldsymbol{G}_{\delta\boldsymbol{v}}\boldsymbol{F}_{\boldsymbol{x}}^{-T} \\ & + J_{\boldsymbol{x}}G_{\delta\boldsymbol{v}}G_{\boldsymbol{u}}\boldsymbol{F}_{\boldsymbol{x}}^{-T} - J_{\boldsymbol{x}}\mathrm{tr}\left(\boldsymbol{G}_{\delta v}\boldsymbol{G}_{\boldsymbol{u}}\right)\boldsymbol{F}_{\boldsymbol{x}}^{-T}; \end{aligned} \tag{39a}$$

$$D^2J\left[\delta\boldsymbol{v};\boldsymbol{u}\right] = J_{\boldsymbol{x}}G_{\delta\boldsymbol{v}}G_{\boldsymbol{u}} - J_{\boldsymbol{x}}\mathrm{tr}\left(\boldsymbol{G}_{\delta v}\boldsymbol{G}_{\boldsymbol{u}}\right), \tag{39b}$$

where

$$G_{\boldsymbol{u}} = \boldsymbol{F}_{\boldsymbol{x}}^{-T} : \boldsymbol{\nabla}_0\boldsymbol{u}; \qquad \boldsymbol{G}_{\boldsymbol{u}} = \boldsymbol{F}_{\boldsymbol{x}}^{-T}\left(\boldsymbol{\nabla}_0\boldsymbol{u}\right)^T. \tag{40}$$

Comparison of (33) vs. (37a), (34) vs. (39a), (35) vs. (37b) and (39b) demonstrates very clearly the simplification introduced as a result of using the new tensor cross product algebra.

3. Polyconvex elasticity

3.1. The strain energy

Polyconvexity is now well accepted as a useful mathematical requirement that must be satisfied by admissible strain energy functions used to describe elastic materials in the large strain regime. Essentially, the strain energy Ψ per unit undeformed volume must be a function of the deformation gradient $\boldsymbol{F}_{\boldsymbol{x}}$ (25a) via a convex multi-variable function W as:

$$\Psi\left(\boldsymbol{F}_{\boldsymbol{x}}\right) = W\left(\boldsymbol{F}, \boldsymbol{H}, J\right), \tag{41}$$

where W is convex with respect to its 19 independent variables, namely, J and the 3×3 components of $\boldsymbol{F}$ and $\boldsymbol{H}$, which are related to $\boldsymbol{F}_{\boldsymbol{x}}$ via the geometric compatibility equations (see equations (25)-(26)). Moreover, invariance with respect to rotations in the material configuration implies that W must be independent of the rotational components of $\boldsymbol{F}$ and $\boldsymbol{H}$. This is typically achieved by ensuring that W depends on $\boldsymbol{F}$ and $\boldsymbol{H}$ via the symmetric tensors $\boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{H}^T\boldsymbol{H}$, respectively. For example, a general compressible Mooney-Rivlin material can be described by an energy function of the type:

$$W_{MR}(\boldsymbol{F}, \boldsymbol{H}, J) = \alpha \boldsymbol{F} : \boldsymbol{F} + \beta \boldsymbol{H} : \boldsymbol{H} + f\left(J\right), \tag{42}$$

where α and β are positive material parameters and f denotes a convex function of J. It is clear therefore that W_{MR} is convex with respect to all of its variables. The condition of vanishing energy at the initial reference configuration can be established by ensuring that $f\left(1\right) = -\left(3\alpha + 3\beta\right)$ or by adding an appropriate constant to W_{MR}. Doing this, however, has no practical effect on the resulting formulation as this will be driven by derivatives of the strain energy. Appropriate values for α and β and suitable functions f will be found in the sections below.

In the classical manner, the strain energy function W_{MR} (42) can also be re-written in terms of the invariants $\{I_1, I_2, I_3\}$ of the right Cauchy-Green deformation tensor $\boldsymbol{C}_{\boldsymbol{x}} := \boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{F}_{\boldsymbol{x}}$ as

$$\widetilde{\Psi}_{MR}\left(\boldsymbol{C}_{\boldsymbol{x}}\right) = \alpha I_1 + \beta I_2 + g\left(I_3\right), \tag{43}$$

with

$$I_1 := \operatorname{tr} \boldsymbol{C}_{\boldsymbol{x}}; \qquad I_2 := \left(\det \boldsymbol{C}_{\boldsymbol{x}}\right)\left(\operatorname{tr}\boldsymbol{C}_{\boldsymbol{x}}{}^{-1}\right); \qquad I_3 := \det \boldsymbol{C}_{\boldsymbol{x}}, \tag{44}$$

where g is not necessarily a convex function of $\det\left(\boldsymbol{C}_{\boldsymbol{x}}\right)$, unlike its counterpart $f(J)$, convex in J.

3.2. Conjugate stresses and the first Piola-Kirchhoff tensor

The three 'independent' (see *Remark 2*) strain measures $\boldsymbol{F}$, $\boldsymbol{H}$ and J have conjugate stresses $\boldsymbol{\Sigma_F}$, $\boldsymbol{\Sigma_H}$ and Σ_J defined by:

$$\boldsymbol{\Sigma_F}(\boldsymbol{F}, \boldsymbol{H}, J) := \frac{\partial W}{\partial \boldsymbol{F}}; \tag{45a}$$

$$\boldsymbol{\Sigma_H}(\boldsymbol{F}, \boldsymbol{H}, J) := \frac{\partial W}{\partial \boldsymbol{H}}; \tag{45b}$$

$$\Sigma_J(\boldsymbol{F}, \boldsymbol{H}, J) := \frac{\partial W}{\partial J}. \tag{45c}$$

For instance, for the case of a Mooney-Rivlin material (42)

$$\boldsymbol{\Sigma_F} = 2\alpha \boldsymbol{F}; \quad \boldsymbol{\Sigma_H} = 2\beta \boldsymbol{H}; \quad \Sigma_J = f'(J). \tag{46}$$

The set of conjugate stresses defined in (45) enables the directional derivative of the stain energy to be expressed as

$$DW[\delta\boldsymbol{F}, \delta\boldsymbol{H}, \delta J] = \boldsymbol{\Sigma_F} : \delta\boldsymbol{F} + \boldsymbol{\Sigma_H} : \delta\boldsymbol{H} + \Sigma_J \delta J. \tag{47}$$

In order to develop a relationship between these conjugate stresses and the more standard first Piola-Kirchhoff stress tensor $\boldsymbol{P}$, recall that:

$$D\Psi[\delta\boldsymbol{v}] = \boldsymbol{P} : \boldsymbol{\nabla}_0 \delta\boldsymbol{v}; \quad \boldsymbol{P} = \frac{\partial \Psi(\boldsymbol{F_x})}{\partial \boldsymbol{F_x}}. \tag{48}$$

With the help of equations (45) and (47), the chain rule and equations (33) and (35) it is possible to express the virtual internal work as

$$\begin{aligned} \boldsymbol{P} : \boldsymbol{\nabla}_0 \delta\boldsymbol{v} &= D\Psi[\delta\boldsymbol{v}] \\ &= DW[D\boldsymbol{F}[\delta\boldsymbol{v}], D\boldsymbol{H}[\delta\boldsymbol{v}], DJ[\delta\boldsymbol{v}]] \\ &= \boldsymbol{\Sigma_F} : D\boldsymbol{F}[\delta\boldsymbol{v}] + \boldsymbol{\Sigma_H} : D\boldsymbol{H}[\delta\boldsymbol{v}] + \Sigma_J DJ[\delta\boldsymbol{v}] \\ &= \boldsymbol{\Sigma_F} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} + \boldsymbol{\Sigma_H} : (\boldsymbol{F_x} \times \boldsymbol{\nabla}_0\delta\boldsymbol{v}) + \Sigma_J(\boldsymbol{H_x} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}) \\ &= (\boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F_x} + \Sigma_J \boldsymbol{H_x}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v}, \end{aligned} \tag{49}$$

which leads to the evaluation of the first Piola-Kirchhoff tensor as

$$\boldsymbol{P} = \boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F_x} + \Sigma_J \boldsymbol{H_x}. \tag{50}$$

In the continuum context where the geometric compatibility conditions are exactly enforced, namely, $\boldsymbol{F} \equiv \boldsymbol{F}_{\boldsymbol{x}}$, $\boldsymbol{H} \equiv \boldsymbol{H}_{\boldsymbol{x}}$ and $J \equiv J_{\boldsymbol{x}}$ (see *Remark 2*), the above equation becomes

$$\boxed{\boldsymbol{P} = \boldsymbol{\Sigma}_{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H}} \tag{51}$$

By using equation (50) for the simple compressible Mooney-Rivlin material

$$\boldsymbol{P} = 2\alpha \boldsymbol{F} + 2\beta \boldsymbol{H} \times \boldsymbol{F}_{\boldsymbol{x}} + f'(J)\, \boldsymbol{H}_{\boldsymbol{x}}. \tag{52}$$

The condition of a stress-free initial configuration, where $\boldsymbol{F} = \boldsymbol{H} = \boldsymbol{I}$ and $J = 1$, together with property (14) of the tensor cross product, leads to the following constraint on the material parameters α, β and $f(J)$

$$f'(1) = -2\alpha - 4\beta. \tag{53}$$

Alternatively, for the strain energy in equation (43), the associated second Piola-Kirchhoff $\boldsymbol{S}$ can be obtained in the classical sense $\left(\boldsymbol{S} = 2\frac{\partial \widetilde{\Psi}(\boldsymbol{C}_{\boldsymbol{x}})}{\partial \boldsymbol{C}_{\boldsymbol{x}}}\right)$ [4] as

$$\boldsymbol{S} = 2\alpha \boldsymbol{I} + 2\beta I_3 \left[\left(\operatorname{tr} \boldsymbol{C}_{\boldsymbol{x}}^{-1}\right) \boldsymbol{C}_{\boldsymbol{x}}^{-1} - \boldsymbol{C}_{\boldsymbol{x}}^{-1}\boldsymbol{C}_{\boldsymbol{x}}^{-1}\right] + 2g'(I_3) I_3 \boldsymbol{C}_{\boldsymbol{x}}^{-1}. \tag{54}$$

Finally, the first Piola-Kirchhoff stress tensor can now be obtained via the classical push forward operation, $\boldsymbol{P} = \boldsymbol{F}_{\boldsymbol{x}} \boldsymbol{S}$

$$\boldsymbol{P} = 2\alpha \boldsymbol{F}_{\boldsymbol{x}} + 2\beta I_3 \left[\left(\operatorname{tr} \boldsymbol{C}_{\boldsymbol{x}}^{-1}\right) \boldsymbol{F}_{\boldsymbol{x}}^{-T} - \boldsymbol{F}_{\boldsymbol{x}}^{-T} \boldsymbol{C}_{\boldsymbol{x}}^{-1}\right] + 2g'(I_3) I_3 \boldsymbol{F}_{\boldsymbol{x}}^{-T} \tag{55}$$

which leads to a lengthier expression in comparison with (52).

3.3. Complementary energy

The convexity of the function $W(\boldsymbol{F}, \boldsymbol{H}, J)$ with respect to its variables ensures that the relationship between $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J\}$ is one to one and invertible. Using the reverse relationships, it is therefore possible to define a convex complementary energy function by means of a Legendre transform as

$$\Upsilon(\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J) = \boldsymbol{\Sigma}_{\boldsymbol{F}} : \boldsymbol{F} + \boldsymbol{\Sigma}_{\boldsymbol{H}} : \boldsymbol{H} + \Sigma_J J - W(\boldsymbol{F}, \boldsymbol{H}, J), \tag{56}$$

so that the reverse constitutive equations are derived as

$$\boldsymbol{F} = \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}}; \quad \boldsymbol{H} = \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}}; \quad J = \frac{\partial \Upsilon}{\partial \Sigma_J}. \tag{57}$$

Note that in contrast to the geometric compatibility conditions (26) for $\boldsymbol{F}$, $\boldsymbol{H}$ and J, above equations (57) represent the constitutive equations for these deformation terms. For instance, in the particular case of a Mooney-Rivlin material

$$\Upsilon_{MR}\left(\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J\right) = \frac{1}{4\alpha}\boldsymbol{\Sigma_F} : \boldsymbol{\Sigma_F} + \frac{1}{4\beta}\boldsymbol{\Sigma_H} : \boldsymbol{\Sigma_H} + g\left(\Sigma_J\right), \tag{58}$$

where the complementary function g is defined by the Legendre transform

$$g\left(\Sigma_J\right) = \Sigma_J J\left(\Sigma_J\right) - f\left(J\left(\Sigma_J\right)\right) \tag{59}$$

and the relationship $J\left(\Sigma_J\right)$ is obtained inverting equation (46c), that is, $J\left(f'\left(x\right)\right) = x$. Note that if either α or β is zero, the corresponding term in the complementary energy also vanishes. For instance, the case $\beta = 0$ corresponds to a compressible neo-Hookean material, for which

$$\Upsilon_{NH}\left(\boldsymbol{\Sigma_F}, J\right) = \frac{1}{4\alpha}\boldsymbol{\Sigma_F} : \boldsymbol{\Sigma_F} + g\left(\Sigma_J\right); \quad \boldsymbol{\Sigma_H} = \boldsymbol{0}. \tag{60}$$

Remark 3: The complementary energy defined above does not coincide with the more traditional definition of complementary energy $\Psi^*(\boldsymbol{P}) = \boldsymbol{P} : \boldsymbol{F_x} - \Psi(\boldsymbol{F_x})$. It is in fact easy to show that

$$\begin{aligned}
\boldsymbol{P} : \boldsymbol{F_x} &= \left(\boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F_x} + \Sigma_J \boldsymbol{H_x}\right) : \boldsymbol{F_x} \\
&= \boldsymbol{\Sigma_F} : \boldsymbol{F_x} + \boldsymbol{\Sigma_H} : \left(\boldsymbol{F_x} \times \boldsymbol{F_x}\right) + \Sigma_J \boldsymbol{H_x} : \boldsymbol{F_x} \\
&= \boldsymbol{\Sigma_F} : \boldsymbol{F_x} + 2\boldsymbol{\Sigma_H} : \boldsymbol{H_x} + 3\Sigma_J J_{\boldsymbol{x}} \\
&\neq \boldsymbol{\Sigma_F} : \boldsymbol{F} + \boldsymbol{\Sigma_H} : \boldsymbol{H} + \Sigma_J J
\end{aligned} \tag{61}$$

and therefore $\Upsilon(\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J) \neq \Psi^*(\boldsymbol{P})$ even in the continuum context when $\boldsymbol{F} \equiv \boldsymbol{F_x}$, $\boldsymbol{H} \equiv \boldsymbol{H_x}$ and $J \equiv J_{\boldsymbol{x}}$. Note that only in the exceptional cases where the relation $\boldsymbol{P}(\boldsymbol{F_x})$ is invertible, it is possible to carry out the Legendre transform in order to obtain $\Psi^*(\boldsymbol{P})$ (see [35], [36]).

Remark 4: In the case of thermoelasticity, the strain energy is also a convex function of the entropy η, and the temperature θ is given by

$$\theta = \frac{\partial W(\boldsymbol{F}, \boldsymbol{H}, J, \eta)}{\partial \eta} \tag{62}$$

and the complementary energy function which will now depend on the temperature can be interpreted as a generalised Gibbs energy function defined as

$$\Upsilon\left(\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J, \theta\right) = \boldsymbol{\Sigma_F} : \boldsymbol{F} + \boldsymbol{\Sigma_H} : \boldsymbol{H} + \Sigma_J J + \eta\theta - W\left(\boldsymbol{F}, \boldsymbol{H}, J, \eta\right). \tag{63}$$

3.4. Stress based compatibility conditions and equilibrium

In linear elasticity it is well known that the stress tensor field must satisfy a set of differential compatibility conditions usually known as Beltrami-Mitchell equations [37]. These conditions ensure that the stress tensor can be derived from a displacement field. In the large strain case, it is also possible to derive a set of relationships that the above conjugate stresses have to satisfy in order to ensure that they correspond to an actual deformation process, that there exist a mapping $\boldsymbol{x} = \boldsymbol{\phi}(\boldsymbol{X})$ such that

$$\frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}} = \boldsymbol{F_x}; \quad \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}} = \boldsymbol{H_x}; \quad \frac{\partial \Upsilon}{\partial \Sigma_J} = J_{\boldsymbol{x}}. \tag{64}$$

These conditions can be enforced weakly in the context of a mixed computational formulation using appropriate variational principles as described below in Section 6. Alternatively, an equivalent set of constraints for the conjugate stresses can be derived as

$$\begin{aligned} \text{CURL}\left(\frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}}\right) &= \boldsymbol{0}; \\ 2\frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}} - \left(\frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}} \boldsymbol{\times} \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}}\right) &= \boldsymbol{0}; \\ 3\frac{\partial \Upsilon}{\partial \Sigma_J} - \left(\frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}} : \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}}\right) &= 0. \end{aligned} \tag{65}$$

These constraints together with the equilibrium equations provide a full set of equations for the augmented set of stresses $\boldsymbol{\Sigma_F}$, $\boldsymbol{\Sigma_H}$, Σ_J. Equilibrium can be enforced in the conventional manner in a Lagrangian setting by means of the divergence of the first Piola-Kirchhoff tensor as [4]

$$\boldsymbol{f}_0 + \mathrm{DIV}\,\boldsymbol{P} = \boldsymbol{0}, \tag{66}$$

Simple algebra using equation (51) and the vector cross product defined by equation (7) gives an expression in terms of the conjugate stresses as

$$\boldsymbol{f}_0 + \mathrm{DIV}\,\boldsymbol{\Sigma_F} + (\mathrm{CURL}\,\boldsymbol{\Sigma_H}) \times \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}} + \boldsymbol{\nabla}_0 \Sigma_J : \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}} = \boldsymbol{0}. \tag{67}$$

This differential equilibrium equation must be complemented by appropriate boundary conditions. Traction boundary conditions on $\partial_t V$ imply

$$\boldsymbol{PN} = \boldsymbol{\Sigma_F N} + \left(\boldsymbol{\Sigma_H} \boldsymbol{\times} \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_F}}\right)\boldsymbol{N} + \Sigma_J \frac{\partial \Upsilon}{\partial \boldsymbol{\Sigma_H}} \boldsymbol{N} = \boldsymbol{t}_0. \tag{68}$$

3.5. Tangent elasticity operator

A tangent elasticity operator will be required in order to ensure quadratic convergence of a Newton-Raphson type of solution process. This is typically evaluated in terms of a fourth order tangent elasticity tensor defined by

$$D^2\Psi\,[\delta\boldsymbol{v};\boldsymbol{u}] = \boldsymbol{\nabla}_0\delta\boldsymbol{v} : D\boldsymbol{P}\,[\boldsymbol{u}] = \boldsymbol{\nabla}_0\delta\boldsymbol{v} : \boldsymbol{\mathcal{C}} : \boldsymbol{\nabla}_0\boldsymbol{u}; \quad \boldsymbol{\mathcal{C}} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F_x}} = \frac{\partial^2 \Psi}{\partial \boldsymbol{F_x} \partial \boldsymbol{F_x}}. \tag{69}$$

Use of equation (50), following a chain rule derivation similar to that of equation (49) and making use of equations (33) and (34) for the derivatives of $\boldsymbol{H}$, yields after simple algebra

$$\begin{aligned} D^2\Psi\,[\delta\boldsymbol{v};\boldsymbol{u}] &= \boldsymbol{\nabla}_0\delta\boldsymbol{v} : D\boldsymbol{P}\,[\boldsymbol{u}] \\ &= \boldsymbol{\nabla}_0\delta\boldsymbol{v} : D\boldsymbol{\Sigma_F}\,[\boldsymbol{u}] + (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \boldsymbol{\times} \boldsymbol{F_x}) : D\boldsymbol{\Sigma_H}\,[\boldsymbol{u}] + (\boldsymbol{\nabla}_0\delta\boldsymbol{v} : \boldsymbol{H_x})\, D\Sigma_J\,[\boldsymbol{u}] \\ &\quad + (\boldsymbol{\Sigma_H} + \Sigma_J \boldsymbol{F_x}) : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \boldsymbol{\times} \boldsymbol{\nabla}_0\boldsymbol{u})\,. \end{aligned} \tag{70}$$

In general, conjugate stresses $\{\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J\}$ will be functions of the strain variables $\{\boldsymbol{F}, \boldsymbol{H}, J\}$ and the resulting tangent operator can be written as

$$\boxed{\begin{aligned} D^2\Psi\,[\delta\boldsymbol{v};\boldsymbol{u}] &= \left[(\boldsymbol{\nabla}_0\delta\boldsymbol{v}) : \quad (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \boldsymbol{\times} \boldsymbol{F_x}) : \quad (\boldsymbol{\nabla}_0\delta\boldsymbol{v} : \boldsymbol{H_x})\right] [\mathbb{H}_W] \begin{bmatrix} : (\boldsymbol{\nabla}_0\boldsymbol{u}) \\ : (\boldsymbol{\nabla}_0\boldsymbol{u} \boldsymbol{\times} \boldsymbol{F_x}) \\ (\boldsymbol{\nabla}_0\boldsymbol{u} : \boldsymbol{H_x}) \end{bmatrix} \\ &\quad + (\boldsymbol{\Sigma_H} + \Sigma_J \boldsymbol{F_x}) : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \boldsymbol{\times} \boldsymbol{\nabla}_0\boldsymbol{u}) \end{aligned}} \tag{71}$$

where the Hessian operator $\mathbb{H}_W$ denotes the symmetric positive definite operator containing the second derivatives of $W(\boldsymbol{F}, \boldsymbol{H}, J)$

$$[\mathbb{H}_W] = \begin{bmatrix} \frac{\partial^2 W}{\partial \boldsymbol{F} \partial \boldsymbol{F}} & \frac{\partial^2 W}{\partial \boldsymbol{F} \partial \boldsymbol{H}} & \frac{\partial^2 W}{\partial \boldsymbol{F} \partial J} \\ \frac{\partial^2 W}{\partial \boldsymbol{H} \partial \boldsymbol{F}} & \frac{\partial^2 W}{\partial \boldsymbol{H} \partial \boldsymbol{H}} & \frac{\partial^2 W}{\partial \boldsymbol{H} \partial J} \\ \frac{\partial^2 W}{\partial J \partial \boldsymbol{F}} & \frac{\partial^2 W}{\partial J \partial \boldsymbol{H}} & \frac{\partial^2 W}{\partial J \partial J} \end{bmatrix}. \tag{72}$$

Once again, in the context of strong enforcement of the geometric compatibility conditions (26), the terms $\boldsymbol{F}_{\boldsymbol{x}}$ and $\boldsymbol{H}_{\boldsymbol{x}}$ in above equation (71) can be replaced by $\boldsymbol{F}$ and $\boldsymbol{H}$, respectively.

Note that the first term in equation (71) is necessarily positive for $\delta \boldsymbol{v} = \boldsymbol{u}$ and therefore buckling can only be induced by the "initial stress" term $(\boldsymbol{\Sigma}_{\boldsymbol{H}} + \Sigma_J \boldsymbol{F}_{\boldsymbol{x}}) : (\boldsymbol{\nabla}_0 \delta \boldsymbol{v} \times \boldsymbol{\nabla}_0 \boldsymbol{u})$. In effect, the above expression for the elasticity tensor separates the material dependencies or physics of the problem (encapsulated in the Hessian tensor) from the geometry dependencies included via the initial stress term.

Remark 5: Equation (71) makes it easy to highlight the relationship between policonvexity and ellipticity. Ellipticity is equivalent to rank-one convexity and requires that the double contraction of the elasticity tensor by an arbitrary rank-one tensor $\boldsymbol{v} \otimes \boldsymbol{V}$ should be positive, that is,

$$(\boldsymbol{v} \otimes \boldsymbol{V}) : \boldsymbol{\mathcal{C}} : (\boldsymbol{v} \otimes \boldsymbol{V}) > 0. \tag{73}$$

Taking $\boldsymbol{\nabla}_0 \delta \boldsymbol{v} = \boldsymbol{\nabla}_0 \boldsymbol{u} = \boldsymbol{v} \otimes \boldsymbol{V}$ in equation (71) makes the initial stress term vanish since,

$$\boldsymbol{\nabla}_0 \delta \boldsymbol{v} \times \boldsymbol{\nabla}_0 \boldsymbol{u} = (\boldsymbol{v} \otimes \boldsymbol{V}) \times (\boldsymbol{v} \otimes \boldsymbol{V}) = (\boldsymbol{v} \times \boldsymbol{v}) \otimes (\boldsymbol{V} \times \boldsymbol{V}) = \boldsymbol{0}. \tag{74}$$

This leaves only the contribution from the first positive definite term in equation (71). It is therefore easy to note that polyconvexity implies ellipticity [21].

It is helpful to consider the simple case of a compressible Mooney-Rivlin material for which the off-diagonal terms of the Hessian operator vanish and the tangent elastic operator becomes

$$\begin{aligned} D^2\Psi_{MR}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= 2\alpha\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{\nabla}_0\boldsymbol{u} + 2\beta\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{F}_{\boldsymbol{x}}\right):\left(\boldsymbol{\nabla}_0\boldsymbol{u}\times\boldsymbol{F}_{\boldsymbol{x}}\right) \\ &+ f''\left(J_{\boldsymbol{x}}\right)\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{H}_{\boldsymbol{x}}\right)\left(\boldsymbol{\nabla}_0\boldsymbol{u}:\boldsymbol{H}_{\boldsymbol{x}}\right) + \left(\boldsymbol{\Sigma}_{\boldsymbol{H}} + \Sigma_J\boldsymbol{F}_{\boldsymbol{x}}\right):\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{\nabla}_0\boldsymbol{u}\right). \end{aligned} \tag{75}$$

It is now possible to derive appropriate values for the material parameters α, β and the function $f\left(J_{\boldsymbol{x}}\right)$ by ensuring that at the reference configuration the above operator coincides with the classic linear elasticity operator, which is typically expressed in terms of the Lamé coefficients $\{\lambda,\mu\}$ as

$$D^2\Psi_{LIN}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] = \lambda\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{I}\right)\left(\boldsymbol{\nabla}_0\boldsymbol{u}:\boldsymbol{I}\right)+\mu\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{\nabla}_0\boldsymbol{u}+\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\right)^T:\boldsymbol{\nabla}_0\boldsymbol{u}\right). \tag{76}$$

Substituting $\boldsymbol{F}_{\boldsymbol{x}} = \boldsymbol{H}_{\boldsymbol{x}} = \boldsymbol{I}$; $J_{\boldsymbol{x}} = 1$ into equation (75), making repeated use of property (13) for the tensor cross product and taking into account the zero initial stress condition (53), gives after lengthy but simple algebra

$$\begin{aligned} D^2\Psi_{MR}\left[\delta\boldsymbol{v};\boldsymbol{u}\right]\big|_{\boldsymbol{I}} &= (2\alpha+2\beta)\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{\nabla}_0\boldsymbol{u}+\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\right)^T:\boldsymbol{\nabla}_0\boldsymbol{u}\right) \\ &+ \left(f''(1)-2\alpha\right)\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{I}\right)\left(\boldsymbol{\nabla}_0\boldsymbol{u}:\boldsymbol{I}\right). \end{aligned} \tag{77}$$

Identifying coefficients leads to the condition relating α, β to μ

$$\alpha+\beta = \frac{\mu}{2} \tag{78}$$

and the condition for the second derivative of f at the origin

$$f''(1) = \lambda + 2\alpha. \tag{79}$$

A commonly used expression for f that satisfies these requirements is

$$f(J) = -4\beta J - 2\alpha\ln J + \frac{\lambda}{2\varepsilon^2}\left(J^{\varepsilon}+J^{-\varepsilon}\right); \quad \varepsilon\geq 1. \tag{80}$$

3.6. A modified Mooney-Rivlin material model

It is interesting to observe that the strain energy expressed in terms of the full set of kinematic variables $\boldsymbol{F}$, $\boldsymbol{H}$ and J is not a unique function. That is, the same physical strain energy $\Psi(\boldsymbol{F}_{\boldsymbol{x}})$ can be expressed by a set of different functions $W(\boldsymbol{F}, \boldsymbol{H}, J)$. For instance, the addition of multiples of the function $\boldsymbol{F} : \boldsymbol{H} - 3J$, which vanishes for geometrically compatible variables, has no effect on the actual physical strain energy described and therefore

$$\Psi(\boldsymbol{F_x}) = W(\boldsymbol{F}, \boldsymbol{H}, J) + \xi(\boldsymbol{F} : \boldsymbol{H} - 3J) = W_\xi(\boldsymbol{F}, \boldsymbol{H}, J), \tag{81}$$

where ξ can be an arbitrary constant provided that the resulting function W_ξ is still convex in its variables. For instance, in the case of Mooney-Rivlin materials (see (42)), it is easy to show that convexity is still maintained for values of ξ such that $\alpha\beta \geq \frac{\xi^2}{4}$. Is is easy to show that the addition of the above term does no alter the first Piola Kirchhoff stress tensor but leads to modified conjugate stresses as

$$\boldsymbol{\Sigma}^\xi_{\boldsymbol{F}} = \boldsymbol{\Sigma}_{\boldsymbol{F}} + \xi\boldsymbol{H}; \quad \boldsymbol{\Sigma}^\xi_{\boldsymbol{H}} = \boldsymbol{\Sigma}_{\boldsymbol{H}} + \xi\boldsymbol{F}; \quad \Sigma^\xi_J = \Sigma_J - 3\xi. \tag{82}$$

It is now possible to adjust the value of ξ so that some or all the conjugate stresses at the initial configuration vanish. As an interesting example, consider the case of a Mooney-Rivlin material for which $\alpha = \beta = \mu/4$. Choosing $\xi = -\mu/2$ leads to the following polyconvex strain energy function

$$W^\xi_{MR}(\boldsymbol{F}, \boldsymbol{H}, J) = \frac{\mu}{4}(\boldsymbol{F}-\boldsymbol{H}) : (\boldsymbol{F}-\boldsymbol{H}) + \frac{\mu}{2}(J - \ln J) + \frac{\lambda}{2\epsilon^2}\left(J^\epsilon + J^{-\epsilon}\right). \tag{83}$$

It is easy to show that all the conjugate stresses in this model vanish at the initial configuration. In addition, the term $(\boldsymbol{F} - \boldsymbol{H})$ has a clear physical interpretation as distortion given that when applied to a reference vector, it measures the difference between the mapped fibre and area vectors.

3.7. Nearly incompressible Mooney-Rivlin material

Very often it is convenient or even necessary to separate the distortional component from the volumetric response of the material. This is invariably the case when attempting to model either nearly-incompressible or truly incompressible solids. Typically, this is achieved by separating the strain energy into isochoric and volumetric components [38], $\hat{\Psi}$ and U, respectively, as

$$\Psi\left(\boldsymbol{F_x}\right) = \hat{\Psi}\left(\boldsymbol{F_x}\right) + U\left(J_{\boldsymbol{x}}\right); \quad \hat{\Psi}\left(\boldsymbol{F_x}\right) = \Psi\left(J_{\boldsymbol{x}}^{-1/3}\boldsymbol{F_x}\right). \tag{84}$$

The first term in this energy expression leads to the deviatoric component of the Piola-Kirchhoff tensor and the derivative of the function U accounts for the pressure p. In the context of polyconvex elasticity, it is also possible to construct a similar decomposition in the form

$$W(\boldsymbol{F}, \boldsymbol{H}, J) = \hat{W}(\boldsymbol{F}, \boldsymbol{H}, J) + U(J). \tag{85}$$

For the purpose of deriving the conditions that need to be satisfied in order to ensure that the U-term alone accounts for the pressure p, it is necessary to restrict the derivation to the exact continuum context where geometrically compatibility conditions are enforced exactly. Recall first that the pressure itself is obtained from the first Piola-kirchhoff tensor via the contraction

$$p = \frac{1}{3}J^{-1}\boldsymbol{P} : \boldsymbol{F}. \tag{86}$$

Note that the sign convention used above is positive pressure in tension, negative in compression. Substituting the relationship between the Piola-Kirchhoff stress tensor and the conjugate stresses given by equation (51), yields a relationship between the pressure and the conjugate stresses as

$$\begin{aligned} p &= \frac{1}{3}J^{-1}\left(\boldsymbol{\Sigma}_{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} \times \boldsymbol{F} + \Sigma_J \boldsymbol{H}\right) : \boldsymbol{F} \\ &= \frac{1}{3}J^{-1}\left(\boldsymbol{\Sigma}_{\boldsymbol{F}} : \boldsymbol{F} + 2\boldsymbol{\Sigma}_{\boldsymbol{H}} : \boldsymbol{H} + 3\Sigma_J J\right), \end{aligned} \tag{87}$$

where property (15) has been made use of. Substituting the constitutive relationships (45) and decomposition (85) into this equation for the pressure gives

$$p = \frac{1}{3}J^{-1}\left(\frac{\partial \hat{W}}{\partial \boldsymbol{F}} : \boldsymbol{F} + 2\frac{\partial \hat{W}}{\partial \boldsymbol{H}} : \boldsymbol{H} + 3J\frac{\partial \hat{W}}{\partial J}\right) + U'(J). \tag{88}$$

Therefore the condition that $\hat{W}$ needs to satisfy in order to ensure a correct decomposition into volumetric and deviatoric components is

$$\frac{\partial \hat{W}}{\partial \boldsymbol{F}} : \boldsymbol{F} + 2\frac{\partial \hat{W}}{\partial \boldsymbol{H}} : \boldsymbol{H} + 3J\frac{\partial \hat{W}}{\partial J} = 0. \tag{89}$$

In order to fulfil this requirement, it is sufficient for $\hat{W}$ to satisfy the following mixed homogeneous condition (refer to [4], page 168)

$$\hat{W}\left(\alpha\boldsymbol{F}, \alpha^2\boldsymbol{H}, \alpha^3 J\right) = \hat{W}(\boldsymbol{F}, \boldsymbol{H}, J). \tag{90}$$

Differentiating this equation with respect to α at $\alpha = 1$, quickly leads to condition (89). A simple way to ensure that this requirement is satisfied would be to construct $\hat{W}$ in terms of the isochoric components of $\boldsymbol{F}$ and $\boldsymbol{H}$

$$\hat{W}(\boldsymbol{F}, \boldsymbol{H}) = W\left(\hat{\boldsymbol{F}}, \hat{\boldsymbol{H}}, 1\right), \tag{91}$$

where the isochoric components could be defined in the usual fashion [4]

$$\hat{\boldsymbol{F}} = (\det \boldsymbol{F})^{-1/3} \boldsymbol{F}; \quad \hat{\boldsymbol{H}} = (\det \boldsymbol{H})^{-1/3} \boldsymbol{H}. \tag{92}$$

Unfortunately, the resulting strain energy function constructed in this manner will not be convex with respect to $\boldsymbol{F}$ and $\boldsymbol{H}$. Alternative expressions can be derived by re-defining the isochoric components of $\boldsymbol{F}$ and $\boldsymbol{H}$ as

$$\hat{\boldsymbol{F}} = J^{-1/3} \boldsymbol{F}; \quad \hat{\boldsymbol{H}} = J^{-2/3} \boldsymbol{H}. \tag{93}$$

Or, alternatively, noting that $\boldsymbol{F} : \boldsymbol{H} = 3J$ (refer to property (15))

$$\hat{\boldsymbol{F}} = \left(\frac{1}{3}\boldsymbol{F} : \boldsymbol{H}\right)^{-1/3} \boldsymbol{F}; \quad \hat{\boldsymbol{H}} = \left(\frac{1}{3}\boldsymbol{F} : \boldsymbol{H}\right)^{-2/3} \boldsymbol{H}. \tag{94}$$

For instance, in the case of the Mooney-Rivlin material, an equivalent polyconvex isochoric energy function is obtained as [32]

$$\hat{W}(\boldsymbol{F}, \boldsymbol{H}, J) = \eta J^{-2/3} (\boldsymbol{F} : \boldsymbol{F}) + \gamma J^{-2} (\boldsymbol{H} : \boldsymbol{H})^{3/2} \tag{95}$$

where η and γ are two positive material parameters of a similar nature to parameters α and β appearing in equation (42). The most commonly used expression for the volumetric strain energy component $U(J)$ is given by

$$U(J) = \frac{1}{2}\kappa (J-1)^2. \tag{96}$$

Note that the dependency of the isochoric strain energy function $\hat{W}$ with respect to J implies that the pressure p and the conjugate stress Σ_J are not identical. They are in fact related by

$$\Sigma_J = \hat{\Sigma}_J + p; \quad \hat{\Sigma}_J = \frac{\partial \hat{W}}{\partial J}; \quad p = U'(J). \tag{97}$$

Finally, the tangent elastic operator of this nearly incompressible model can be derived in a manner similar to equation (71) to give

$$\boxed{\begin{aligned} D^2\Psi\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= D^2\hat{\Psi}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] + U''\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{H}_{\boldsymbol{x}}\right)\left(\boldsymbol{\nabla}_0\boldsymbol{u}:\boldsymbol{H}_{\boldsymbol{x}}\right) \\ D^2\hat{\Psi}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= \left[(\boldsymbol{\nabla}_0\delta\boldsymbol{v}): \quad (\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{F}_{\boldsymbol{x}}): \quad (\boldsymbol{\nabla}_0\delta\boldsymbol{v}:\boldsymbol{H}_{\boldsymbol{x}})\right]\left[\mathbb{H}_{\hat{W}}\right]\begin{bmatrix} :(\boldsymbol{\nabla}_0\boldsymbol{u}) \\ :(\boldsymbol{\nabla}_0\boldsymbol{u}\times\boldsymbol{F}_{\boldsymbol{x}}) \\ (\boldsymbol{\nabla}_0\boldsymbol{u}:\boldsymbol{H}_{\boldsymbol{x}}) \end{bmatrix} \\ &\quad + \left(\boldsymbol{\Sigma}_{\boldsymbol{H}} + \Sigma_J\boldsymbol{F}_{\boldsymbol{x}}\right):\left(\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{\nabla}_0\boldsymbol{u}\right) \end{aligned}} \tag{98}$$

4. Material and Spatial Descriptions

4.1. The Second Piola-Kirchhoff tensor

The formulation developed so far has been expressed in terms of the main kinematic variables $\boldsymbol{F}$, $\boldsymbol{H}$ and J. However, material frame indifference implies that the dependency of the strain energy with respect to $\boldsymbol{F}$, $\boldsymbol{H}$ must be via the right Cauchy-Green tensor $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$ and its co-factor $\boldsymbol{G} = \boldsymbol{H}^T\boldsymbol{H}$. Similarly to the definition of $\{\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\}$ (25), it is possible to define analogous strain measures

$$\boldsymbol{C}_{\boldsymbol{x}} := \left(\boldsymbol{\nabla}_0\boldsymbol{x}\right)^T\left(\boldsymbol{\nabla}_0\boldsymbol{x}\right); \tag{99a}$$

$$\boldsymbol{G}_{\boldsymbol{x}} := \frac{1}{2}\boldsymbol{C}_{\boldsymbol{x}}\times\boldsymbol{C}_{\boldsymbol{x}}; \tag{99b}$$

$$C_{\boldsymbol{x}} := \frac{1}{3}\boldsymbol{G}_{\boldsymbol{x}}:\boldsymbol{C}_{\boldsymbol{x}} = \det\boldsymbol{C}_{\boldsymbol{x}}, \tag{99c}$$

where a similar set of geometric compatibility conditions to (26) would be

$$\boldsymbol{C} = \boldsymbol{C}_{\boldsymbol{x}}; \tag{100a}$$

$$\boldsymbol{G} = \boldsymbol{G}_{\boldsymbol{x}}; \tag{100b}$$

$$C = C_{\boldsymbol{x}}, \tag{100c}$$

where, for consistency, $C = J^2$ is being used instead of J as the variable describing the volumetric change. Re-expressing first the strain energy Ψ in terms of the right Cauchy-Green tensor, $\Psi(\boldsymbol{F}_{\boldsymbol{x}}) \equiv \tilde{\Psi}(\boldsymbol{C}_{\boldsymbol{x}})$ (see equation (43)), it is possible to re-write the strain energy as a multi-variable function $\tilde{W}$ as

$$\tilde{\Psi}\left(\boldsymbol{C}_{\boldsymbol{x}}\right) = \tilde{W}\left(\boldsymbol{C}, \boldsymbol{G}, C\right). \tag{101}$$

Notice that in above equation (101), $\tilde{W}$ is expressed in terms of 13 independent variables, namely, C and the six independent components of $\boldsymbol{C}$ and $\boldsymbol{G}$, which are related to $\boldsymbol{C_x}$ via the geometric compatibility equations (see equations (99)-(100)). Note, however, that the function $\tilde{W}$ need not to be strictly convex with respect to its variables. For instance, in the case of a Mooney-Rivlin material, $\tilde{W}$ is linear with respect to both $\boldsymbol{C}$ and $\boldsymbol{G}$ as,

$$\tilde{W}_{MR}\left(\boldsymbol{C}, \boldsymbol{G}, C\right) = \alpha \boldsymbol{C} : \boldsymbol{I} + \beta \boldsymbol{G} : \boldsymbol{I} + \tilde{f}\left(C\right); \quad \tilde{f}\left(C\right) = f\left(\sqrt{C}\right). \tag{102}$$

Using the work conjugacy expression between the second Piola-Kirchhoff $\boldsymbol{S}$ and the right Cauchy-Green tensor $\boldsymbol{C}$ given by

$$D\tilde{\Psi}\left[\delta \boldsymbol{v}\right] = \boldsymbol{S} : \frac{1}{2} D\boldsymbol{C_x}\left[\delta \boldsymbol{v}\right]; \quad \boldsymbol{S} = 2\frac{\partial \tilde{\Psi}\left(\boldsymbol{C_x}\right)}{\partial \boldsymbol{C_x}} \tag{103}$$

and defining the conjugate stresses to $\boldsymbol{C}$, $\boldsymbol{G}$ and C as

$$\boldsymbol{\Sigma_C} := 2\frac{\partial \tilde{W}}{\partial \boldsymbol{C}}; \tag{104a}$$

$$\boldsymbol{\Sigma_G} := 2\frac{\partial \tilde{W}}{\partial \boldsymbol{G}}; \tag{104b}$$

$$\Sigma_C := 2\frac{\partial \tilde{W}}{\partial C}, \tag{104c}$$

enables an expression for the second Piola-Kirchhoff tensor to be derived using the same steps employed in equation (49) for the derivation of the first Piola-Kirchhoff tensor to give

$$\boldsymbol{S} = \boldsymbol{\Sigma_C} + \boldsymbol{\Sigma_G} \times \boldsymbol{C_x} + \Sigma_C \boldsymbol{G_x}, \tag{105}$$

where in the continuum context, where the geometric compatibility conditions (100) are satisfied strongly, the above equations can be re-expressed as

$$\boxed{\boldsymbol{S} = \boldsymbol{\Sigma_C} + \boldsymbol{\Sigma_G} \times \boldsymbol{C} + \Sigma_C \boldsymbol{G}} \tag{106}$$

Comparing equation (51) and (106), using the chain rule to relate derivatives with respect to $\boldsymbol{F}$, $\boldsymbol{H}$ and J to derivatives with respect to $\boldsymbol{C}$, $\boldsymbol{G}$ and C it is possible to establish the relationships

$$\boldsymbol{\Sigma_F} = \boldsymbol{F}\boldsymbol{\Sigma_C}; \quad \boldsymbol{\Sigma_H} = \boldsymbol{H}\boldsymbol{\Sigma_G}; \quad \Sigma_J = J\Sigma_C. \tag{107}$$

For the particular case of a Mooney-Rivlin material, expression (105) becomes

$$\boldsymbol{S} = 2\alpha\boldsymbol{I} + 2\beta\boldsymbol{I} \times \boldsymbol{C}_{\boldsymbol{x}} + \tilde{f}'(C)\,\boldsymbol{G}_{\boldsymbol{x}}. \tag{108}$$

4.2. The material elasticity tensor

It is also possible to derive the total Lagrangian elasticity tensor in terms of the Hessian matrix of $\tilde{W}$ following similar steps to those employed in previous sections. For this purpose, note first that the Lagrangian elasticity tensor $\boldsymbol{\mathcal{C}}_M$ is usually defined via the second directional derivative of the strain energy expressed in terms of the right Cauchy-Green tensor as:

$$\begin{aligned} D^2\tilde{\Psi}(\boldsymbol{C}_{\boldsymbol{x}})[\delta\boldsymbol{v};\delta\boldsymbol{u}] &= \frac{1}{2}D\boldsymbol{C}_{\boldsymbol{x}}[\delta\boldsymbol{v}] : \boldsymbol{\mathcal{C}}_M : \frac{1}{2}D\boldsymbol{C}_{\boldsymbol{x}}[\boldsymbol{u}] + \boldsymbol{S} : \frac{1}{2}D^2\boldsymbol{C}_{\boldsymbol{x}}[\delta\boldsymbol{v};\boldsymbol{u}] \\ &= \left(\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\delta\boldsymbol{v}\right) : 2\frac{\partial\boldsymbol{S}}{\partial\boldsymbol{C}_{\boldsymbol{x}}} : \left(\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\boldsymbol{u}\right) + \boldsymbol{S}_{\boldsymbol{x}} : \left[(\boldsymbol{\nabla}_0\delta\boldsymbol{v})^T(\boldsymbol{\nabla}_0\boldsymbol{u})\right]. \end{aligned} \tag{109}$$

Note that due to the symmetry of both $\boldsymbol{S}$ and $\boldsymbol{C}$, it is only necessary to consider one of the two terms making up the differential of the right Cauchy-Green tensor. The first term in the above equation can be related to the Hessian of the strain energy functional $\tilde{W}$ using similar steps to those employed above to derive equation (71). Similar algebra eventually leads to

$$\begin{aligned} D^2\Psi\,[\delta\boldsymbol{v};\boldsymbol{u}] &= \begin{bmatrix}\delta\boldsymbol{C} : & \delta\boldsymbol{G} : & \delta C\end{bmatrix}[\mathbb{H}_{\tilde{W}}]\begin{bmatrix}:\Delta\boldsymbol{C}\\ :\Delta\boldsymbol{G}\\ \Delta C\end{bmatrix} + \boldsymbol{S} : \left[(\boldsymbol{\nabla}_0\delta\boldsymbol{v})^T(\boldsymbol{\nabla}_0\boldsymbol{u})\right] \\ &\quad + (\boldsymbol{\Sigma}_{\boldsymbol{G}} + \Sigma_C\boldsymbol{C}_{\boldsymbol{x}}) : \left[(\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\delta\boldsymbol{v}) \times (\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\boldsymbol{u})\right], \end{aligned} \tag{110}$$

where the derivatives of $\boldsymbol{C}$, $\boldsymbol{G}$ and C are (refer to equations (99)-(100))

$$\delta\boldsymbol{C} = \boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\delta\boldsymbol{v}; \quad \delta\boldsymbol{G} = \boldsymbol{C}_{\boldsymbol{x}} \times (\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\delta\boldsymbol{v}); \quad \delta C = \boldsymbol{G}_{\boldsymbol{x}} : (\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\delta\boldsymbol{v}) \tag{111}$$

and similarly,

$$\Delta\boldsymbol{C} = \boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\boldsymbol{u}; \quad \Delta\boldsymbol{G} = \boldsymbol{C}_{\boldsymbol{x}} \times (\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\boldsymbol{u}); \quad \Delta C = \boldsymbol{G}_{\boldsymbol{x}} : (\boldsymbol{F}_{\boldsymbol{x}}{}^T\boldsymbol{\nabla}_0\boldsymbol{u}). \tag{112}$$

Note that since these expressions multiply symmetric tensors only one component of these derivatives, rather than the full symmetric expression, has been used. Note also that for the Mooney-Rivlin material model most of the terms of the Hessian matrix vanish:

$$[\mathbb{H}_{\tilde{W}_{MR}}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{f}''(C) \end{bmatrix}. \tag{113}$$

In the classical manner, it is possible to obtain the Lagrangian elasticity tensor $\boldsymbol{\mathcal{C}}_M$ [4] via differentiation of equation (54) as:

$$\begin{aligned} \boldsymbol{\mathcal{C}}_M =& \left[4\beta I_3 \mathrm{tr}\, \boldsymbol{C_x}^{-1} + 4g''(I_3)I_3^2 + 4g'(I_3)I_3\right] \boldsymbol{C_x}^{-1} \otimes \boldsymbol{C_x}^{-1} \\ & - 4\beta I_3 \left[\boldsymbol{C_x}^{-1} \otimes \left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right) + \left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right) \otimes \boldsymbol{C_x}^{-1}\right] \\ & - 4\left[\beta \mathrm{tr}\, \boldsymbol{C_x}^{-1} + g'(I_3)\right] I_3 \boldsymbol{\mathcal{I}} - 4\beta I_3 \boldsymbol{\mathcal{J}}, \end{aligned} \tag{114}$$

where

$$\boldsymbol{\mathcal{I}} = -\frac{\partial \boldsymbol{C_x}^{-1}}{\partial \boldsymbol{C_x}}; \qquad \boldsymbol{\mathcal{J}} = \frac{\partial \left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right)}{\partial \boldsymbol{C_x}}, \tag{115}$$

with components given for a symmetric tensor $\boldsymbol{C_x}$ [4] as

$$(\boldsymbol{\mathcal{I}})_{IJKL} = \frac{1}{2}\left[(\boldsymbol{C_x})^{-1}_{IK}(\boldsymbol{C_x})^{-1}_{JL} + (\boldsymbol{C_x})^{-1}_{IL}(\boldsymbol{C_x})^{-1}_{JK}\right] \tag{116}$$

and

$$\begin{aligned} (\boldsymbol{\mathcal{J}})_{IJKL} = -\frac{1}{2}\Big[& \left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right)_{IK}(\boldsymbol{C_x})^{-1}_{JL} + \left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right)_{IL}(\boldsymbol{C_x})^{-1}_{JK} \\ & + (\boldsymbol{C_x})^{-1}_{IK}\left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right)_{JL} + (\boldsymbol{C_x})^{-1}_{IL}\left(\boldsymbol{C_x}^{-1}\boldsymbol{C_x}^{-1}\right)_{JK}\Big]. \end{aligned} \tag{117}$$

4.3. The Kirchhoff and Cauchy stress tensors

In addition to the first and second Piola-Kirchhoff stresses, it is necessary to derive expressions for the Cauchy and Kirchhoff stresses as often these tensors are needed in order to express plasticity models or simply to display solution results. Such expressions can be relatively easily derived from the standard relationship between these tensors [4]

$$J_{\boldsymbol{x}}\boldsymbol{\sigma} : \boldsymbol{\nabla}\delta\boldsymbol{v} = \boldsymbol{P} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} = \boldsymbol{P} : \left[(\boldsymbol{\nabla}\delta\boldsymbol{v})\,\boldsymbol{F_x}\right], \tag{118}$$

with $\boldsymbol{\nabla} := \frac{\partial}{\partial \boldsymbol{x}}$. Substituting equation (50) for the first Piola-Kirchhoff tensor and recalling that $\boldsymbol{H_x}\boldsymbol{F_x}^T \equiv J_{\boldsymbol{x}}\boldsymbol{I}$ gives,

$$J_{\boldsymbol{x}}\boldsymbol{\sigma} = \boldsymbol{\tau} = \boldsymbol{\Sigma_F}\boldsymbol{F_x}^T + (\boldsymbol{\Sigma_H} \times \boldsymbol{F_x})\,\boldsymbol{F_x}^T + J_{\boldsymbol{x}}\Sigma_J\boldsymbol{I}. \tag{119}$$

The second term in the right hand side of above expression can be transformed with the help of property (22) of the tensor cross product by taking $\boldsymbol{B} := \boldsymbol{F_x}$, $\boldsymbol{A}_1 := J_{\boldsymbol{x}}^{-1}\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T$ and $\boldsymbol{A}_2 := \boldsymbol{I}$ as follows,

$$(\boldsymbol{A}_1\boldsymbol{B}) \times (\boldsymbol{A}_2\boldsymbol{B}) = (J_{\boldsymbol{x}}^{-1}\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T\boldsymbol{F_x}) \times \boldsymbol{F_x} = \boldsymbol{\Sigma_H} \times \boldsymbol{F_x}; \tag{120a}$$

$$(\boldsymbol{A}_1 \times \boldsymbol{A}_2)\mathrm{Cof}\boldsymbol{B} = \left[(J_{\boldsymbol{x}}^{-1}\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T) \times \boldsymbol{I}\right]\boldsymbol{H_x} = \left[(\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T) \times \boldsymbol{I}\right]\boldsymbol{F_x}^{-T}, \tag{120b}$$

where equation (2) has been used for the last step in both equations (120a)-(120b). Multiplication by $\boldsymbol{F_x}^T$ on (120a)-(120b) renders:

$$(\boldsymbol{\Sigma_H} \times \boldsymbol{F_x})\,\boldsymbol{F_x}^T = (\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T) \times \boldsymbol{I}, \tag{121}$$

thus giving an expression for the Kirchhoff stresses as:

$$J_{\boldsymbol{x}}\boldsymbol{\sigma} = \boldsymbol{\tau} = \boldsymbol{\Sigma_F}\boldsymbol{F_x}^T + (\boldsymbol{\Sigma_H}\boldsymbol{H_x}^T) \times \boldsymbol{I} + J_{\boldsymbol{x}}\Sigma_J\boldsymbol{I}, \tag{122}$$

or introducing the notation:

$$\boldsymbol{\tau_F} = \boldsymbol{\Sigma_F}\boldsymbol{F_x}^T; \quad \boldsymbol{\tau_H} = \boldsymbol{\Sigma_H}\boldsymbol{H_x}^T; \quad \tau_J = J_{\boldsymbol{x}}\Sigma_J \tag{123}$$

gives,

$$J_{\boldsymbol{x}}\boldsymbol{\sigma} = \boldsymbol{\tau} = \boldsymbol{\tau_F} + \boldsymbol{\tau_H} \times \boldsymbol{I} + \tau_J\boldsymbol{I}. \tag{124}$$

In the continuum context, where geometric compatibility is satisfied exactly,

$$\boldsymbol{\tau_F} = \boldsymbol{\Sigma_F}\boldsymbol{F}^T; \quad \boldsymbol{\tau_H} = \boldsymbol{\Sigma_H}\boldsymbol{H}^T; \quad \tau_J = J\Sigma_J \tag{125}$$

above equation (124) becomes

$$\boxed{J\boldsymbol{\sigma} = \boldsymbol{\tau} = \boldsymbol{\tau_F} + \boldsymbol{\tau_H} \times \boldsymbol{I} + \tau_J\boldsymbol{I}} \tag{126}$$

and given that W is a function $\boldsymbol{F}$ and $\boldsymbol{H}$ via $\boldsymbol{F}^T\boldsymbol{F}$ and $\boldsymbol{H}^T\boldsymbol{H}$ the above Kirchhoff stress components will be symmetric. Note that, if geometric compatibility is not exactly satisfied, this will not be the case and may lead to non-symmetric Cauchy and Kirchhoff stress tensors.

For the particular case of the Mooney-Rivlin model under consideration, above equation (124) leads after simple algebra to:

$$J_{\boldsymbol{x}}\boldsymbol{\sigma} = \boldsymbol{\tau} = 2\alpha\,\boldsymbol{b} + 2\beta\,\boldsymbol{g}\boldsymbol{\times}\boldsymbol{I} + J_{\boldsymbol{x}}f'(J)\boldsymbol{I}; \quad \boldsymbol{b} = \boldsymbol{F_x}\boldsymbol{F_x}^T; \quad \boldsymbol{g} = \boldsymbol{H_x}\boldsymbol{H_x}^T. \tag{127}$$

Alternatively, in the standard manner [4], postmultiplication of equation (55) by $\boldsymbol{F_x}^T$ leads to the following expression for the Kirchhoff stress tensor:

$$\boldsymbol{\tau} = 2\alpha\boldsymbol{b} + 2\beta\left[I_3\left(\operatorname{tr}\boldsymbol{b}^{-1}\right)\boldsymbol{I} - I_3\boldsymbol{b}^{-1}\right] + 2g'(I_3)I_3\boldsymbol{I}. \tag{128}$$

4.4. The spatial elasticity tensor

In the context of a spatial description, it is usually necessary to derive a spatial or Eulerian elasticity tensor which relates the second derivative of the strain energy to the spatial gradients of virtual velocities and displacements. For this purpose, equation (71) for the tangent elasticity operator is transformed with the help of the chain rule, which provides a relationship between material and spatial gradients, namely $\boldsymbol{\nabla}_0\boldsymbol{a} = (\boldsymbol{\nabla}\boldsymbol{a})\boldsymbol{F_x}$ for any field $\boldsymbol{a}$, and the repeated use of property (22). After simple algebra this leads to:

$$\begin{aligned}
D^2\Psi\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= \left[(\boldsymbol{\nabla}\delta\boldsymbol{v})\,\boldsymbol{F_x} : \quad (\boldsymbol{\nabla}\delta\boldsymbol{v}\boldsymbol{\times}\boldsymbol{I})\,\boldsymbol{H_x} : \quad (\boldsymbol{\nabla}\delta\boldsymbol{v} : \boldsymbol{I})\,J_{\boldsymbol{x}}\right]\left[\mathbb{H}_W\right]\begin{bmatrix} : (\boldsymbol{\nabla}\boldsymbol{u})\,\boldsymbol{F_x} \\ : (\boldsymbol{\nabla}\boldsymbol{u}\boldsymbol{\times}\boldsymbol{I})\,\boldsymbol{H_x} \\ (\boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{I})\,J_{\boldsymbol{x}} \end{bmatrix} \\
&\quad + (\boldsymbol{\Sigma_H} + \Sigma_J\boldsymbol{F_x}) : \left[(\boldsymbol{\nabla}\delta\boldsymbol{v}\boldsymbol{\times}\boldsymbol{\nabla}\boldsymbol{u})\,\boldsymbol{H_x}\right] \\
&= \left[(\boldsymbol{\nabla}\delta\boldsymbol{v}) : \quad (\boldsymbol{\nabla}\delta\boldsymbol{v}\boldsymbol{\times}\boldsymbol{I}) : \quad \operatorname{div}\delta\boldsymbol{v}\right]\boldsymbol{\phi}_*\left[\mathbb{H}_W\right]\begin{bmatrix} : (\boldsymbol{\nabla}\boldsymbol{u}) \\ : (\boldsymbol{\nabla}\boldsymbol{u}\boldsymbol{\times}\boldsymbol{I}) \\ \operatorname{div}\boldsymbol{u} \end{bmatrix} \\
&\quad + (\boldsymbol{\tau_H} + \tau_J\boldsymbol{I}) : (\boldsymbol{\nabla}\delta\boldsymbol{v}\boldsymbol{\times}\boldsymbol{\nabla}\boldsymbol{u}),
\end{aligned} \tag{129}$$

where $\boldsymbol{\phi}_*[\mathbb{H}_W]$ denotes the appropriate push forward of the components of the Hessian operator with either $\boldsymbol{F}$, $\boldsymbol{H}$ or J. Specifically, in component form, this operator is defined for a generic material as:

$$\boldsymbol{\phi}_*[\mathbb{H}_W] = \begin{bmatrix} [\boldsymbol{F}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial F_{iI}\partial F_{kJ}}\,[\boldsymbol{F}_{\boldsymbol{x}}]_{lJ} & [\boldsymbol{F}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial F_{iI}\partial H_{kJ}}\,[\boldsymbol{H}_{\boldsymbol{x}}]_{lJ} & [\boldsymbol{F}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial F_{iI}\partial J}\,J_{\boldsymbol{x}} \\ [\boldsymbol{H}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial H_{iI}\partial F_{kJ}}\,[\boldsymbol{F}_{\boldsymbol{x}}]_{lJ} & [\boldsymbol{H}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial H_{iI}\partial H_{kJ}}\,[\boldsymbol{H}_{\boldsymbol{x}}]_{lJ} & [\boldsymbol{H}_{\boldsymbol{x}}]_{jI}\,\frac{\partial^2 W}{\partial H_{iI}\partial J}\,J_{\boldsymbol{x}} \\ J_{\boldsymbol{x}}\frac{\partial^2 W}{\partial J\partial F_{kJ}}\,[\boldsymbol{F}_{\boldsymbol{x}}]_{lJ} & J_{\boldsymbol{x}}\frac{\partial^2 W}{\partial J\partial H_{kJ}}\,[\boldsymbol{H}_{\boldsymbol{x}}]_{lJ} & J_{\boldsymbol{x}}\frac{\partial^2 W}{\partial J\partial J}\,J_{\boldsymbol{x}} \end{bmatrix}, \tag{130}$$

which for the particular case of Mooney-Rivlin becomes:

$$\boldsymbol{\phi}_*[\mathbb{H}_W] = \begin{bmatrix} 2\alpha b_{jl}\delta_{ik} & 0 & 0 \\ 0 & 2\beta g_{jl}\delta_{ik} & 0 \\ 0 & 0 & J^2 f''(J) \end{bmatrix}. \tag{131}$$

Substituting this expression into equation (129) gives after simple algebra:

$$\begin{aligned} D^2\Psi_{MR}\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= 2\alpha\left(\boldsymbol{\nabla}\delta\boldsymbol{v}\right)\boldsymbol{b} : \boldsymbol{\nabla}\boldsymbol{u} + 2\beta\left(\boldsymbol{\nabla}\delta\boldsymbol{v}\times\boldsymbol{I}\right)\boldsymbol{g} : \left(\boldsymbol{\nabla}\boldsymbol{u}\times\boldsymbol{I}\right) \\ &\quad + f''\left(J\right)J^2\,\mathrm{div}\delta\boldsymbol{v}\,\mathrm{div}\boldsymbol{u} + \left(\boldsymbol{\tau}_{\boldsymbol{H}} + \tau_J\boldsymbol{I}\right) : \left(\boldsymbol{\nabla}\delta\boldsymbol{v}\times\boldsymbol{\nabla}\boldsymbol{u}\right). \end{aligned} \tag{132}$$

5. Isotropic and transversely isotropic elasticity

5.1. Isotropic elasticity

In the particular case of isotropic elasticity, the expression for the energy density functional can be established through the invariants I_1, I_2 and I_3 of the right Cauchy-Green strain tensor (44). A re-definition of the above invariants, more suitable in the case of a isotropic polyconvex energy functional, is given by

$$I_1 := \boldsymbol{F} : \boldsymbol{F}; \qquad I_2 := \boldsymbol{H} : \boldsymbol{H}; \qquad I_3 := J^2, \tag{133}$$

leading to a representation of the energy density functional as:

$$W(\boldsymbol{F}, \boldsymbol{H}, J) = w(I_1, I_2, I_3). \tag{134}$$

In order to obtain simple expressions for the first Piola-Kirchhoff and elasticity tensors directly in terms of the derivatives of the function w rather than W, and making use of the directional derivative equations (32) to (36), note that the first and second derivatives of the invariants are given by

$$\begin{aligned} DI_1[\delta\boldsymbol{v}] &= 2\boldsymbol{F} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; & (135)\\ DI_2[\delta\boldsymbol{v}] &= 2(\boldsymbol{F}_{\boldsymbol{x}}\times\boldsymbol{H}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; & (136)\\ DI_3[\delta\boldsymbol{v}] &= 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; & (137)\\ D^2I_1[\delta\boldsymbol{v};\boldsymbol{u}] &= 2\boldsymbol{\nabla}_0\delta\boldsymbol{v} : \boldsymbol{\nabla}_0\boldsymbol{u}; & (138)\\ D^2I_2[\delta\boldsymbol{v};\boldsymbol{u}] &= 2(\boldsymbol{F}_{\boldsymbol{x}}\times\boldsymbol{\nabla}_0\delta\boldsymbol{v}) : (\boldsymbol{F}_{\boldsymbol{x}}\times\boldsymbol{\nabla}_0\boldsymbol{u}) + 2\boldsymbol{H} : (\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{\nabla}_0\boldsymbol{u}); & (139)\\ D^2I_3[\delta\boldsymbol{v};\boldsymbol{u}] &= 2(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v})(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\boldsymbol{u}) + 2J\boldsymbol{F}_{\boldsymbol{x}} : (\boldsymbol{\nabla}_0\delta\boldsymbol{v}\times\boldsymbol{\nabla}_0\boldsymbol{u}). & (140) \end{aligned}$$

The first Piola-Kirchhoff tensor can be derived either using directly the first three equations above which enable the internal virtual work to be written as

$$\boldsymbol{P} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} = [\mathbb{D}_w] \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{H}) : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \\ 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \end{bmatrix}; \quad [\mathbb{D}_w] = \left[\frac{\partial w}{\partial I_1}, \frac{\partial w}{\partial I_2}, \frac{\partial w}{\partial I_3} \right] \tag{141}$$

thereby leading to

$$\boldsymbol{P} = 2\frac{\partial w}{\partial I_1}\boldsymbol{F} + 2\frac{\partial w}{\partial I_2}\boldsymbol{H} \times \boldsymbol{F}_{\boldsymbol{x}} + 2\frac{\partial w}{\partial I_3} J\boldsymbol{H}_{\boldsymbol{x}}. \tag{142}$$

Alternatively, it is also possible to obtain the same equation for the first Piola-Kirchhoff sress tensor via the work conjugate stresses $\boldsymbol{\Sigma}_{\boldsymbol{F}}$, $\boldsymbol{\Sigma}_{\boldsymbol{H}}$ and Σ_J and using the chain rule to give

$$\boldsymbol{\Sigma}_{\boldsymbol{F}} = 2\frac{\partial w}{\partial I_1}\boldsymbol{F}; \qquad \boldsymbol{\Sigma}_{\boldsymbol{H}} = 2\frac{\partial w}{\partial I_2}\boldsymbol{H}; \qquad \Sigma_J = 2\frac{\partial w}{\partial I_3}J. \tag{143}$$

Introducing these equations into equation (50) leads immediately to equation (142). The tangent elasticity operator can be formulated by differentiating again equation (141), which after simple algebra using the second derivatives of the invariants given above leads to

$$\begin{aligned} D^2\Psi\,[\delta \boldsymbol{v}; \boldsymbol{u}] = & \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{H}) : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \\ 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v} \end{bmatrix}^T [\mathbb{H}_w] \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0 \boldsymbol{u} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{H}) : \boldsymbol{\nabla}_0 \boldsymbol{u} \\ 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0 \boldsymbol{u} \end{bmatrix} \\ & + [\mathbb{D}_w] \begin{bmatrix} 2\boldsymbol{\nabla}_0 \delta \boldsymbol{v} : \boldsymbol{\nabla}_0 \boldsymbol{u} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0 \delta \boldsymbol{v}) : (\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0 \boldsymbol{u}) \\ 2(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0 \delta \boldsymbol{v})(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0 \boldsymbol{u}) \end{bmatrix} \\ & + [\mathbb{D}_w] \begin{bmatrix} 0 \\ 2\boldsymbol{H} : (\boldsymbol{\nabla}_0 \delta \boldsymbol{v} \times \boldsymbol{\nabla}_0 \boldsymbol{u}) \\ 2J\boldsymbol{F}_{\boldsymbol{x}} : (\boldsymbol{\nabla}_0 \delta \boldsymbol{v} \times \boldsymbol{\nabla}_0 \boldsymbol{u}) \end{bmatrix}, \end{aligned} \tag{144}$$

with

$$[\mathbb{H}_w] = \begin{bmatrix} \frac{\partial^2 w}{\partial I_1 \partial I_1} & \frac{\partial^2 w}{\partial I_1 \partial I_2} & \frac{\partial^2 w}{\partial I_1 \partial I_3} \\ \frac{\partial^2 w}{\partial I_2 \partial I_1} & \frac{\partial^2 w}{\partial I_2 \partial I_2} & \frac{\partial^2 w}{\partial I_2 \partial I_3} \\ \frac{\partial^2 w}{\partial I_3 \partial I_1} & \frac{\partial^2 w}{\partial I_3 \partial I_2} & \frac{\partial^2 w}{\partial I_3 \partial I_3} \end{bmatrix}. \tag{145}$$

Note that the sum of the first two terms needs to be positive definite for materials with a polyconvex strain energy function, as the last term represents the geometrical term depicted in equation (71).

5.2. Transversely isotropic materials

It is possible to derive similar expressions for anisotropic materials by extending the range of invariants taken into account. An example of particular interest in many bioengineering applications is that of transversely isotropic materials [39], [40]. In such cases the strain energy $w_{tr}(I_1, I_2, I_3, I_4, I_5)$ can be expressed as a function of two further invariants, which with the current notation can be defined as

$$I_4 := \boldsymbol{FN} \cdot \boldsymbol{FN}; \quad I_5 := \boldsymbol{HN} \cdot \boldsymbol{HN}, \tag{146}$$

where $\boldsymbol{N}$ is a unit material vector defining the direction of transverse isotropy. The first and second derivatives of these two new invariants can be obtained using standard algebra, making use of the directional derivative equations (32) to (34), and the properties of the tensor cross product to give

$$\begin{aligned}
DI_4[\delta\boldsymbol{v}] &= 2(\boldsymbol{FN} \otimes \boldsymbol{N}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; && (147)\\
DI_5[\delta\boldsymbol{v}] &= 2\left[\boldsymbol{F_x} \boldsymbol{\times} (\boldsymbol{HN} \otimes \boldsymbol{N})\right] : \boldsymbol{\nabla}_0\delta\boldsymbol{v}; && (148)\\
D^2 I_4[\delta\boldsymbol{v}; \boldsymbol{u}] &= 2(\boldsymbol{\nabla}_0\delta\boldsymbol{v})\boldsymbol{N} \cdot (\boldsymbol{\nabla}_0\boldsymbol{u})\boldsymbol{N}; && (149)\\
D^2 I_5[\delta\boldsymbol{v}; \boldsymbol{u}] &= 2(\boldsymbol{F_x} \boldsymbol{\times} \boldsymbol{\nabla}_0\delta\boldsymbol{v})\boldsymbol{N} \cdot (\boldsymbol{F_x} \boldsymbol{\times} \boldsymbol{\nabla}_0\boldsymbol{u})\boldsymbol{N} && (150)\\
&+ 2(\boldsymbol{HN} \otimes \boldsymbol{N}) : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \boldsymbol{\times} \boldsymbol{\nabla}_0\boldsymbol{u}). && (151)
\end{aligned}$$

The above expressions enable the internal virtual energy to be expressed in terms of the vector $[\mathbb{D}_{w_{tr}}]$ containing the derivatives of $w_{tr}(I_1, I_2, I_3, I_4, I_5)$ with respect to the 5 invariants as

$$\boldsymbol{P} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} = [\mathbb{D}_{w_{tr}}] \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2(\boldsymbol{F_x} \boldsymbol{\times} \boldsymbol{H}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2J\boldsymbol{H_x} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2(\boldsymbol{FN} \otimes \boldsymbol{N}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2\left[\boldsymbol{F_x} \boldsymbol{\times} (\boldsymbol{HN} \otimes \boldsymbol{N})\right] : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \end{bmatrix} \tag{152}$$

and therefore the first Piola-Kirchhoff tensor emerges as

$$\begin{aligned} \boldsymbol{P} &= 2\frac{\partial w}{\partial I_1}\boldsymbol{F} + 2\frac{\partial w}{\partial I_2}\boldsymbol{H} \times \boldsymbol{F}_{\boldsymbol{x}} + 2\frac{\partial w}{\partial I_3} J\boldsymbol{H}_{\boldsymbol{x}} \\ &+ 2\frac{\partial w}{\partial I_4}(\boldsymbol{FN} \otimes \boldsymbol{N}) + 2\frac{\partial w}{\partial I_5}(\boldsymbol{HN} \otimes \boldsymbol{N}) \times \boldsymbol{F}_{\boldsymbol{x}}. \end{aligned} \tag{153}$$

Finally, the tangent elastic operator can be expressed in terms of the 5×5 Hessian matrix of the function $[\mathbb{H}_{w_{tr}}]$ using the second derivatives of the two new invariants given in equations (150) and (151). After simple algebra, this leads to

$$\begin{aligned} D^2\Psi\left[\delta\boldsymbol{v};\boldsymbol{u}\right] &= \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{H}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2(\boldsymbol{FN} \otimes \boldsymbol{N}) : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \\ 2\left[\boldsymbol{F}_{\boldsymbol{x}} \times (\boldsymbol{HN} \otimes \boldsymbol{N})\right] : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \end{bmatrix}^T [\mathbb{H}_{w_{tr}}] \begin{bmatrix} 2\boldsymbol{F} : \boldsymbol{\nabla}_0\boldsymbol{u} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{H}) : \boldsymbol{\nabla}_0\boldsymbol{u} \\ 2J\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\boldsymbol{u} \\ 2(\boldsymbol{FN} \otimes \boldsymbol{N}) : \boldsymbol{\nabla}_0\boldsymbol{u} \\ 2\left[\boldsymbol{F}_{\boldsymbol{x}} \times (\boldsymbol{HN} \otimes \boldsymbol{N})\right] : \boldsymbol{\nabla}_0\boldsymbol{u} \end{bmatrix} \\ &+ [\mathbb{D}_{w_{tr}}] \begin{bmatrix} 2\boldsymbol{\nabla}_0\delta\boldsymbol{v} : \boldsymbol{\nabla}_0\boldsymbol{u} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0\delta\boldsymbol{v}) : (\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0\boldsymbol{u}) \\ 2(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v})(\boldsymbol{H}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\boldsymbol{u}) \\ 2(\boldsymbol{\nabla}_0\delta\boldsymbol{v})\boldsymbol{N} \cdot (\boldsymbol{\nabla}_0\boldsymbol{u})\boldsymbol{N} \\ 2(\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0\delta\boldsymbol{v})\boldsymbol{N} \cdot (\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{\nabla}_0\boldsymbol{u})\boldsymbol{N} \end{bmatrix} \\ &+ [\mathbb{D}_{w_{tr}}] \begin{bmatrix} 0 \\ 2\boldsymbol{H} : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \times \boldsymbol{\nabla}_0\boldsymbol{u}) \\ 2J\boldsymbol{F}_{\boldsymbol{x}} : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \times \boldsymbol{\nabla}_0\boldsymbol{u}) \\ 0 \\ 2(\boldsymbol{HN} \otimes \boldsymbol{N}) : (\boldsymbol{\nabla}_0\delta\boldsymbol{v} \times \boldsymbol{\nabla}_0\boldsymbol{u}) \end{bmatrix}. \end{aligned} \tag{154}$$

6. Variational formulations

This section shows how the proposed tensor cross product algebra can facilitate the formulation of various mixed variational formulations [41, 42] in order to establish the static equilibrium and compatibility equations. The section starts reviewing the standard displacement based variational principle. This provides a useful background for comparison with mixed and complementary energy variational principles presented later in the section.

6.1. Standard displacement based variational principle

The solution of large strain elastic problems is often expressed by means of the total energy minimisation variational principle as

$$\Pi(\boldsymbol{x}^*) = \inf_{\boldsymbol{x}\in\mathcal{X}} \left\{ \int_V \Psi\left(\boldsymbol{F}_{\boldsymbol{x}}\right) dV - \int_V \boldsymbol{f}_0 \cdot \boldsymbol{x}\, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x}\, dA \right\}, \tag{155}$$

where $\boldsymbol{x}^*$ denotes the exact solution and $\mathcal{X}$ the appropriate Sobolev space of functions satisfying the relevant displacement boundary conditions. The strain energy function in this potential can be replaced by the convex function $W(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}})$, where the geometrically compatible strain measures were defined in (25). The stationary condition of this functional leads to the principle of virtual work (or power), commonly written as

$$D\Pi\left[\delta\boldsymbol{v}\right] = \int_V \boldsymbol{P}_{\boldsymbol{x}} : \boldsymbol{\nabla}_0\delta\boldsymbol{v}\, dV - \int_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v}\, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v}\, dA = 0; \quad \forall \delta\boldsymbol{v} \in \mathcal{X}_0. \tag{156}$$

In this expression, the first Piola-Kirchhoff tensor $\boldsymbol{P}_{\boldsymbol{x}}$ is evaluated in the standard fashion using equation (51) in terms of the gradient of the deformation $\boldsymbol{F}_{\boldsymbol{x}}$ as

$$\boldsymbol{P}_{\boldsymbol{x}} = \boldsymbol{\Sigma}_{\boldsymbol{F}}^{\boldsymbol{x}} + \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\boldsymbol{x}} \times \boldsymbol{F}_{\boldsymbol{x}} + \Sigma_J^{\boldsymbol{x}} \boldsymbol{H}_{\boldsymbol{x}}, \tag{157}$$

where the superscript $\boldsymbol{x}$ in the above stresses indicates that they are evaluated in terms of the geometric deformation gradient as

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{F}}^{\boldsymbol{x}} &= \boldsymbol{\Sigma}_{\boldsymbol{F}}\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right); \\ \boldsymbol{\Sigma}_{\boldsymbol{H}}^{\boldsymbol{x}} &= \boldsymbol{\Sigma}_{\boldsymbol{H}}\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right); \\ \Sigma_J^{\boldsymbol{x}} &= \Sigma_J\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right). \end{aligned} \tag{158}$$

An iterative Newton-Raphson process to converge towards the solution is usually established by solving a linearized system for the increment $\boldsymbol{u}$ as

$$D^2\Pi\left[\delta\boldsymbol{v}; \boldsymbol{u}\right] = -D\Pi\left(\boldsymbol{x}_k\right)\left[\delta\boldsymbol{v}\right]; \quad \boldsymbol{x}_{k+1} = \boldsymbol{x}_k + \boldsymbol{u}, \tag{159}$$

where, in the absence of follower forces, the second derivative of the total energy functional is given by

$$D^2\Pi\left[\delta\boldsymbol{v}; \boldsymbol{u}\right] = \int_V D^2\Psi\left[\boldsymbol{\nabla}_0\delta\boldsymbol{v}, \boldsymbol{\nabla}_0\boldsymbol{u}\right] dV, \tag{160}$$

where the tangent operator is evaluated using equation (71).

6.2. Mixed Variational Principle

An equivalent but alternative expression for the total energy variational principle can be written in terms of the geometry and strain variables as a constrained minimisation problem in the form

$$\Pi(\boldsymbol{x}^*) = \inf_{\substack{\boldsymbol{x},\,\boldsymbol{F},\,\boldsymbol{H},\,J,\ \text{s.t.}\\ \boldsymbol{F}=\boldsymbol{F}_x,\\ \boldsymbol{H}=\boldsymbol{H}_x,\\ J=J_x}} \left\{ \int_V W\left(\boldsymbol{F},\boldsymbol{H},J\right)\,dV - \int_V \boldsymbol{f}_0\cdot\boldsymbol{x}\,dV - \int_{\partial_t V} \boldsymbol{t}_0\cdot\boldsymbol{x}\,dA \right\}. \tag{161}$$

Using a standard Lagrange multiplier approach to enforce the compatibility constraints gives the following augmented mixed variational principle

$$\Pi_M(\boldsymbol{x}^*,\boldsymbol{F}^*,\boldsymbol{H}^*,J^*,\boldsymbol{\Sigma_F}^*,\boldsymbol{\Sigma}^*_{\boldsymbol{H}},\Sigma^*_J) = \inf_{\boldsymbol{x},\boldsymbol{F},\boldsymbol{H},J} \left\{ \sup_{\boldsymbol{\Sigma_F},\boldsymbol{\Sigma_H},\Sigma_J} \left\{ \int_V W(\boldsymbol{F},\boldsymbol{H},J)\,dV \right.\right.$$
$$+ \int_V \boldsymbol{\Sigma_F} : (\boldsymbol{F_x}-\boldsymbol{F})\,dV + \int_V \boldsymbol{\Sigma_H} : (\boldsymbol{H_x}-\boldsymbol{H})\,dV + \int_V \Sigma_J(J_{\boldsymbol{x}}-J)\,dV$$
$$\left.\left. - \int_V \boldsymbol{f}_0\cdot\boldsymbol{x}\,dV - \int_{\partial_t V} \boldsymbol{t}_0\cdot\boldsymbol{x}\,dA \right\}\right\}. \tag{162}$$

This expression belongs to the general class of Hu-Washizu type of mixed variational principles [43] which have been widely used for the development of enhanced finite element formulations [4]. Note that the stress variables $\{\boldsymbol{\Sigma_F},\boldsymbol{\Sigma_H},\Sigma_J\}$ in this expression, at this stage, are simply Lagrange multipliers and are as yet unconnected to the strain variables. Both stress and strain variables belong to appropriate Sobolev function spaces, which generally require simple piecewise continuity and are unrestricted on the boundaries.

The stationary condition of the above augmented Lagrangian with respect to the first variable enforces equilibrium in the form of the principle of virtual work as

$$D_1\Pi_M\left[\delta\boldsymbol{v}\right] = \int_V \boldsymbol{P}_M : \boldsymbol{\nabla}_0\delta\boldsymbol{v}\,dV - \int_V \boldsymbol{f}_0\cdot\delta\boldsymbol{v}\,dV - \int_{\partial_t V} \boldsymbol{t}_0\cdot\delta\boldsymbol{v}\,dA = 0, \tag{163}$$

where the first Piola-Kirchhoff stress now emerges as

$$\boldsymbol{P}_M = \boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \boldsymbol{\times} \boldsymbol{F_x} + \Sigma_J \boldsymbol{H_x}, \tag{164}$$

which is an identical expression to that of equation (50). The stationary conditions with respect to the three strain variables enforce the constitutive relationships between the stresses and the derivatives of the strain energy in a weak form

$$D_{2,3,4}\Pi_M\left[\delta\boldsymbol{F},\delta\boldsymbol{H},\delta J\right]=\int\limits_V\left(\frac{\partial W}{\partial\boldsymbol{F}}-\boldsymbol{\Sigma}_{\boldsymbol{F}}\right):\delta\boldsymbol{F}\,dV+\int\limits_V\left(\frac{\partial W}{\partial\boldsymbol{H}}-\boldsymbol{\Sigma}_{\boldsymbol{H}}\right):\delta\boldsymbol{H}\,dV+\int\limits_V\left(\frac{\partial W}{\partial J}-\Sigma_J\right)\delta J\,dV=0. \tag{165}$$

Finally, the stationary conditions with respect to the stress variables enforce the geometric compatibility conditions between strains and geometry

$$D_{5,6,7}\Pi_M\left[\delta\boldsymbol{\Sigma}_{\boldsymbol{F}},\delta\boldsymbol{\Sigma}_{\boldsymbol{H}},\delta\Sigma_J\right]=\int\limits_V\delta\boldsymbol{\Sigma}_{\boldsymbol{F}}:\left(\boldsymbol{F}_{\boldsymbol{x}}-\boldsymbol{F}\right)dV+\int\limits_V\delta\boldsymbol{\Sigma}_{\boldsymbol{H}}:\left(\boldsymbol{H}_{\boldsymbol{x}}-\boldsymbol{H}\right)dV+\int\limits_V\delta\Sigma_J\left(J_{\boldsymbol{x}}-J\right)dV. \tag{166}$$

The second derivatives of the above functional required for a nonlinear Newton-Raphson solution process are given in Reference [2] in the context of a finite element implementation.

The set of equations derived in this section enables the use of different spaces for each of the problem variables. This level of flexibility may be useful but it is costly in a computational context given the large number of unknowns generated in the process. An alternative approach that significantly reduces the number of problem variables is presented in the next section.

6.3. *Mixed Complementary Energy Principle*

In order to derive a variational principle in terms of the complementary energy, recall first the mixed variational principle (162) with a different ordering of terms

$$\Pi_M(\boldsymbol{x}^*, \boldsymbol{F}^*, \boldsymbol{H}^*, J^*, \boldsymbol{\Sigma_F}^*, \boldsymbol{\Sigma}^*_{\boldsymbol{H}}, \Sigma^*_J) = \inf_{\boldsymbol{x},\boldsymbol{F},\boldsymbol{H},J} \left\{ \sup_{\boldsymbol{\Sigma_F},\boldsymbol{\Sigma_H},\Sigma_J} \{ \right.$$
$$- \int_V [\boldsymbol{\Sigma_F} : \boldsymbol{F} + \boldsymbol{\Sigma_H} : \boldsymbol{H} + \Sigma_J J - W(\boldsymbol{F}, \boldsymbol{H}, J)] \, dV - \int_V \boldsymbol{f}_0 \cdot \boldsymbol{x} \, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x} \, dA$$
$$\left. + \int_V \boldsymbol{\Sigma_F} : \boldsymbol{F_x} \, dV + \int_V \boldsymbol{\Sigma_H} : \boldsymbol{H_x} \, dV + \int_V \Sigma_J J_{\boldsymbol{x}} \, dV \right\}\Bigg\}. \tag{167}$$

Comparing the term in square brackets in the first integral with the definition of the complementary energy given by equation (56), enables a mixed complementary variational principle to be established as[3]

$$\Pi_C(\boldsymbol{x}^*, \boldsymbol{\Sigma_F}^*, \boldsymbol{\Sigma}^*_{\boldsymbol{H}}, \Sigma^*_J) = \inf_{\boldsymbol{x}} \left\{ \sup_{\boldsymbol{\Sigma_F},\boldsymbol{\Sigma_H},\Sigma_J} \left\{ - \int_V \Upsilon(\boldsymbol{\Sigma_F}, \boldsymbol{\Sigma_H}, \Sigma_J) \, dV \right.\right.$$
$$\left.\left. + \int_V \boldsymbol{\Sigma_F} : \boldsymbol{F_x} \, dV + \int_V \boldsymbol{\Sigma_H} : \boldsymbol{H_x} \, dV + \int_V \Sigma_J J_{\boldsymbol{x}} \, dV - \int_V \boldsymbol{f}_0 \cdot \boldsymbol{x} \, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x} \, dA \right\}\right\}. \tag{168}$$

This represents a Helinger-Reissner type of variational principle [4]. The stationary condition of this principle with respect to its first variable, the geometry, enforces equilibrium in a manner identical to equations (163) and (164), that is,

$$D_1\Pi_C[\delta\boldsymbol{v}] = D_1\Pi_M[\delta\boldsymbol{v}] = \int_V \boldsymbol{P}_M : \boldsymbol{\nabla}_0\delta\boldsymbol{v} \, dV - \int_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v} \, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v} \, dA = 0$$
$$\boldsymbol{P}_C = \boldsymbol{P}_M = \boldsymbol{\Sigma_F} + \boldsymbol{\Sigma_H} \times \boldsymbol{F_x} + \Sigma_J \boldsymbol{H_x}. \tag{169}$$

Similarly, the stationary conditions with respect to stresses, enforce the

[3]Note that this step relies on the strong duality property of the mixed functional which allows the order of the inf and sup operations with respect to strains and stresses to be swapped. This is the case here given the convexity of the strain energy function.

geometric compatibility conditions, now expressed as,

$$
\begin{aligned}
D_{2,3,4}\Pi_C\left[\delta\boldsymbol{\Sigma}_{\boldsymbol{F}}, \delta\boldsymbol{\Sigma}_{\boldsymbol{H}}, \delta\Sigma_J\right] = & \int\limits_V \delta\boldsymbol{\Sigma}_{\boldsymbol{F}} : \left(\boldsymbol{F}_{\boldsymbol{x}} - \frac{\partial\Upsilon}{\partial\boldsymbol{\Sigma}_{\boldsymbol{F}}}\right) dV \\
& + \int\limits_V \delta\boldsymbol{\Sigma}_{\boldsymbol{H}} : \left(\boldsymbol{H}_{\boldsymbol{x}} - \frac{\partial\Upsilon}{\partial\boldsymbol{\Sigma}_{\boldsymbol{H}}}\right) dV \\
& + \int\limits_V \delta\Sigma_J \left(J_{\boldsymbol{x}} - \frac{\partial\Upsilon}{\partial\Sigma_J}\right) dV.
\end{aligned} \tag{170}
$$

The second derivatives of this complementary functional and its use in the context of finite element discretisations is discussed in detail in Reference [2].

6.4. Variational principles for incompressible and nearly incompressible models

Many applications of practical importance rely on the decomposition of the strain energy into isochoric and volumetric components. For such cases, it is possible to modify the variational formulations above in such a way that different approaches are used for the isochoric and volumetric components. In particular, it is often useful to follow a standard displacement based formulation for the isochoric component and a mixed approach for the volumetric terms [5]. In the present framework, this leads to the following hybrid mixed variational principle

$$
\begin{aligned}
\hat{\Pi}_M(\boldsymbol{x}^*, J^*, p^*) = \inf_{\boldsymbol{x},J}\Bigg\{\sup_{p}\Bigg\{ & \int\limits_V \hat{W}\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right) dV \\
& + \int\limits_V U\left(J\right) dV + \int\limits_V p\left(J_{\boldsymbol{x}} - J\right) dV \\
& - \int\limits_V \boldsymbol{f}_0 \cdot \boldsymbol{x}\, dV - \int\limits_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x}\, dA \Bigg\}\Bigg\},
\end{aligned} \tag{171}
$$

where $\hat{W}$ and U are the isochoric and volumetric components of the strain energy defined in Section 3.7. Note that, in general $\hat{W}$, will be a direct function of the volume ratio. This volume ratio is expressed differently in the two terms making up the strain energy: it is directly evaluated from the geometry in the isochoric strain energy, whereas it is expressed as an independent variable J in the volumetric component. The third integral term above enforces the compatibility between these two measures. The particular case of full incompressibility can be obtained by simply taking $J = 1$ in the above expression to give

$$\hat{\Pi}^I_M(\boldsymbol{x}^*, p^*) = \inf_{\boldsymbol{x},J} \left\{ \sup_p \left\{ \int_V \hat{W}\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right) dV + \int_V p\left(J_{\boldsymbol{x}} - 1\right) dV \right.\right.$$
$$\left.\left. - \int_V \boldsymbol{f}_0 \cdot \boldsymbol{x}\, dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x}\, dA \right\}\right\}. \tag{172}$$

The stationary conditions of these hybrid functionals are evaluated in the same fashion as above. For instance, the first derivative with respect to geometry gives the principle of virtual work as

$$D_1\hat{\Pi}_M\left[\delta\boldsymbol{v}\right] = D_1\hat{\Pi}^I_M\left[\delta\boldsymbol{v}\right] = \int_V \boldsymbol{P}_I : \boldsymbol{\nabla}_0\delta\boldsymbol{v}\, dV - \int_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v}\, dV$$
$$- \int_{\partial_t V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v}\, dA = 0, \tag{173}$$

where the first Piola-Kirchoff stress tensor is now evaluated as

$$\boldsymbol{P}_I = \boldsymbol{\Sigma}^{\boldsymbol{x}}_{\boldsymbol{F}} + \boldsymbol{\Sigma}^{\boldsymbol{x}}_{\boldsymbol{H}} \times \boldsymbol{F}_{\boldsymbol{x}} + \Sigma_J \boldsymbol{H}_{\boldsymbol{x}}; \quad \Sigma_J = \hat{\Sigma}^{\boldsymbol{x}}_J + p \tag{174}$$

and the last term in (174) indicates that the volumetric conjugate stress includes a component due to the independent variable p as well as a contribution due to the isochoric strain energy function as

$$\hat{\Sigma}^{\boldsymbol{x}}_J = \frac{\partial\hat{W}\left(\boldsymbol{F}_{\boldsymbol{x}}, \boldsymbol{H}_{\boldsymbol{x}}, J_{\boldsymbol{x}}\right)}{\partial J_{\boldsymbol{x}}}. \tag{175}$$

The first derivative with respect to J enforces the volumetric component of the constitutive model as

$$D_2\hat{\Pi}_M\left[\delta J\right] = \int_V \left(U'(J) - p\right)\delta J\, dV = 0. \tag{176}$$

Finally, the stationary condition with respect to the pressure enforces geometric compatibility between J and $J_{\boldsymbol{x}}$ as

$$\begin{aligned} D_3\hat{\Pi}_M\,[\delta p] &= \int_V (J_{\boldsymbol{x}} - J)\,\delta p\,dV = 0; \\ D_2\hat{\Pi}_M^I\,[\delta p] &= \int_V (J_{\boldsymbol{x}} - 1)\,\delta p\,dV = 0. \end{aligned} \tag{177}$$

6.5. Alternative mixed variational principles

The enforcement of the geometrical compatibility constraints in the mixed principles presented above can be formulated in a variety of forms. For instance, the constraint for the area map $\boldsymbol{H}$ can be expressed directly in terms of $\boldsymbol{F}_{\boldsymbol{x}}$ or indirectly in terms of $\boldsymbol{F}$. Similarly, J can be related to $\det \boldsymbol{F}$ or to $\frac{1}{3}\boldsymbol{H} : \boldsymbol{F}$ or even to $\frac{1}{3}\boldsymbol{H} : \boldsymbol{F}_{\boldsymbol{x}}$. In this way, alternative variational principles may be constructed. As an example of the resulting type of functional consider the expression

$$\begin{aligned} \tilde{\Pi}_M(\boldsymbol{x}^*, \boldsymbol{F}^*, \boldsymbol{H}^*, J^*, \boldsymbol{\Gamma}_{\boldsymbol{F}}^*, \boldsymbol{\Gamma}_{\boldsymbol{H}}^*, \Gamma_J^*) = \inf_{\boldsymbol{x},\boldsymbol{F},\boldsymbol{H},J} \Bigg\{ \sup_{\boldsymbol{\Gamma}_{\boldsymbol{F}},\boldsymbol{\Gamma}_{\boldsymbol{H}},\Gamma_J} \Bigg\{ &\int_V W(\boldsymbol{F}, \boldsymbol{H}, J)\,dV \\ &+ \int_V \boldsymbol{\Gamma}_{\boldsymbol{F}} : (\boldsymbol{F}_{\boldsymbol{x}} - \boldsymbol{F})\,dV \\ &+ \int_V \boldsymbol{\Gamma}_{\boldsymbol{H}} : (\tfrac{1}{2}\boldsymbol{F} \times \boldsymbol{F}_{\boldsymbol{x}} - \boldsymbol{H})\,dV \\ &+ \int_V \Gamma_J(\tfrac{1}{3}\boldsymbol{H} : \boldsymbol{F}_{\boldsymbol{x}} - J)\,dV \\ &- \int_V \boldsymbol{f}_0 \cdot \boldsymbol{x}\,dV - \int_{\partial_t V} \boldsymbol{t}_0 \cdot \boldsymbol{x}\,dA \Bigg\}\Bigg\}. \end{aligned} \tag{178}$$

Note that the stress variables $\{\boldsymbol{\Gamma}_{\boldsymbol{F}}, \boldsymbol{\Gamma}_{\boldsymbol{H}}, \Gamma_J\}$ in this expression are simply Lagrange multipliers and will generally not coincide with the conjugate stresses $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J\}$ as shown below.

The stationary condition of the above Lagrangian with respect to the first variable enforces equilibrium in the form of the principle of virtual work as

$$D_1\tilde{\Pi}_M\left[\delta\boldsymbol{v}\right] = \int\limits_V \tilde{\boldsymbol{P}}_M : \boldsymbol{\nabla}_0\delta\boldsymbol{v}\,dV - \int\limits_V \boldsymbol{f}_0 \cdot \delta\boldsymbol{v}\,dV - \int\limits_{\partial_t V} \boldsymbol{t}_0 \cdot \delta\boldsymbol{v}\,dA = 0, \tag{179}$$

where the first Piola-Kirchhoff stress now emerges as

$$\tilde{\boldsymbol{P}}_M = \boldsymbol{\Gamma}_{\boldsymbol{F}} + \tfrac{1}{2}\boldsymbol{\Gamma}_{\boldsymbol{H}} \times \boldsymbol{F} + \tfrac{1}{3}\Gamma_J\boldsymbol{H}. \tag{180}$$

The stationary conditions with respect to the three strain variables enforce the constitute relationships between the stress multipliers and the derivatives of the strain energy in a weak form

$$\begin{aligned} D_{2,3,4}\tilde{\Pi}_M\left[\delta\boldsymbol{F}, \delta\boldsymbol{H}, \delta J\right] = &\int\limits_V \left(\frac{\partial W}{\partial \boldsymbol{F}} - \boldsymbol{\Gamma}_{\boldsymbol{F}} + \tfrac{1}{2}\boldsymbol{\Gamma}_{\boldsymbol{H}} \times \boldsymbol{F}_{\boldsymbol{x}}\right) : \delta\boldsymbol{F}\,dV \\ &+ \int\limits_V \left(\frac{\partial W}{\partial \boldsymbol{H}} - \boldsymbol{\Gamma}_{\boldsymbol{H}} + \tfrac{1}{3}\Gamma_J\boldsymbol{F}_{\boldsymbol{x}}\right) : \delta\boldsymbol{H}\,dV \\ &+ \int\limits_V \left(\frac{\partial W}{\partial J} - \Gamma_J\right)\delta J\,dV = 0. \end{aligned} \tag{181}$$

Note that for sufficiently rich function spaces the above equation gives relationships between the stress Lagrange multipliers and the conjugate stresses as

$$\Gamma_J = \Sigma_J; \quad \Sigma_J = \frac{\partial W}{\partial J}; \tag{182}$$

$$\boldsymbol{\Gamma}_{\boldsymbol{H}} = \boldsymbol{\Sigma}_{\boldsymbol{H}} + \tfrac{1}{3}\Sigma_J\boldsymbol{F}_{\boldsymbol{x}}; \quad \boldsymbol{\Sigma}_{\boldsymbol{H}} = \frac{\partial W}{\partial \boldsymbol{H}}; \tag{183}$$

$$\boldsymbol{\Gamma}_{\boldsymbol{F}} = \boldsymbol{\Sigma}_{\boldsymbol{F}} + \tfrac{1}{2}\boldsymbol{\Sigma}_{\boldsymbol{H}} \times \boldsymbol{F}_{\boldsymbol{x}} + \tfrac{1}{6}\Sigma_J\,\boldsymbol{F}_{\boldsymbol{x}} \times \boldsymbol{F}_{\boldsymbol{x}}; \quad \boldsymbol{\Sigma}_{\boldsymbol{F}} = \frac{\partial W}{\partial \boldsymbol{F}}. \tag{184}$$

Substituting these relationships into equation (180) gives a hybrid relationship for the first Piola-Kirchhoff stress tensor as:

$$\tilde{\boldsymbol{P}}_M = \boldsymbol{\Gamma}_{\boldsymbol{F}} + \boldsymbol{\Sigma}_{\boldsymbol{H}} \times \bar{\boldsymbol{F}} + \Sigma_J\bar{\boldsymbol{H}}. \tag{185}$$

where the average fibre and area maps are:

$$\bar{\boldsymbol{F}} = \tfrac{1}{2}(\boldsymbol{F} + \boldsymbol{F}_{\boldsymbol{x}}); \tag{186}$$

$$\bar{\boldsymbol{H}} = \tfrac{1}{3}(\boldsymbol{H} + \boldsymbol{F}_{\boldsymbol{x}} \times \bar{\boldsymbol{F}}). \tag{187}$$

Finally, the stationary conditions with respect to the stress variables enforce the geometric compatibility conditions between strains and geometry

$$
\begin{aligned}
D_{5,6,7}\tilde{\Pi}_M\left[\delta\boldsymbol{\Sigma}_{\boldsymbol{F}}, \delta\boldsymbol{\Sigma}_{\boldsymbol{H}}, \delta\Sigma_J\right] = & \int\limits_V \delta\boldsymbol{\Sigma}_{\boldsymbol{F}} : \left(\boldsymbol{F}_{\boldsymbol{x}} - \boldsymbol{F}\right)\, dV \\
& + \int\limits_V \delta\boldsymbol{\Gamma}_{\boldsymbol{H}} : \left(\tfrac{1}{2}\boldsymbol{F} \times \boldsymbol{F}_{\boldsymbol{x}} - \boldsymbol{H}\right)\, dV \\
& + \int\limits_V \delta\Gamma_J \left(\tfrac{1}{3}\boldsymbol{H} : \boldsymbol{F}_{\boldsymbol{x}} - J\right)\, dV.
\end{aligned} \tag{188}
$$

7. Concluding remarks

This paper has provided a novel approach to formulate polyconvex large strain elasticity using a simplified algebra provided by the tensor cross product, originally presented by de Boer [1] and recently re-introduced by Bonet et al. in [2] and [3]. The key novel contributions of the work presented here are:

- The use of the tensor cross product and its properties to define the area map, its derivatives and the derivatives of the volume map, which leads to much simpler algebra to that commonly used in the past for large strain elasticity.
- The definition of stresses $\{\boldsymbol{\Sigma}_{\boldsymbol{F}}, \boldsymbol{\Sigma}_{\boldsymbol{H}}, \Sigma_J\}$ conjugate to the main extended kinematic variable set $\{\boldsymbol{F}, \boldsymbol{H}, J\}$, which are elegantly related to the classical stress tensors, and the introduction of a convex complementary strain energy functional in terms of this new set of conjugate stresses.
- The derivation of compatibility and equilibrium equations for the conjugate set of stresses.
- The development of a new set of formulae for material and spatial elasticity tensor in a manner that clearly separates physical components from geometrical dependencies.
- The application of the proposed methodology for isotropic and transversely isotropic constitutive models where the strain energy can be expressed as functions of a set of invariants

- The development of a series of mixed and complementary variational principles which enforce equilibrium in the form of a principle of virtual work together with the geometric compatibility constraints in a weak form.

Throughout the paper, the simple case of both compressible and nearly incompressible Mooney-Rivlin materials has been used as an example of application of the methodology proposed. Future work will consider the extension of the present framework to electromechanical phenomena.

Acknowledgements

The authors acknowledge the financial support provided by the Sêr Cymru National Research Network for Advanced Engineering and Materials. The second author acknowledges the financial support received through "The Leverhulme Prize" awarded by The Leverhulme Trust, UK.

References

[1] R. de Boer, Vektor- und Tensorrechnung für Ingenieure, Springer-Verlag, 1982.

[2] J. Bonet, A. J. Gil, R. Ortigosa, A computational framework for polyconvex large strain elasticity, Computer Methods in Applied Mechanics and Engineering 283 (2015) 1061 – 1094.

[3] J. Bonet, A. J. Gil, C. H. Lee, M. Aguirre, R. Ortigosa, A first order hyperbolic framework for large strain computational solid dynamics. Part I: Total Lagrangian isothermal elasticity, Computer Methods in Applied Mechanics and Engineering 283 (2015) 689 – 732.

[4] J. Bonet, R. D. Wood, Nonlinear Continuum Mechanics for Finite Element Analysis, Cambridge University Press, second edition, 2008.

[5] E. A. de Souza Neto, D. Perić, D. R. J. Owen, Computational Methods for Plasticity. Theory and Applications, John Wiley and Sons, 2008.

[6] T. J. R. Hughes, The finite element method: Linear static and dynamic finite element analysis, Dover Publications, 2000.

[7] T. Belytschko, W. K. Liu, B. Moran, Nonlinear finite elements for continua and structures, John Wiley and Sons, 2000.

[8] J. Bonet, A. J. Gil, R. D. Wood, R. Said, R. V. Curtis, Simulating superplastic forming, Computer Methods in Applied Mechanics and Engineering 195 (2006) 6580–6603.

[9] A. J. Gil, R. V. Curtis, J. Bonet, T. Coward, Finite element superplastic forming (FE-SPF) of patient-specific maxillofacial prostheses, International Journal for Numerical Methods in Biomedical Engineering 26 (2010) 139–155.

[10] O. C. Zienkiewicz, J. Rojek, R. L. Taylor, M. Pastor, Triangles and tetrahedra in explicit dynamic codes for solids, International Journal for Numerical Methods in Engineering 43 (1998) 565–583.

[11] M. W. Gee, C. R. Dohrmann, S. W. Key, W. A. Wall, A uniform nodal strain tetrahedron with isochoric stabilization, International Journal for Numerical Methods in Engineering 78 (2009) 429–443.

[12] A. J. Gil, Structural analysis of prestressed Saint Venant-Kirchhoff hyperelastic membranes subjected to moderate strains, Computers & Structures 84 (2006) 1012 – 1028.

[13] A. J. Gil, J. Bonet, Finite element analysis of partly wrinkled reinforced prestressed membranes, Computational Mechanics 40 (2007) 595 – 615.

[14] A. J. Gil, J. Bonet, Finite element analysis of prestressed structural membranes, Finite Elements in Analysis and Design 42 (2006) 683 – 697.

[15] J. Bonet, Large strain viscoelastic constitutive models, International Journal of Solids and Structures 38 (2001) 2953 – 2968.

[16] R. Ortigosa, A. J. Gil, J. Bonet, A computational framework for polyconvex large strain elasticity for geometrically exact beam theory, Computational Mechanics (2015).

[17] R. Hill, On uniqueness and stability in the theory of finite elastic strain, Journal of the Mechanics and Physics of Solids 5 (1957) 229–241.

[18] B. D. Coleman, W. Noll, On the thermostatics of continuous media, Archive for Rational Mechanics and Analysis 4 (1959) 97–128.

[19] P. G. Ciarlet, Mathematical Elasticity. Volume 1: Three Dimensional Elasticity, Elsevier, 1988.

[20] O. Gonzalez, A. M. Stuart, A first course in Continuum Mechanics, Cambridge University Press, 2008.

[21] J. E. Marsden, T. J. R. Hughes, Mathematical foundations of elasticity, Dover Civil and Mechanical Engineering, 1994.

[22] J. M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, Archive for Rational Mechanics and Analysis 63 (1977) 337–403.

[23] J. M. Ball, Some open problems in Elasticity, in: P. Newton, P. Holmes, A. Weinstein (Eds.), Geometry, Mechanics and Dynamics, Springer-Verlag, 2002, pp. 3–59.

[24] J. M. Ball, F. Murat, $W^{1,p}$-quasiconvexity and variational problems for multiple integrals, Journal of Functional Analysis 58 (1984) 225 – 253.

[25] J. M. Ball, Energy-minimising configurations in nonlinear elasticity, in: Proceedings of the International Congress of Mathematicians, Warsaw.

[26] K. Zhang, A construction of quasiconvex functions with linear growth at infinity, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 97 (1992) 313–326.

[27] B. Dacorogna, Direct Methods in the Calculus of Variations, Springer, 2008.

[28] P. Ciarlet, Existence theorems in intrinsic nonlinear elasticity, Journal des mathématiques pures et appliqués 94 (2010) 229–243.

[29] J. Schröder, P. Neff, Invariant formulation of hyperelastic transverse isotropy based on polyconvex free energy functions, International Journal of Solids and Structures 40 (2003) 401–445.

[30] J. Schröder, P. Neff, V. Ebbing, Anisotropic polyconvex energies on the basis of crystallographic motived structural tensors, Journal of the Mechanics and Physics of Solids 56 (2008) 3486–3506.

[31] J. Schröder, P. Neff, V. Ebbing, Polyconvex energies for trigonal, tetragonal and cubic symmetry groups, in: IUTAM-Symposium on variational concepts with applications to the mechanics of materials : Proceedings of the IUTAM-Symposium.

[32] J. Schröder, P. Wriggers, D. Balzani, A new mixed finite element based on different approximations of the minors of deformation tensors, Computer Methods in Applied Mechanics and Engineering 200 (2011) 3583–3600.

[33] J. Schröder, Anisotropic polyconvex energies, in: J. Schröder, P. Neff (Eds.), Poly-, quasi- and rank-one convexity in Applied Mechanics, volume 516 of CISM Courses and Lectures, volume 516 of CISM Courses and Lectures, Springer-Verlag, 2010, pp. 53–105.

[34] N. Kambouchev, R. Radovitzky, J. Fernández, Anisotropic materials which can be modeled by polyconvex strain energy functions, in: 47^{th} AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, volume AIAA 2006-2250.

[35] R. Ogden, Non-linear Elastic Deformations, Dover Civil and Mechanical Engineering Series, Dover Publications, 1997.

[36] D. Gao, Duality Principles in Nonconvex Systems: Theory, Methods and Applications, Nonconvex Optimization and Its Applications, Springer, 2000.

[37] S. Timoshenko, Theory of Elasticity, Engineering societies monographs, McGraw-Hill, 1934.

[38] C. Sansour, On the physical assumptions underlying the volumetric-isochoric split and the case of anisotropy, European Journal of Mechanics - A/Solids 27 (2008) 28–39.

[39] D. Balzani, P. Neff, J. Schröder, G. Holzapfel, A polyconvex framework for soft biological tissues. adjustment to experimental data, International Journal of Solids and Structures 43 (2006) 6052 – 6070.

[40] J. Bonet, A. Burton, A simple orthotropic, transversely isotropic hyperelastic constitutive equation for large strain computations, Computer Methods in Applied Mechanics and Engineering 162 (1998) 151 – 164.

[41] J. C. Simo, F. Armero, Geometrically non-linear enhanced strain mixed methods and the method of incompatible modes, International Journal for Numerical Methods in Engineering 33 (1992) 1413–1449.

[42] J. C. Simo, F. Armero, R. Taylor, Improved versions of assumed enhanced strain tri-linear elements for 3D finite deformation problems, Computational Methods for Applied Mechanical Engineering 110 (1993) 359–386.

[43] K. Washizu, Variational methods in elasticity and plasticity, Pergamon Press, Oxford, 1975.