Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

GLOBAL DYNAMICS OF A PIECE-WISE EPIDEMIC MODEL WITH SWITCHING VACCINATION STRATEGY

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(Communicated by the associate editor name)

Abstract. A piece-wise epidemic model is proposed to describe a switching vaccination program such that it is implemented once the number of people exposed to a disease-causing virus reaches a critical level. We also examine a perturbed system to represent variation or uncertainties in interventions. By using generalized Jacobian theory, Lyapunov constants for a non-smooth vector field and generalized Dulac's criterion, we theoretically analyzed the global dynamic behaviors of the original piece-wise system and the perturbed version. The main results show that as the critical value varies, the piece-wise system will stabilize at the disease-free equilibrium or at the endemic states for the two subsystems or at a generalized equilibrium which is a novel global attractor for non-smooth systems. The perturbed system exhibits new global attractors including a pseudo-focus of parabolic-parabolic (PP) type, a pseudo-equilibrium and a crossing cycle surrounding a sliding mode region. Our findings demonstrate that we can either eradicate an infectious disease by increasing the vaccination rate or by stabilizing the number of infected individuals at a previously given level conditional upon a suitable critical level and parameters.

1. **Introduction.**

Infectious disease remains a major threat to public health around the world [3, 25]. Therefore, designing an effective prevention and control strategy to fight against epidemic outbreaks is a vital task for government and public health officers. Because of the global eradication of smallpox in May 1980, vaccination has gained in prominence as a strategy for the elimination of such diseases such as measles, hepatitis, parotitis, smallpox, and phthisis. Investigation of the impact of

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2010 *Mathematics Subject Classification.* Primary: 92D30, 92B05; Secondary:34C05.

Key words and phrases. piece-wise epidemic model, vaccination, generalized equilibrium, global dynamics, perturbed system, limit cycle.

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vaccines based on mathematical models has received much attention in recent years [1, 8, 30, 13, 12, 21, 20].

Continuous and impulsive vaccination have been proposed as the two main modeling approaches. The former strategy, relying on ordinary differential equations, may lead to epidemic eradication if the vaccination ratio is sufficiently high [1, 8, 30]. The latter refers to repeated application of vaccine at fixed moments so impulsive differential equations are appropriate for it [13, 12, 21, 20]. Infectious diseases can also be eradicated by choosing suitable vaccine dosing intervals [12]. Impulsive vaccination strategy is more realistic for describing the scheduled immunization programme than a continuous vaccination strategy. However, neither of these two strategies consider limited medical resources or represent variations in vaccination strategies when facing emerging infectious diseases. For example, during the 2009 A/H1N1 pandemic, the vaccination strategy in mainland China varied according to provinces or population ages, due to a lack of vaccine against A/H1N1 [10, 11]. Ideally, those who are exposed to the virus should be effectively vaccinated and the vaccination strategy should depend on the number of individuals who are exposed to the virus (susceptibles). However, little attention has been paid to modeling switching vaccination strategies and its effect on dynamic behaviors of infectious diseases, which is the scope of this study.

Our main idea for this study is to propose a piece-wise epidemic model to describe a switching vaccination strategy such that it is implemented only when the number of those exposed to disease-causing virus is greater than a critical level, and there are no vaccinations otherwise. Note that this type of vaccination strategy belongs to a so-called threshold policy (TP) [19, 24, 18] and systems subject to such a policy are called switched systems. Switched systems serve as models for a large number of problems in subjects ranging from mechanics and electrical engineering to biology [16, 23, 2, 15] and epidemiology [31, 32, 28]. Here we examine whether the proposed vaccination strategy affects the evolution of infectious disease in comparison with a general vaccination without switching. To this end, we analyzed the global dynamics of the piece-wise system with switching vaccination strategy on the basis of qualitative theories for non-smooth systems. Moreover, some more interesting and complicated biological phenomena are observed if the system admits a small perturbation.

The rest of our paper is organized as follows. We propose a piece-wise epidemic model with switching vaccination strategy and present some preliminaries in the following section. Section 3 is devoted to the dynamics of two subsystems. The global behavior of the switched system is rigorously investigated in section 4 and a perturbed system is further examined in section 5. In the last section, we present some concluding remarks on the results.

2. **Piece-wise epidemic model with vaccination and preliminaries.**

Consider a population that is divided into three types: susceptible, infective and recovered. Let S, I and V be the number of susceptible, infective and recovered individuals, respectively. Then the model equations take the following form:

$$\begin{cases} \dfrac{dS(t)}{dt} &= A - \dfrac{kSI}{1+\omega I} - \delta S - H(S), \\ \dfrac{dI(t)}{dt} &= \dfrac{kSI}{1+\omega I} - \delta I - \alpha I, \\ \dfrac{dV(t)}{dt} &= H(S) - \delta V, \end{cases} \tag{1}$$

with vaccination policy function

$$H(S) = \frac{\varepsilon h(S - S_c)}{h_m + \varepsilon(S - S_c)} \tag{2}$$

and

$$\varepsilon = \begin{cases} 0, & \sigma(S, I) < 0, \\ 1, & \sigma(S, I) \geq 0. \end{cases} \tag{3}$$

Note that function $\sigma(S, I) = S - S_c$ determines whether vaccination is implemented or not, and parameter S_c is the threshold of susceptibles. Vaccination policy function $H(S)$ represents when no vaccination strategy is adopted if the number of susceptible individuals is below the critical level S_c. In contrast, once the number of susceptibles in the population reaches and exceeds the level S_c, the vaccination strategy is implemented and governed by a nonlinear function in $S - S_c$, given by $H(S) = h(S-S_c)/(h_m+(S-S_c))$. Here h gives the maximum vaccination per unit of time, and h_m is the number of those susceptible individuals $S - S_c$ that yields a 50% chance of the maximum vaccination level being reached (i.e. $H(S_c+h_m) = h/2$) and measures how soon saturation occurs. This vaccination represents a logistic response to the increase in $S - S_c$, that is, when $S - S_c \ll h_m$, H grows linearly with $S - S_c$; when $S - S_c \gg h_m$, H approaches a steady state h, showing the effect of saturation. Parameter A represents the recruitment rate of the population, δ is the natural death rate and α denotes the disease-related death rate, respectively. It is reasonable to assume $A > h$ since the recruitment rate is usually greater than the maximum number of vaccinated individuals.

We assume that the immunity to the virus of vaccinated individuals will persist. Taking into account the 'psychological' effects, we adopt a saturated incidence rate in this model. Indeed, we choose the incidence rate as $kI/(1 + \omega I)$, where kI denotes the infection force of a disease, and $1/(1 + \omega I)$ reflects an inhibition effect resulting from the reducing contact rate as the population size of infected individuals increases. All other parameters are positive constants. Model (1) with (2) and (3) is a description of a dynamical system subject to a threshold policy (TP), which is referred to as a switched system [16]. In particular, it involves a coupling between smooth dynamics of two distinct subsystems with or without implementation of the vaccination strategy and the discrete switching events from the dynamics of one subsystem to the other dynamics. It is a special and simple case of variable structure control in the control literature.

Note that the vaccinated class does not influence the dynamics of the first and second equations of model (1), so we only need to consider the reduced system

$$\begin{cases} \dfrac{dS(t)}{dt} &= A - \dfrac{kSI}{1 + \omega I} - \delta S - H(S), \\ \dfrac{dI(t)}{dt} &= \dfrac{kSI}{1 + \omega I} - \delta I - \alpha I. \end{cases} \tag{4}$$

Let $N = S + I$, and it follows from the definition of $H(S)$ that $\frac{dN}{dt} = A - \delta N - \alpha I - H(S) \leq A - \delta N$, which indicates that any trajectory of system (4) will attain the region Ω ultimately, where

$$\Omega \doteq \Big\{(S, I) \in R_+^2 \mid S + I \leq \frac{A}{\delta}\Big\}.$$

Moreover, since $\frac{dS}{dt}|_{S=0} = A > 0$ and $\frac{dI}{dt}|_{I=0} = 0$, every trajectory of system (4) initiating from some point in Ω remains in the region Ω forever. Hence, Ω is an attraction region of system (4).

Obviously, the (S, I) phase plane is divided into two parts: $G_1 = \{(S, I) \in R_+^2 \mid \sigma(S, I) < 0\}$, $G_2 = \{(S, I) \in R_+^2 \mid \sigma(S, I) \geq 0\}$. We denote the switching boundary by

$$\Sigma = \{(S, I) \in R_+^2 \mid \sigma(S, I) = 0\}. \tag{5}$$

Let vector $Z = (S, I)^T$ and

$$F(Z) = \left(A - \frac{kSI}{1 + \omega I} - \delta S - H(S),\quad \frac{kSI}{1 + \omega I} - \delta I - \alpha I\right)^T,$$

then system (4) can be rewritten as the following switched system

$$\dot{Z} = F(Z) = \begin{cases} F_1(Z), & \sigma(S, I) < 0, \\ F_2(Z), & \sigma(S, I) \geq 0 \end{cases} \tag{6}$$

with

$$\begin{aligned} F_1(Z) &= \left(A - \frac{kSI}{1 + \omega I} - \delta S,\quad \frac{kSI}{1 + \omega I} - \delta I - \alpha I\right)^T \doteq (f_{11}(Z), f_{12}(Z))^T, \\ F_2(Z) &= \left(A - \frac{kSI}{1 + \omega I} - \delta S - \frac{h(S - S_c)}{h_m + (S - S_c)},\quad \frac{kSI}{1 + \omega I} - \delta I - \alpha I\right)^T \\ &\doteq (f_{21}(Z), f_{22}(Z))^T. \end{aligned}$$

It is worth noting that the vector field defined by system (6) is continuous. Further, if it is locally Lipschitz-continuous, the trajectory of (6) initiating from any point in R_+^2 exists and is unique. Indeed, we provide a brief examination of the local Lipschtiz-continuity of system (6) in Appendix A.

We call system (6) defined in region G_1 system S_1 and that defined in region G_2 system S_2.

In the following, we introduce two types of equilibria, which will play an important role later in the discussion of this paper.

Definition 2.1. An equilibrium Z^* of system (6) is said to be real if it lies in the region governed by the structure that it originates from, whereas it is called virtual if it is located in another region. Both the real equilibria and virtual equilibria are called regular equilibria.

Definition 2.2. An equilibrium $Z^* = (S_c, I^*)$ is said to be a generalized equilibrium of (6) if $f_{11}(Z^*)f_{21}(Z^*) \leq 0$.

It is known that the main characteristics of trajectories for a smooth system near a non-degenerate equilibrium point of focus type is that it turns around the point. For non-smooth systems, such points are replaced by a type of generalized equilibrium point, i.e. the so-called 'pseudo-focus' point, which consists of four possible types, i.e. focus-focus type (denoted by FF), focus-parabolic type (denoted by FP), parabolic-focus type (denoted by PF) and parabolic-parabolic type (denoted by PP). In the following, we give a brief introduction to the pseudo-foci of FF and PP type of system (6) which will be used in the rest of this paper and the detailed description about other types of pseudo-foci can be found in the literature [6, 9].

Definition 2.3. Let Z^* be a generalized equilibrium point of system (6) and trajectories near it are oriented counter-clockwise.

- Z^* is said to be of FF type if it is a focus for both systems S_1 and S_2.

- Z^* is said to be of PP type if solutions for both systems S_1 and S_2 have a parabolic contact at Z^* with Σ.

Definition 2.4. (i) Let $Z^* = (S_c, I^*)$ be a pseudo-focus of FF type for system (6). It is said to be elementary if Z^* is elementary as an equilibrium point for both systems S_1 and S_2.
(ii) Let $Z^* = (S_c, I^*)$ be a pseudo-focus of PP type for system (6). It is said to be elementary if

$$f_{i1}(Z^*) = 0, \quad \frac{\partial f_{i1}(Z^*)}{\partial I} f_{i2}(Z^*) \neq 0, \quad i = 1, 2.$$

In terms of Filippov theory [7], there are three types of sliding modes, namely, transversal sliding mode, attracting sliding mode and repulsing sliding mode. Since system (6) is continuous, attracting and repulsing sliding modes are excluded and only a transversal sliding mode is allowed. This demonstrates that the vector field is directed from one side to the other at switching boundary $S = S_c$. The trajectories will cross switching boundary $S = S_c$ and the trajectory initiating from one specified point is unique.

3. **Dynamics of two subsystems.** In this section, we examine the existence of all possible equilibria and their local stability. We initially consider the equilibria of system S_1, the basic reproduction number of which is $R_{01} = Ak/(\delta(\delta+\alpha))$. It is easy to get that the disease free equilibrium $E_{01} = (A/\delta, 0)$ is a locally stable node for $R_{01} < 1$, while it is a saddle for $R_{01} > 1$. According to Definition 2.1, E_{01} is a real equilibrium (denoted by E_{01}^r) for $S_c > A/\delta$, and it is a virtual equilibrium (denoted by E_{01}^v) for $S_c < A/\delta$. When $R_{01} > 1$, the endemic equilibrium $E_1 = (S_1^*, I_1^*)$ is feasible and locally asymptotically stable with

$$S_1^* = \frac{A\omega + \delta + \alpha}{k + \omega\delta}, \qquad I_1^* = \frac{Ak - \delta(\delta + \alpha)}{(\delta + \alpha)(k + \omega\delta)}.$$

It is a real or virtual equilibrium (represented by E_1^r or E_1^v) for $S_c > S_1^*$ or $S_c < S_1^*$, respectively. Furthermore, it is a node or a focus if $\eta_1 \geq 0$ or $\eta_1 < 0$, where

$$\eta_1 = \frac{\{\delta + [k + \omega(2\delta + \alpha)]I_1^*\}^2}{(1 + \omega I_1^*)^2} - 4\det(J_1(S_1^*, I_1^*))$$

and

$$J_1(S, I) = \begin{pmatrix} -\delta - \dfrac{kI}{1 + \omega I} & -\dfrac{kS}{(1 + \omega I)^2} \\ \dfrac{kI}{1 + \omega I} & -(\delta + \alpha) + \dfrac{kS}{(1 + \omega I)^2} \end{pmatrix}.$$

Consequently, we have

Proposition 1. *For system S_1, disease-free equilibrium $E_{01} = (A/\delta, 0)$ is a locally asymptotically stable node for $R_{01} < 1$ and a saddle for $R_{01} > 1$. Unique endemic equilibrium $E_1 = (S_1^*, I_1^*)$ is feasible and locally asymptotically stable if $R_{01} > 1$. Further, E_1 is a stable node for $\eta_1 \geq 0$ and a stable focus for $\eta_1 < 0$.*

Now we will investigate all possible equilibria of system S_2. It is worth noting that virtual equilibrium points locate in their opposite region, so they cannot be attained although they are locally stable. That is because the dynamics change once the trajectories cross the switching boundary $\sigma(S, I) = 0$. Therefore, we only focus on the real equilibria of system S_2 in the following, but for completeness we also analyze the virtual equilibria in Appendix A.

For system S_2, the disease free equilibria satisfy the equations

$$\begin{cases} A - \delta S - \dfrac{h(S-S_c)}{h_m + (S-S_c)} = 0, \\ I = 0. \end{cases} \tag{7}$$

The first equation of (7) is equivalent to

$$\delta S^2 - [A - h + \delta(S_c - h_m)]\, S - [hS_c - A(S_c - h_m)] = 0, \tag{8}$$

solving which yields

$$S_{02} = \frac{\delta S_c - \delta h_m + A - h + \sqrt{(\delta S_c - \delta h_m + A - h)^2 + 4\delta(Ah_m + hS_c - AS_c)}}{2\delta},$$
$$S_{03} = \frac{\delta S_c - \delta h_m + A - h - \sqrt{(\delta S_c - \delta h_m + A - h)^2 + 4\delta(Ah_m + hS_c - AS_c)}}{2\delta}.$$

We denote $E_{02} = (S_{02}, 0), E_{03} = (S_{03}, 0)$ in the following.

Taking $x = S - S_c$, the first equation of (7) reads

$$\frac{hh_m}{x + h_m} = \delta x + \delta S_c + h - A. \tag{9}$$

It follows that $E_{02} = (S_{02}, 0)$ is real if a positive root for equation (9) exists and vice versa; $E_{03} = (S_{03}, 0)$ is always virtual if it is well defined. Equation (9) possesses a positive root if and only if $g_1(0) > g_2(0)$ (i.e., $S_c < A/\delta$). As a conclusion, there exists a unique real disease free equilibrium (denoted by $E_{02}^r = (S_{02}, 0)$) for system S_2 provided $S_c < A/\delta$. For the existence of virtual equilibria (denoted by $E_{02}^v = (S_{02}, 0), E_{03}^v = (S_{03}, 0)$), see Appendix A.

Further, E_{02} is stable for $R_{02} < 1$ and unstable for $R_{02} > 1$, where the basic reproduction number $R_{02} = kS_{02}/(\delta + \alpha)$.

Next we consider the existence of endemic equilibria for system S_2, which satisfy the equations

$$\begin{cases} S = \dfrac{(\delta+\alpha)(1+\omega I)}{k}, \\ (A-h) - \delta S - (\delta+\alpha)I + \dfrac{hh_m}{h_m + (S - S_c)} = 0. \end{cases} \tag{10}$$

Substituting the first one of (10) into the second one yields

$$\frac{hh_m k^2}{\omega(\delta+\alpha)I + (\delta+\alpha) + k(h_m - S_c)} = (\delta+\alpha)(k+\omega\delta)I + \delta(\delta+\alpha) + k(h - A), \tag{11}$$

which is equivalent to

$$\omega(\delta+\alpha)^2(k+\omega\delta)I^2 + \Phi_1(S_c)I + \Phi_2(S_c) = 0 \tag{12}$$

with

$$\Phi_1(S_c) = \omega(\delta+\alpha)\,[\delta(\delta+\alpha) + k(h-A)] + (\delta+\alpha)(k+\omega\delta)\,[(\delta+\alpha) + kh_m - kS_c]\,,$$
$$\Phi_2(S_c) = (\delta+\alpha-kS_c)\,[\delta(\delta+\alpha) + k(h-A)] + kh_m\,[\delta(\delta+\alpha) - Ak]\,.$$

We denote $E_2 = (S_2^*, I_2^*), E_3 = (S_3^*, I_3^*)$, where

$$\begin{aligned} I_2^* &= \frac{-\Phi_1(S_c) + \sqrt{\Phi_1^2(S_c) - 4\omega(\delta+\alpha)^2(k+\omega\delta)\Phi_2(S_c)}}{2\omega(\delta+\alpha)^2(k+\omega\delta)}, \\ I_3^* &= \frac{-\Phi_1(S_c) - \sqrt{\Phi_1^2(S_c) - 4\omega(\delta+\alpha)^2(k+\omega\delta)\Phi_2(S_c)}}{2\omega(\delta+\alpha)^2(k+\omega\delta)}, \\ S_2^* &= \frac{(\delta+\alpha)(1+\omega I_2^*)}{k}, \qquad S_3^* = \frac{(\delta+\alpha)(1+\omega I_3^*)}{k}. \end{aligned} \tag{13}$$

It follows from the first equation of (10) that $S > S_c$ is equivalent to $I > I_c$, where

$$I_c = \frac{kS_c - (\delta+\alpha)}{\omega(\delta+\alpha)}.$$

Let $x = I - I_c$, and equation (11) can be written as

$$g_3(x) = g_4(x), \tag{14}$$

where

$$\begin{aligned} g_3(x) &= \frac{hh_m k^2}{\omega(\delta+\alpha)x + h_m k}, \\ g_4(x) &= (\delta+\alpha)(k+\omega\delta)x + \frac{k(k+\omega\delta)}{\omega}S_c + \frac{\omega k(h-A) - k(\delta+\alpha)}{\omega}. \end{aligned}$$

Equation (11) possesses a root $I_i^*, i = 2,3$ with $I_i^* > \max\{I_c, 0\}$ if and only if equation (14) possesses a root x^* satisfying $x^* > \max\{0, -I_c\}$. For $I_c \geq 0$, we require $g_3(0) > g_4(0)$ to ensure $x^* > I_c > 0$, while for $I_c < 0$, we require $g_3(-I_c) > g_4(-I_c)$ to ensure $x^* > -I_c > 0$. There are two cases (including $I_c \geq 0$ and $I_c < 0$) to consider in the following and we initially investigate the case $I_c \geq 0$.

Note that

$$S_c \geq \frac{\delta+\alpha}{k} \Rightarrow I_c \geq 0, \quad S_c < S_1^* \Rightarrow g_3(0) > g_4(0).$$

Further, if $R_{01} > 1, \delta(\delta+\alpha) + k(h-A) > 0$, the following inequalities hold

$$S_{c0} < \frac{\delta+\alpha}{k} < S_1^* < \frac{A}{\delta}, \quad S_{c1} > S_{c0} \tag{15}$$

with

$$\begin{aligned} S_{c0} &= h_m + \frac{\delta+\alpha}{k} - \frac{hh_m k}{\delta(\delta+\alpha) + k(h-A)}, \\ S_{c1} &= h_m + \frac{\delta+\alpha}{k} + \frac{\omega\left[\delta(\delta+\alpha) + k(h-A)\right]}{k(k+\omega\delta)}, \end{aligned}$$

which will be useful in the rest of this work.

A similar discussion yields

$$\frac{\delta+\alpha}{k} < S_1^* < \frac{A}{\delta}, \quad S_{c0} > \frac{\delta+\alpha}{k}, \quad S_{c1} < S_{c0} \tag{16}$$

are true for $R_{01} > 1, \delta(\delta+\alpha) + k(h-A) < 0$.

Then we conclude that there is only one real endemic equilibrium (denoted by E_2^r) for system S_2 if

$$R_{01} > 1, \quad \frac{\delta+\alpha}{k} \leq S_c < S_1^*.$$

Now we turn to examine the case $I_c < 0$. In such scenarios, an equilibrium is real provided it is well defined. It follows from (14) that there is a unique endemic equilibrium (i.e. E_2) when $g_3(-I_c) > g_4(-I_c)$ and no endemic equilibrium exists when $g_3(-I_c) \leq g_4(-I_c)$. Furthermore, if

$$\delta(\delta+\alpha) + k(h-A) \leq 0, \quad S_c < h_m + \frac{\delta+\alpha}{k}$$

or

$$\delta(\delta+\alpha) + k(h-A) > 0, \quad S_c > S_{c0},$$

then $g_3(-I_c) > g_4(-I_c)$ holds. We also note that

$$S_c < \frac{\delta+\alpha}{k} \Rightarrow I_c < 0.$$

Therefore, based on (15) and (16), if

$$R_{01} > 1, \quad \delta(\delta+\alpha) + k(h-A) \le 0, \quad S_c < \frac{\delta+\alpha}{k}$$

or

$$R_{01} > 1, \quad \delta(\delta+\alpha) + k(h-A) > 0, \quad S_{c0} < S_c < \frac{\delta+\alpha}{k}$$

are true, system S_2 possesses a unique real endemic equilibrium (i.e. E_2^r).

In addition to the existence of the real endemic equilibrium (i.e. E_2^r) for system S_2, we address the existence of virtual endemic equilibria (denoted by E_2^v, E_3^v) for system S_2 in detail in Appendix A.

Further, endemic equilibrium E_2 is a stable node for $\eta_2 \ge 0$ and a stable focus for $\eta_2 < 0$, where

$$\eta_2 = \left[\frac{\delta + kI_2^* + \omega(2\delta+\alpha)I_2^*}{1+\omega I_2^*} + \frac{hh_m}{(S_2^* - S_c + h_m)^2}\right]^2 - 4\det(J_2(S_2^*, I_2^*))$$

and

$$J_2(S,I) = \begin{pmatrix} -\delta - \dfrac{kI}{1+\omega I} - \dfrac{hh_m}{(S-S_c+h_m)^2} & -\dfrac{kS}{(1+\omega I)^2} \\ \dfrac{kI}{1+\omega I} & -(\delta+\alpha) + \dfrac{kS}{(1+\omega I)^2} \end{pmatrix}.$$

Summarizing the above discussion, we obtain the following result.

Proposition 2. *For system S_2, there exists a real disease-free equilibrium $E_{02} = (S_{02}, 0)$ if $S_c < A/\delta$, which is a locally asymptotically stable node for $R_{02} < 1$ and a saddle for $R_{02} > 1$. If conditions*

$$\delta(\delta+\alpha) + k(h-A) \le 0, \quad S_c < S_1^* \tag{17}$$

or

$$\delta(\delta+\alpha) + k(h-A) > 0, \quad S_{c0} < S_c < S_1^* \tag{18}$$

hold true, there is a unique real endemic equilibrium $E_2 = (S_2^*, I_2^*)$ which is locally asymptotically stable. Furthermore, E_2 is a stable node for $\eta_2 \ge 0$ and a stable focus for $\eta_2 < 0$.

Remark 1. (i) It is impossible that all equilibria of system S_1 and S_2 are virtual or real. In particular, if E_{01} (E_1) is real, then E_{02} (E_2) is virtual and hence no disease-free equilibrium (endemic equilibrium) of system S_2 is real; if E_{01} (E_1) is virtual, then disease-free equilibrium E_{02} (endemic equilibrium E_2) of system S_2 is real.
(ii) It follows from formula (13) that I_2^* varies with variation of threshold value S_c and further

$$\operatorname{sgn}\left\{\frac{\delta I_2^*}{dS_c}\right\} = \operatorname{sgn}\left\{(k+\omega\delta)S_c - \nu + \sqrt{\left[(k+\omega\delta)S_c - \nu\right]^2 + 4\omega h h_m(k+\omega\delta)}\right\},$$

where $\nu = \omega(A-h) + (\delta+\alpha) + h_m(k+\omega\delta)$. Then we get $\dfrac{dI_2^*}{dS_c} > 0$ and so I_2^* is monotonically increasing with respect to variable S_c.

4. **Global dynamics.** The objective of this section is to perform a global qualitative analysis of system (6). To investigate the richness of the dynamics that switched system (6) can exhibit, we consider all possible combinations of parameters in the following. The basic transmission rate k and threshold level S_c are selected to build the bifurcation diagram and all other parameters are specified as given in legend of Figure 1. We define five critical curves in parameter plane $S_c - k$ as follows:

$$L_1 = \left\{ (S_c, k) \mid k = \frac{\delta(\delta+\alpha)}{A} \right\}, \qquad L_2 = \left\{ (S_c, k) \mid S_c = \frac{A}{\delta} \right\},$$
$$L_3 = \left\{ (S_c, k) \mid S_c = S_1^* \right\}, \qquad L_4 = \left\{ (S_c, k) \mid k = \frac{\delta(\delta+\alpha)}{A-h} \right\},$$
$$L_5 = \left\{ (S_c, k) \mid S_c = S_{c0} \right\}.$$

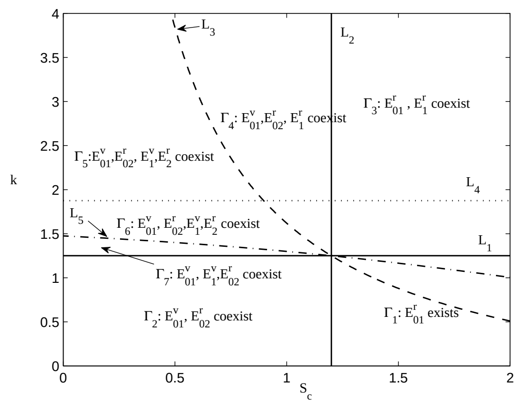

FIGURE 1. Bifurcation set for switched model (6) with respect to basic transmission rate k and threshold level S_c. We fix all other parameters as follows: $A = 0.6, \omega = 1.2, \delta = 0.5, h = 0.2, h_m = 1.2, \alpha = 1$. Let Γ_1 be the region bounded by lines $L_1, L_2, S_c = 2$ and S_c-axis; Γ_2 be the region bounded by lines L_1, L_2 and the two axes; Γ_3 be the region enclosed by lines $L_1, L_2, S_c = 2$ and line $k = 4$; Γ_4 be the region bounded by lines L_2, L_3 and line $k = 4$; Γ_5 be the region delimited by lines $L_3, L_4, k = 4$ and $k-$axis; Γ_6 be the region bounded by lines L_3, L_4, L_5 and $k-$axis; Γ_7 be the region bounded by lines L_1, L_5 and $k-$axis.

Solid line L_1 divides the parameter space into two parts in terms of whether endemic equilibrium E_1 of system S_1 exists. Solid line L_2 also divides the parameter space into two parts, in terms of whether disease-free equilibrium E_{01} is real. Dash-dotted line L_5 divides the region bounded by lines $L_2, k = 4$, S_c-axis and $k-$axis into two parts, and the upper region is for $R_{02} > 1$ while the lower region is for $R_{02} < 1$. Dashed-line L_3 divides the region bounded by lines $L_2, L_5, k = 4$ and $k-$axis into two parts in terms of whether endemic equilibrium E_2^r exists. Dotted line L_4 is plotted to delimit the regions where inequalities (17) and (18) are satisfied in the parameter space.

Before showing the global behavior of trajectories of system (6), we first give the following lemma.

Lemma 4.1. *$R_{02} < 1$ if and only if $\delta(\delta+\alpha)+k(h-A)>0$ and $S_c < S_{c0}$.*

In this section we examine the global stability of all possible equilibria including regular equilibria and generalized equilibria. To realize this purpose, we initially need to preclude the existence of limit cycles in attraction region Ω. Note that the vector field defined by system (6) is locally Lipschitz- continuous in Ω but not in C^1, which means that the classical Dulac's criterion cannot be applied to the system. Hence, we shall use the generalized Dulac's criterion to exclude the existence of closed orbits in Ω, which is addressed in [22] and is for locally Lipschitz-continuous planar systems

$$\frac{dZ}{dt} = \Upsilon(Z), \quad Z \in R^2, \tag{19}$$

where $\Upsilon = (\Upsilon_1, \Upsilon_2)$ is a locally Lipschitz-continuous vector field. For convenience, we give a brief introduction of it as follows.

Lemma 4.2. *Let D be a simply connected, bounded and open subset of R^2, and suppose there exists a constant $c > 0$ and a C^1 function $\chi : R^2 \to R$ such that*

$$div(\chi(Z)\Upsilon(Z)) \le -c \quad a.e.\ in \quad D,$$

then every compact limit set of (19) in D consists of equilibria.

Applying Lemma 4.2 to system (6) yields the following conclusion.

Lemma 4.3. *There is no closed orbit in attraction region Ω for system (6).*

Proof. We consider a C^1 function in Ω defined by

$$\chi(S, I) = \frac{1}{SI}.$$

Then for system S_1, we have

$$\begin{aligned} \text{div } \big(\chi(S,I)F_1(S,I)\big) &= \frac{\partial(\chi(S,I)f_{11}(S,I))}{\partial S} + \frac{\partial(\chi(S,I)f_{12}(S,I))}{\partial I} \\ &= -\frac{A}{S^2 I} - \frac{\omega k}{(1+\omega I)^2} \\ &\le -\frac{\omega k \delta^2}{(\delta+\omega A)^2}. \end{aligned}$$

A similar discussion to that for system S_2 yields

$$\begin{aligned} &\text{div } \big(\chi(S,I)F_2(S,I)\big) = \frac{\partial(\chi(S,I)f_{21}(S,I))}{\partial S} + \frac{\partial(\chi(S,I)f_{22}(S,I))}{\partial I} \\ = \quad & -\frac{A}{S^2 I} - \frac{\omega k}{(1+\omega I)^2} - \frac{hh_m}{(h_m+S-S_c)^2 SI} + \frac{h(S-S_c)}{(h_m+S-S_c)S^2 I} \\ \le \quad & -\frac{1}{S^2 I}(A-h) - \frac{\omega k}{(1+\omega I)^2} - \frac{hh_m}{(h_m+S-S_c)^2 SI} \\ \le \quad & -\frac{\omega k \delta^2}{(\delta+\omega A)^2}. \end{aligned}$$

Taking $c = \frac{\omega k \delta^2}{(\delta+\omega A)^2}$, we have

$$\text{div } (\chi(S,I)F(S,I)) \le -c$$

for all $(S, I) \in \Omega$. Hence, the generalized Dulac's criteria can be applied to system (6) in region $\text{Int}(\Omega)$, and it follows that no closed orbit exists in the region. Furthermore, it is not difficult to rule out the existence of a closed orbit on the boundary of Ω and so no closed orbit exists in attraction region Ω. This completes the proof. □

We consider the following two cases according to whether R_{01} is greater than unity.

Case A $R_{01} < 1$ (i.e. $Ak < \delta(\delta+\alpha)$).

In this case, endemic equilibrium E_1 is not feasible due to $R_{01} < 1$ and neither does E_2 since conditions (17) or (18) do not hold true. The disease-free equilibrium E_{01} does exist and it may be real or virtual for $S_c > A/\delta$ or $S_c < A/\delta$, whereas equilibrium E_{02} is real provided it is well defined. On the basis of Lemma 4.3, we know that there is no limit cycle in the attraction region. We can also exclude the existence of limit cycles of system (6) in attraction region Ω by computing the divergence of system (6) based on distribution theory in this case. A detailed introduction about the method is addressed in [17]. Based on the feature of stable disease-free equilibria, we consider the following three subcases.

(A_1) $S_c > A/\delta$ (region Γ_1 shown in Figure 1).

In this scenario, E_{01} is the unique real disease-free equilibrium (denoted by E_{01}^r) and by Proposition 1, we know E_{01}^r is locally asymptotically stable. The nonexistence of limit cycles in the attraction region has been obtained in Lemma 4.3, so E_{01}^r is a global attractor, as shown in Figure 2 (a).

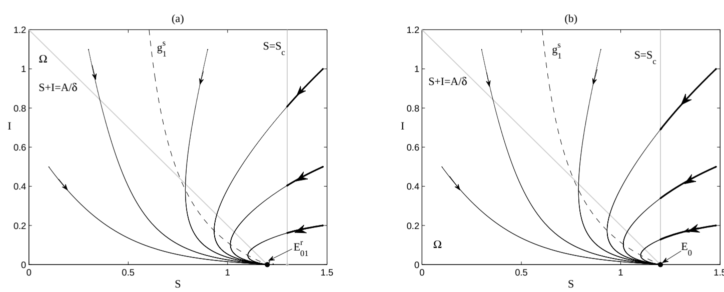

FIGURE 2. Phase plane S-I of non-smooth epidemic model (6) for case A, showing the distinct asymptotic equilibrium for different parameter sets. Vertical isocline g_1^s is plotted for system S_1. The curves represent the orbits in the phase plane indicating the asymptotic equilibrium. Parameter values are: $A = 0.6, \omega = 1.2, \delta = 0.5, h = 0.2, h_m = 1.2, \alpha = 1$ and $k = 1, S_c = 1.3$ (a); $k = 1, S_c = 1.2$ (b).

(A_2) $S_c < A/\delta$ (region Γ_2 shown in Figure 1).

It follows from Definition 2.1 that E_{01} is virtual (denoted by E_{01}^v) and E_{02} is the unique real disease-free equilibrium (denoted by E_{02}^r) in this case. Since $R_{02} < R_{01} < 1$, equilibrium E_{02}^r is locally asymptotically stable. Lemma 4.3 excludes the existence of limit cycles in the attraction region, and hence E_{02}^r is globally asymptotically stable.

(A_3) $S_c = A/\delta$ (region $\Gamma_1 \cap \Gamma_2$ given in Figure 1).

In this case, two disease-free equilibria E_{01} and E_{02} collide together and we denote this by $E_0 = (S_0, 0) = (A/\delta, 0)$, which is actually a generalized equilibrium according to Definition 2.2. It is difficult to study the stability of generalized equilibrium E_0 since the vector field defined by (6) is not C^1 and the classical approach using the Jacobian for smooth vector fields cannot be applied. Here, we introduce the generalized Jacobian developed by Clarke [14, 4, 33]. In such scenarios, the generalized Jacobian at E_0 gives

$$J(S_0,0)=\overline{co}\Big\{J_1(S_0,0),J_2(S_0,0)\Big\}=\Big\{(1-p)J_1(S_0,0)+pJ_2(S_0,0)\ \mid\ p\in[0,1]\Big\}.$$

For any $p\in[0,1]$, it reads

$$J(S,I)=\begin{pmatrix} -\delta-\dfrac{kI}{1+\omega I}-\dfrac{phh_m}{(S-S_c+h_m)^2} & -\dfrac{kS}{(1+\omega I)^2} \\ \dfrac{kI}{1+\omega I} & -(\delta+\alpha)+\dfrac{kS}{(1+\omega I)^2} \end{pmatrix}.$$

Direct computation shows that $J(S_0,0)$ possesses two negative real characteristic values, i.e.

$$\lambda_1=-\delta-\frac{ph}{h_m},\qquad \lambda_2=-(\delta+\alpha)+\frac{kA}{\delta}.$$

Hence, generalized equilibrium point E_0 is locally asymptotically stable. It follows from Lemma 4.3 that no limit cycle exists in the attraction region Ω and so E_0 is globally asymptotically stable, as shown in Figure 2 (b).

In summary, we obtain the following result.

Theorem 4.4. *For system (6), if $R_{01}<1$ the disease dies out. In particular, the disease-free equilibrium E_{01}^r, E_{02}^r or E_0 is globally asymptotically stable for $S_c>A/\delta$, $S_c<A/\delta$ or $S_c=A/\delta$, respectively.*

Case B $R_{01}>1$ (i.e. $Ak>\delta(\delta+\alpha)$).

Since $R_{01}>1$, disease-free equilibrium E_{01} is unstable while endemic equilibrium E_1 is locally stable and it is real or virtual for $S_c>S_1^*$ or $S_c<S_1^*$. The existence of real disease-free equilibrium E_{02} depends on whether $S_c<A/\delta$ or not. If one of the conditions (17) and (18) is true, endemic equilibrium E_2^r is feasible and real. We shall address how global attractors vary with the variation of a combination of parameters. For global qualities of system (6) in this scenario, we consider the following four subcases.

(B_1) $S_c>S_1^*$ (i.e. region $\Gamma_3\cup\Gamma_4$ described in Figure 1).

Equilibrium E_1 is the unique real endemic equilibrium (denoted by E_1^r) and locally asymptotically stable in this scenario, while real disease-free equilibrium E_{02}^r is feasible for $S_c<A/\delta$ (region Γ_4) and unfeasible for $S_c\geq A/\delta$ (region Γ_3). According to (15), (16) and Lemma 4.1, one gets $R_{02}>1$, which implies when $S_c<A/\delta$, E_{02}^r is feasible but unstable. Therefore, only endemic equilibrium E_1^r is locally stable. By Lemma 4.3, we easily conclude that endemic equilibrium E_1^r is globally asymptotically stable, as shown in Figure 3 (a).

(B_2) $S_c<S_1^*,R_{02}>1$ (i.e. region $\Gamma_5\cup\Gamma_6$ given in Figure 1).

Endemic state E_1 is virtual (denoted by E_1^v) and disease-free equilibrium E_{02}^r is unstable. It follows from inequalities (15), (16) and Lemma 4.1 that condition (17) (region Γ_5), or condition (18) (region Γ_6) holds in such scenarios, which implies real endemic equilibrium E_2^r is feasible and locally asymptotically stable. Thus, endemic equilibrium E_2^r is globally asymptotically stable, according to Lemma 4.3.

(B_3) $S_c<S_1^*,R_{02}<1$ (i.e. region Γ_7 given in Figure 1).

A similar discussion to that in (B_2) yields that endemic equilibrium E_1 is virtual and endemic state E_2^r is unfeasible. It follows from Proposition 2 and inequalities (15), (16) that disease-free equilibrium E_{02}^r is feasible and locally asymptotically

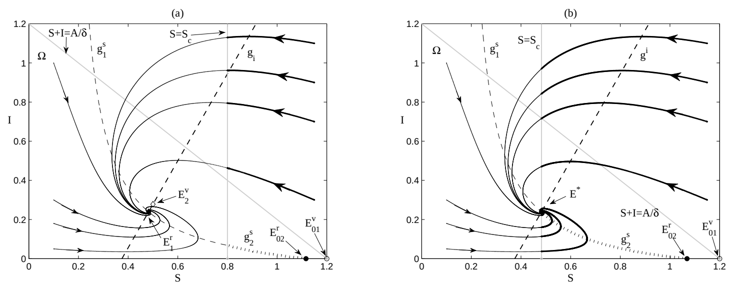

FIGURE 3. Phase plane S-I of non-smooth epidemic model (6) for case B, showing the distinct asymptotical equilibrium for different parameter sets. Vertical isoclines g_1^s and g_2^s are plotted for systems S_1 and S_2, respectively, and g^i denotes the horizontal isocline. The curves represent the orbits in the phase plane indicating the asymptotical equilibrium. Parameter values are: $A = 0.6, \omega = 1.2, \delta = 0.5, h = 0.2, h_m = 1.2, \alpha = 1$ and $k = 4, S_c = 0.8$ (a); $k = 4, S_c = 0.4826$ (b).

stable in this scenario. Therefore, it is globally asymptotically stable by Lemma 4.3.

(B_4) $S_c = S_1^*$ (i.e. region $\Gamma_4 \cap (\Gamma_5 \cup \Gamma_6)$ given in Figure 1).

Two endemic equilibria E_1 and E_2 collide together, and we denote this by E^*, which results in a critical case. It follows that $E^* = (S^*, I^*)$ with $S^* = S_1^*, I^* = I_1^*$, which is a generalized equilibrium in the sense of Definition 2.2. We shall examine it is globally asymptotically stable in the following. To this end, we initially investigate the local stability of E^*.

By implementing a discussion similar to that in subcase (A_3), we derive the generalized Jacobian at E^* as follows

$$\begin{aligned} J(S^*, I^*) &= \overline{co}\{J_1(S^*, I^*), J_2(S^*, I^*)\} = (1-p)J_1(S^*, I^*) + pJ_2(S^*, I^*) \\ &= \begin{pmatrix} -\delta - \dfrac{kI^*}{1+\omega I^*} - \dfrac{ph}{h_m} & -\dfrac{kS^*}{(1+\omega I^*)^2} \\ \dfrac{kI^*}{1+\omega I^*} & -(\delta+\alpha) + \dfrac{kS^*}{(1+\omega I^*)^2} \end{pmatrix}, \end{aligned}$$

with $p \in [0, 1]$. Since

$$\text{tr } (J(S^*, I^*)) = -\frac{\delta + kI^* + \omega(2\delta+\alpha)I^*}{1+\omega I^*} - \frac{ph}{h_m} < 0$$

and

$$\det\ (J(S^*, I^*)) = \left(\delta + \frac{ph}{h_m}\right)\left[(\delta+\alpha) - \frac{kS^*}{(1+\omega I^*)^2}\right] + \frac{(\delta+\alpha)kI^*}{1+\omega I^*} > 0,$$

one gets that the generalized Jacobian J possesses two eigenvalues with negative real parts at point E^* and concludes the following.

Lemma 4.5. *If $R_{01} > 1$ and $S_c = S_1^*$, generalized equilibrium E^* is feasible and locally asymptotically stable.*

By Lemma 4.3, it follows that there is no limit cycle in attraction region Ω, and hence E^* is globally asymptotically stable. In particular, we can identify the type of generalized equilibrium E^*. Denote

$$\overline{\eta}_1 = \frac{\{\delta + [k + \omega(2\delta + \alpha)]I^*\}^2}{(1 + \omega I^*)^2} - 4\det(J_1(S^*, I^*)),$$
$$\overline{\eta}_2 = \left[\frac{\delta + kI^* + \omega(2\delta + \alpha)I^*}{1 + \omega I^*} + \frac{h}{h_m}\right]^2 - 4\det(J_2(S^*, I^*)),$$

where $J_i(S, I), i = 1, 2$ is defined in section 3. (i) When $\overline{\eta}_1 < 0,\ \overline{\eta}_2 < 0$ hold true, then according to Proposition 1 and 2, we know E^* is a focus for both systems S_1 and S_2, so it is a pseudo-focus of FF type in the sense of Definition 2.3 (shown in Figure 3 (b)). It is worth noting that we can also examine the local stability of E^* in such scenarios (i.e. a pseudo-focus of FF type) by presenting the expression of the first Lyapunov constant, according to [6, 9], which is distinct from the one for smooth systems. (ii) When $\overline{\eta}_1 \geq 0,\ \overline{\eta}_2 \geq 0$ hold true, generalized equilibrium point E^* appears as a node for both systems S_1 and S_2 in such scenarios. (iii) When $\overline{\eta}_1\overline{\eta}_2 < 0$ is true, generalized equilibrium point E^* is a focus for system S_1 (S_2), while it is a node for system S_2 (S_1).

Based on the above discussion, we derive the following conclusion.

Theorem 4.6. *For system (6), when $R_{01} > 1$, the disease may either persist or die out. In particular, endemic equilibrium E_1 is globally asymptotically stable if $S_c > S_1^*$; equilibrium E_2 is globally asymptotically stable if $S_c < S_1^*$ and $R_{02} > 1$; disease-free equilibrium E_{02} is globally asymptotically stable if $S_c < S_1^*$ and $R_{02} < 1$; generalized equilibrium E^* is globally asymptotically stable if $S_c = S_1^*$.*

5. A perturbed system and its dynamics.

To model the density-dependent vaccination strategy subject to a threshold policy, we formulate model (1) by ignoring many minor factors such as natural immunity, treatments, migrations and effects of media coverage. However, if we consider these elements as small variables, we can derive a perturbed system which may be a better approximation of the real world. In particular, we can consider the natural acquisition of immunity and treatments on the infected class. Let $\phi(S)$ be the rate of acquiring natural immunity to the virus and $\varphi(I)$ be the treatment rate. Then the perturbed system reads

$$\begin{cases} \dfrac{dS(t)}{dt} &= A - \dfrac{kSI}{1+\omega I} - \delta S - H(S) - \phi(S), \\ \dfrac{dI(t)}{dt} &= \dfrac{kSI}{1+\omega I} - \delta I - \alpha I - \varphi(I), \\ \dfrac{dV(t)}{dt} &= H(S) - \delta V + \phi(S) + \varphi(I) \end{cases} \tag{20}$$

where

$$\phi(S) = \begin{cases} \mu_1, & \sigma(S, I) < 0 \\ \mu_2, & \sigma(S, I) > 0 \end{cases}, \qquad \varphi(I) = \begin{cases} \nu_1, & \sigma(S, I) < 0 \\ \nu_2 I, & \sigma(S, I) > 0 \end{cases}, \tag{21}$$

with $\mu_i > 0, \nu_i > 0 (i = 1, 2)$ sufficiently small. Note that the vaccinated class R does not influence the dynamics of the first two equations of (20), so we only focus only on the following system

$$\begin{cases} \dfrac{dS(t)}{dt} &= A - \dfrac{kSI}{1+\omega I} - \delta S - H(S) - \phi(S), \\ \dfrac{dI(t)}{dt} &= \dfrac{kSI}{1+\omega I} - \delta I - \alpha I - \varphi(I). \end{cases} \tag{22}$$

Denote

$$\begin{aligned}\tilde{F}_1(Z) &= \Big(A-\frac{kSI}{1+\omega I}-\delta S-\mu_1,\quad \frac{kSI}{1+\omega I}-\delta I-\alpha I-\nu_1\Big)^T \doteq (\tilde{f}_{11}(Z),\tilde{f}_{12}(Z))^T,\\ \tilde{F}_2(Z) &= \left(A-\frac{kSI}{1+\omega I}-\delta S-\frac{h(S-S_c)}{h_m+(S-S_c)}-\mu_2,\quad \frac{kSI}{1+\omega I}-\delta I-\alpha I-\nu_2 I\right)^T\\ &\doteq (\tilde{f}_{21}(Z),\tilde{f}_{22}(Z))^T.\end{aligned}$$

Then system (22) becomes

$$\dot{Z}=\begin{cases}\tilde{F}_1(Z), & \sigma(S,I)<0,\\ \tilde{F}_2(Z), & \sigma(S,I)>0.\end{cases} \tag{23}$$

We denote the system defined by $\tilde{F}_i(Z)$ as $S_{pi}, i=1,2$ in the rest of this paper. Since $\mu_i,\nu_i, i=1,2$ are small parameters with small values, system (23) is a perturbed one of system (6). Consequently, the dynamics of system (23) is consistent with that of system (6) in most cases except for critical cases (A_3) and (B_4). We only focus only on the dynamics of system (23) for case (B_4) in the following, and a similar argument for case (A_3) can be made and so we omit it.

Note that the righthand side of system (23) is piecewise continuous, so we take into account its solutions in Filippov's sense, i.e., the solutions of differential inclusions $\dot{Z}(t)\in\tilde{F}(Z(t))$ with

$$\tilde{F}(Z(t))=\bigcap_{\delta>0}\bigcap_{\mu N=0}\overline{\text{co}}(\tilde{F}_1(B(Z;\delta)\backslash N)\cup\tilde{F}_2(B(Z;\delta)\backslash N)),$$

where $\overline{\text{co}}$ represents the convex hull, $B(Z;\delta)$ denotes the open $\delta-$neighbourhood of Z, and μ stands for the Lebesgue measure. By the definition of $\tilde{F}_i, i=1,2$, it follows that $\tilde{F}(Z(t))$ is nonempty, bounded and closed, convex, and it is upper semi-continuous in Z. According to [7], a solution for system (23) with initial value $S(t_0)=S_0, I(t_0)=I_0$ does exist but may not be unique.

Remark 2. As far as perturbation of system (6) is considered, there is more than one form, which means that there are quite a lot of forms for functions $\phi(S)$ and $\varphi(S)$. However, if the coordinates of the generalized equilibrium point (see case (B_4)) remain the same under perturbation, no new phenomena occurs. In particular, if the type of a pseudo-focus including FF type, FP type, PF type and PP type remains unchanged, an attracting or repulsing sliding mode, or limit cycle does not occur. Hence, we consider perturbation (21) in this work.

We can choose perturbed parameters μ_1,ν_1 appropriately such that endemic equilibria $\tilde{E}^v_{11}=(\tilde{S}_{11},\tilde{I}_{11})$ and $\tilde{E}^v_{12}=(\tilde{S}_{12},\tilde{I}_{12})$ of system S_{p1} are virtual, where

$$\begin{aligned}&\tilde{S}_{1i}=\frac{A-\mu_1-\nu_1-(\delta+\alpha)\tilde{I}_{1i}}{\delta},\ i=1,2,\\ &\tilde{I}_{11}=\frac{Ak-\delta(\delta+\alpha)-\nu_1\omega\delta-k(\mu_1+\nu_1)-\sqrt{\xi}}{2(\delta+\alpha)(k+\omega\delta)},\\ &\tilde{I}_{12}=\frac{Ak-\delta(\delta+\alpha)-\nu_1\omega\delta-k(\mu_1+\nu_1)+\sqrt{\xi}}{2(\delta+\alpha)(k+\omega\delta)},\\ &\xi=[Ak-\delta(\delta+\alpha)-\nu_1\omega\delta-k(\nu_1+\mu_1)]^2-4\nu_1\delta(\delta+\alpha)(\omega\delta+k).\end{aligned}$$

Similarly, we can assign a set of appropriate values for μ_2,ν_2 such that endemic equilibrium $\tilde{E}^v_2=(\tilde{S}_2,\tilde{I}_2)$ for system S_{p2} is feasible and virtual, where

$$\tilde{S}_2=\frac{(\delta+\alpha+\nu_2)(1+\omega\tilde{I}_2)}{k},\qquad \tilde{I}_2=\frac{-\tilde{a}_1+\sqrt{\tilde{a}_1^2-4\tilde{a}_0\tilde{a}_2}}{2\tilde{a}_0} \tag{24}$$

and

$$\begin{aligned}
\tilde{a}_0 &= \omega(\delta+\alpha+\nu_2)^2(k+\omega\delta),\\
\tilde{a}_1 &= (\delta+\alpha+\nu_2)(k+\omega\delta)[(\delta+\alpha+\nu_2)+k(h_m-S_c)]+\\
&\quad \omega(\delta+\alpha+\nu_2)[\delta(\delta+\alpha+\nu_2)+k(h-A-\mu_2)],\\
\tilde{a}_2 &= [k(h-A-\mu_2)+\delta(\delta+\alpha+\nu_2)][(\delta+\alpha+\nu_2)+k(h_m-S_c)]-hh_mk^2.
\end{aligned}$$

By (23), equation

$$\tilde{f}_{i1}(S_c, I) = 0, \quad i = 1, 2$$

possesses a unique solution

$$\tilde{I}_{ci} = \frac{A-\mu_i-\delta S_c}{kS_c-\omega(A-\mu_i-\delta S_c)}, \quad i = 1, 2.$$

It follows that trajectories of (23) passing through point $\tilde{A}_i = (S_c, \tilde{I}_{ci}), i = 1, 2$ are tangent to switching boundary $S = S_c$ if $\tilde{A}_i$ is not singular. We consider three cases in the following in terms of the relation of perturbed parameters μ_1 and μ_2.

Case (C_1) $\mu_1 = \mu_2$.

In this case, two points $\tilde{A}_1$ and $\tilde{A}_2$ collide together and we denote this by $\tilde{A}(S_c, \tilde{I}_c)$ with $\tilde{I}_c = (A-\mu_1-\delta S_c)/(kS_c-\omega(A-\mu_1-\delta S_c))$. Furthermore, we can choose appropriate parameter values such that endemic equilibria $\tilde{E}_{11}, \tilde{E}_{12}$ and $\tilde{E}_2$ are all virtual and $\tilde{A}$ is a pseudo-focus of PP type, according to Definition 2.3. In order to determine the global stability of $\tilde{A}$, we initially need to preclude the existence of limit cycles in attraction region Ω. It is worth noting that on the one hand, perturbed system (23) is not continuous any longer, so it cannot be analyzed by applying the generalized Dulac's Theorem; on the other hand, neither can it be completed by calculating the divergence in the sense of distributions. That is because the transversal mode is no longer allowed for every point on the switching boundary due to the existence of pseudo-focus $\tilde{A}$. Thus, we shall establish two lemmas to rule out the existence of any possible limit cycles.

Lemma 5.1. *There is no closed orbit totally within region $G_i, i = 1, 2$.*

Proof. Let $B = 1/(SI)$. In region G_1, we have

$$\frac{\partial(B\tilde{f}_{11})}{\partial S} + \frac{\partial(B\tilde{f}_{12})}{\partial I} = -\frac{A-\mu_1}{S^2I} - \frac{\omega k}{(1+\omega I)^2} + \frac{\nu_1}{SI^2} < 0 \tag{25}$$

for sufficiently small μ_1, ν_1, which indicates that no closed orbit exists in region G_1.

Similarly, in region G_2, one obtains

$$\frac{\partial(B\tilde{f}_{21})}{\partial S} + \frac{\partial(B\tilde{f}_{22})}{\partial I} \leq -\frac{1}{S^2I}(A-h-\mu_2) - \frac{\omega k}{(1+\omega I)^2} - \frac{hh_m}{(h_m+S-S_c)^2SI} < 0 \tag{26}$$

for sufficiently small μ_2 and ν_2, so no closed orbit exists in G_2. Hence, there is no closed orbit totally within region $G_i, i = 1, 2$. This completes the proof. □

Lemma 5.2. *No closed orbit crosses switching boundary $S = S_c$ in attraction region Ω.*

The approach for proving this lemma is only slightly different from that in [31, 29], so we omit it here.

Theorem 5.3. *There is a set of sufficiently small positive constants for perturbed parameters $\mu_1, \mu_2, \nu_1, \nu_2$ such that PP type of pseudo-focus $\tilde{A}$ is globally asymptotically stable.*

Proof. According to the above discussion, we can assign a set of appropriate values for perturbed parameters $\mu_i, \nu_i, i = 1, 2$ such that endemic equilibria $\tilde{E}_{11}, \tilde{E}_{12}$ and $\tilde{E}_2$ are virtual (denoted by $\tilde{E}_{11}^v, \tilde{E}_{12}^v, \tilde{E}_2^v$) and generalized equilibrium point $\tilde{A}(S_c, \tilde{I}_c)$ is of PP type. In the following, we will use the theory developed in [6, 9] to determine the local stability of $\tilde{A}$. Let

$$\begin{aligned} a_{pi} &= \tilde{f}_{i2}(\tilde{A}), \qquad b_{pi} = \frac{\partial \tilde{f}_{i2}(\tilde{A})}{\partial I}, \qquad m_{pi} = \frac{\partial \tilde{f}_{i1}(\tilde{A})}{\partial S} \\ l_{pi} &= \frac{\partial \tilde{f}_{i1}(\tilde{A})}{\partial I}, \qquad n_{pi} = \frac{\partial^2 \tilde{f}_{i1}(\tilde{A})}{2\partial I^2}, \qquad i = 1, 2. \end{aligned}$$

By (23), one gets

$$\begin{aligned} a_{p1} &= \frac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} - (\delta+\alpha)\tilde{I}_c - \nu_1, & a_{p2} &= \frac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} - (\delta+\alpha)\tilde{I}_c - \nu_2\tilde{I}_c, \\ b_{p1} &= -(\delta+\alpha) + \frac{kS_c}{(1+\omega\tilde{I}_c)^2}, & b_{p2} &= -(\delta+\alpha+\nu_2) + \frac{kS_c}{(1+\omega\tilde{I}_c)^2}, \\ m_{p1} &= -\delta - \frac{k\tilde{I}_c}{1+\omega\tilde{I}_c}, & m_{p2} &= -\delta - \frac{k\tilde{I}_c}{1+\omega\tilde{I}_c} - \frac{h}{h_m}, \\ l_{pi} &= -\frac{kS_c}{(1+\omega\tilde{I}_c)^2}, & n_{pi} &= \frac{\omega kS_c}{(1+\omega\tilde{I}_c)^3}, \quad i = 1, 2. \end{aligned}$$

Then it follows that

$$\begin{aligned} w_1 &= \frac{2\left[a_{p1}n_{p1} - (b_{p1}+m_{p1})l_{p1}\right]}{3a_{p1}l_{p1}} \\ &= \frac{\dfrac{2\omega}{1+\omega\tilde{I}_c}\left[\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} - (\delta+\alpha)\tilde{I}_c - \nu_1\right] - 4\delta - 2\alpha + \dfrac{2kS_c}{\left(1+\omega\tilde{I}_c\right)^2} - \dfrac{2k\tilde{I}_c}{1+\omega\tilde{I}_c}}{3\left[-\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} + (\delta+\alpha)\tilde{I}_c + \nu_1\right]} \end{aligned}$$

and

$$\begin{aligned} w_2 &= \frac{2\left[a_{p2}n_{p2} - (b_{p2}+m_{p2})l_{p2}\right]}{3a_{p2}l_{p2}} \\ &= \frac{\dfrac{2\omega}{1+\omega\tilde{I}_c}\left[\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} - (\delta+\alpha+\nu_2)\tilde{I}_c\right] - 4\delta - 2\alpha - 2\nu_2 - \dfrac{2h}{h_m} - \dfrac{2k\tilde{I}_c}{1+\omega\tilde{I}_c} + \dfrac{2kS_c}{\left(1+\omega\tilde{I}_c\right)^2}}{3\left[(\delta+\alpha+\nu_2)\tilde{I}_c - \dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c}\right]}. \end{aligned}$$

Therefore, the first two Lyapunov constants are

$$\begin{aligned} V_1 &= 0, \\ V_2 &= w_1 - w_2 \\ &= \frac{2\left[\dfrac{kS_c}{\left(1+\omega\tilde{I}_c\right)^2} - \dfrac{k\tilde{I}_c}{1+\omega\tilde{I}_c} - (2\delta+\alpha)\right](\nu_2\tilde{I}_c - \nu_1)}{3\left[-\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} + (\delta+\alpha)\tilde{I}_c + \nu_1\right]\left[-\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} + (\delta+\alpha)\tilde{I}_c + \nu_2\tilde{I}_c\right]} + \\ &\qquad \frac{2\left(\nu_2 + \dfrac{h}{h_m}\right)}{3\left[-\dfrac{kS_c\tilde{I}_c}{1+\omega\tilde{I}_c} + (\delta+\alpha)\tilde{I}_c + \nu_2\tilde{I}_c\right]}. \end{aligned}$$

We can select a set of appropriate values for perturbed parameters $\mu_i, \nu_i, i = 1, 2$ such that $V_2 < 0$, and hence we can conclude PP type of pseudo-focus $\tilde{A}$ is locally stable. Furthermore, by Lemma 5.1 and 5.2, there is no limit cycle in attraction region Ω, so $\tilde{A}$ is globally asymptotically stable, as shown in Figure 4 (a).

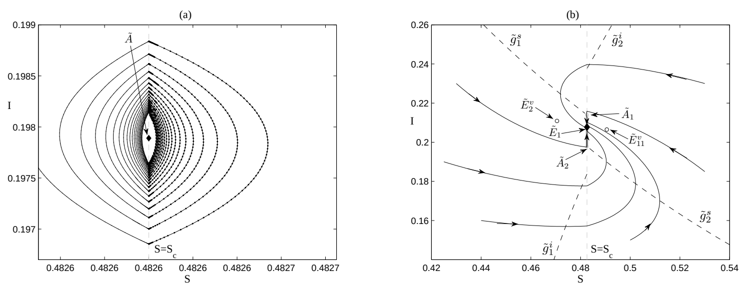

FIGURE 4. Phase plane S-I of perturbed system (23) for Case (C_1) and (C_2). (a) Stability of PP type of pseudo-focus $\tilde{A}$ for case (C_1); (b) Attracting sliding mode domain $(\tilde{A}_2\tilde{A}_1)$ and global stability of pseudo-equilibrium $\tilde{E}_1$ for case (C_2) . Isoclinic lines $\tilde{g}_1^i$ $(\tilde{g}_2^i)$ and $\tilde{g}_1^s$ $(\tilde{g}_2^s)$ are plotted for system S_{p1} (S_{p2}). The curves are plotted to show the asymptotic equilibrium. Parameter values are: $A = 0.6, k = 4, \omega = 1.2, \delta = 0.5, h = 0.2, h_m = 1.2, \alpha = 1, S_c = 0.4826$ and $\mu_1 = 0.05, \mu_2 = 0.05, \nu_1 = 0.025, \nu_2 = 0.002(a); \mu_1 = 0.03, \mu_2 = 0.05, \nu_1 = 0.015, \nu_2 = 0.002(b)$.

□

Case (C_2) $\mu_1 < \mu_2$.

In such scenarios, two points $\tilde{A}_1$ and $\tilde{A}_2$ are distinct and we define

$$L_1^s = \left\{ (S_c, I) \mid \tilde{I}_{c2} \le I \le \tilde{I}_{c1} \right\}.$$

In fact, L_1^s turns out to be the attracting sliding mode domain of system (23), according to theories in [7, 27, 26]. Basically, there are three approaches to determine the sliding mode dynamics, i.e. a method via singular perturbation [5], Utkin's equivalent control method [27] and the well known Filippov's convex method [7]. In the following, we will examine the sliding mode dynamics by applying Filippov's convex method. Before doing that we initially introduce a type of equilibrium which is in the sliding mode domain and plays an important role in the rest of this work.

Definition 5.4. A point Z^* is called a pseudo-equilibrium of system (23) if it is an equilibrium of the attracting or repulsing sliding mode of system, i.e.,

$$\lambda \tilde{F}_1(Z^*) + (1-\lambda)\tilde{F}_2(Z^*) = 0, \quad \sigma(Z^*) = 0, \quad \lambda = \frac{\tilde{F}_2\sigma(Z^*)}{(\tilde{F}_2 - \tilde{F}_1)\sigma(Z^*)}$$

where $\tilde{F}\sigma(Z) = \tilde{F}(Z) \cdot \mathrm{grad}\, \sigma(Z)$ is Lie derivative of σ with respect to vector field $\tilde{F}$ at point Z.

Let $\tilde{F} = q\tilde{F}_1 + (1-q)\tilde{F}_2$, where $q \in [0, 1]$. By (23), we obtain

$$\tilde{F} = \begin{pmatrix} A - \dfrac{kSI}{1+\omega I} - \delta S - q\mu_1 - (1-q)\mu_2 - \dfrac{(1-q)h(S-S_c)}{h_m + (S-S_c)} \\ \dfrac{kSI}{1+\omega I} - (\delta+\alpha)I - q\nu_1 - (1-q)\nu_2 I \end{pmatrix}. \tag{27}$$

It follows from $\tilde{F}\sigma(Z) = 0$ that

$$q = \frac{(A - \mu_2 - \delta S_c)(1 + \omega I) - kS_c I}{(\mu_1 - \mu_2)(1 + \omega I)}. \tag{28}$$

Substituting (28) into (27) gives the sliding mode dynamics of (23) governed by

$$\frac{dI}{dt} = \frac{kS_c I}{1 + \omega I} - (\delta + \alpha + \nu_2)I - \frac{(\nu_1 - \nu_2 I)\big[(A - \mu_2 - \delta S_c)(1 + \omega I) - kS_c I\big]}{(\mu_1 - \mu_2)(1 + \omega I)} \tag{29}$$

with $I \in L_1^s$. Denote

$$\begin{aligned}
\tilde{b}_0 &= \nu_2\omega(A - \mu_2 - \delta S_c) - \nu_2 kS_c - \omega(\mu_1 - \mu_2)(\delta + \alpha + \nu_2),\\
\tilde{b}_1 &= (\mu_1 - \mu_2)(kS_c - \delta - \alpha - \nu_2) + (\nu_2 - \nu_1\omega)(A - \mu_2 - \delta S_c) + \nu_1 kS_c,\\
\tilde{b}_2 &= -\nu_1(A - \mu_2 - \delta S_c).
\end{aligned}$$

Then the equilibrium of (29) is exactly the root of the following equation

$$-\,[\tilde{b}_0 I^2 + \tilde{b}_1 I + \tilde{b}_2] = 0. \tag{30}$$

Therefore, we can choose perturbed parameters $\mu_i, \nu_i, i = 1, 2$ appropriately such that there is an equilibrium $\tilde{I}_1^* \in [\tilde{I}_{c2}, \tilde{I}_{c1}]$ for equation (29) with

$$\tilde{I}_1^* = \frac{-\tilde{b}_1 + \sqrt{\tilde{b}_1^2 - 4\tilde{b}_0\tilde{b}_2}}{2\tilde{b}_0}$$

and $\tilde{b}_0 > 0, \tilde{b}_2 < 0$. Thus system (23) has a pseudo-equilibrium $\tilde{E}_1 = (S_c, \tilde{I}_1^*)$ in terms of Definition 5.4. Denote the right-hand side of (29) as $f(I)$. Then one gets $f(I) > 0$ for $I < \tilde{I}_1^*$ and $f(I) < 0$ for $I > \tilde{I}_1^*$, so $\tilde{E}_1$ is locally asymptotically stable on attracting sliding mode domain L_1^s. To determine the global stability of $\tilde{E}_1$, we also need to preclude the existence of any limit cycles in attraction region Ω. Indeed, there is no limit cycle containing part of sliding segment L_1^s due to the stability of pseudo-equilibrium $\tilde{E}_1$. According to Lemma 5.1, no limit cycle is totally in $G_i, i = 1, 2$. Besides, by implementing a slightly different process to that in [31, 29], we obtain that there is no limit cycle crossing switching boundary $S = S_c$. Then we can conclude that no limit cycle exists in attraction region Ω as shown in Figure 4 (b) and derive the following result.

Theorem 5.5. *There is a set of appropriate values for perturbed parameters μ_1, μ_2, ν_1, ν_2 such that pseudo-equilibrium $\tilde{E}_1$ is globally asymptotically stable.*

Case (C_3) $\mu_1 > \mu_2$.

In such scenarios, there does also exist an attracting sliding mode region

$$L_2^s = \Big\{(S_c, I) : \tilde{I}_{c1} \le I \le \tilde{I}_{c2}\Big\}$$

and (29) determines the sliding mode dynamics on L_2^s. Then the equilibrium of (29) is exactly the root of equation

$$\tilde{b}_0 I^2 + \tilde{b}_1 I + \tilde{b}_2 = 0 \tag{31}$$

in this case, so it is possible to assign a set of values for the perturbed parameters such that $\tilde{b}_0 < 0, \tilde{b}_2 < 0$ and only one pseudo-equilibrium $\tilde{E}_2(S_c, \tilde{I}_2^*)$ is feasible and satisfies $\tilde{E}_2 \in L_2^s$, where

$$\tilde{I}_2^* = \frac{-\tilde{b}_1 - \sqrt{\tilde{b}_1^2 - 4\tilde{b}_0\tilde{b}_2}}{2\tilde{b}_0}.$$

A similar discussion to Case (C_2) yields that pseudo-equilibrium $\tilde{E}_2$ is unstable on sliding mode region L_2^s. Furthermore, on the one hand all possible regular equilibria $\tilde{E}_{11}, \tilde{E}_{12}, \tilde{E}_2$ are virtual conditional upon proper parameter values, so none of them can act as a global attractor. On the other hand we can preclude the existence of any possible limit cycles except for the ones surrounding attracting sliding segment L_2^s. In fact, limit cycles totally within region $G_i, i = 1, 2$ are excluded in Lemma 5.1. Since the vector field to the right of null-isocline $\tilde{g}_1^s$ points to the left, while the vector field to the left of null-isocline $\tilde{g}_2^s$ points to the right as shown in Figure 5 (a), there is no crossing cycle without surrounding attracting sliding segment L_2^s. It is not difficult to get trajectories initiating from point $\tilde{A}_2$ or $\tilde{A}_1$ cannot hit L_2^s again according to Figure 5 (a), which demonstrates no canard cycle (i.e. a

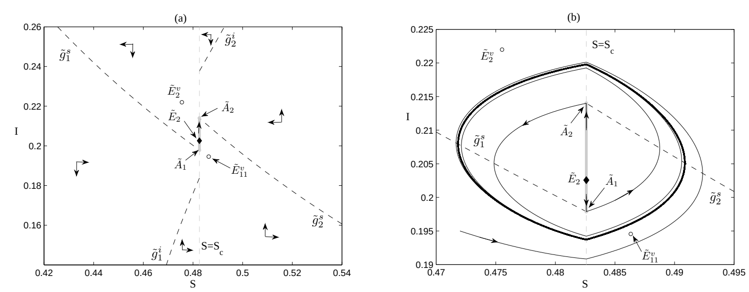

FIGURE 5. Phase plane $S - I$ of Filippov system (23) for Case (C_3). (a) The vector field in different subregions divided by switching boundary $S = S_c$, vertical null-isocline $\tilde{g}_1^s(\tilde{g}_2^s)$ and horizontal null-isocline $\tilde{g}_1^i(\tilde{g}_2^i)$ of system S_{p1} (system S_{p2}); (b) The existence of limit cycle in attraction region Ω. Parameter values are $A = 0.6, k = 4, \omega = 1.2, \delta = 0.5, h = 0.2, h_m = 1.2, \alpha = 1, S_c = 0.4826, \mu_1 = 0.05, \mu_2 = 0.03, \nu_1 = 0.015, \nu_2 = 0.002$.

cycle containing part of L_2^s) exists in attraction region Ω. Therefore, a crossing cycle appears in attraction region Ω and surrounds sliding segment L_2^s, as shown in Figure 5 (b). Then we summarize as follows.

Theorem 5.6. *There exists a set of proper values for perturbed parameters $\mu_1, \mu_2, \nu_1, \nu_2$ such that a crossing cycle surrounding attracting sliding mode region L_2^s is feasible in attraction region Ω.*

Remark 3. Only the transversal sliding mode is admitted for the original system (i.e. system (6)) while the attracting sliding mode besides transversal sliding mode is allowed for the perturbed system (i.e. system (23)). That is because the original system is continuous but the perturbed version of it is piecewise continuous and it is in fact a Filippov system. It is because of this that a pseudo-equilibrium and a crossing cycle occur for the perturbed system.

6. **Discussion.** In this paper, we have proposed a piece-wise SIV model with saturated incidence rate and switching vaccination strategy to represent a vaccination policy that is implemented only when the number of susceptible individuals reaches the threshold level. The resulting model is a continuous system with discrete switching events. We mainly focus on global dynamics of the formulated piece-wise system and obtain the effect of the switching vaccination strategy on the control of infectious diseases. We conclude that this non-smooth system exhibits a much wider

range of dynamical behaviors than the smooth counterpart does. Our main results show that the switching vaccination policy could maintain a disease at previously specified and acceptable level if it is impossible to eradicate.

By carrying out a global qualitative analysis, we have obtained the global structure of the behavior of the proposed switched epidemic model, which exhibits some novel dynamics. Our results demonstrate that if the infectivity of a disease (represented as k) is relatively low (i.e. regions Γ_1, Γ_2 and Γ_7 given in Figure 1), the disease will be eradicated as shown in Figure 2 (a). Otherwise (i.e. region $\bigcup_{j=3}^{6} \Gamma_j$ presented in Figure 1), the system can stabilize at one of the endemic equilibria for two variable systems (i.e. E_1 or E_2), as shown in Figure 3 (a). The most interesting result is that we have examined the stability of the so-called generalized equilibrium points E_0 and E^*, which are newly arisen properties of switched models, compared with the smooth systems [1, 8, 30]. The results obtained here indicate that if the threshold level is specified at critical values A/δ or S_1^*, solutions of the system can approach generalized equilibrium point E_0 or E^*, as shown in Figure 2 (b) and Figure 3 (b), which is typical for the switched system and qualitatively different from the long behavior for smooth systems. The significant challenge in proving their global stability is to exclude the existence of limit cycles and to prove the local stability of generalized equilibrium points, which is technically analyzed by introducing and implementing the theories for non-smooth systems. This the first time that these theories have been applied to solve real world problems.

According to Theorem 4.5, when the basic reproduction number R_{01} is greater than unity, the infectious disease may persist or die out, depending on whether the other basic reproduction number R_{02} is greater or less than unity with other parameters being appropriately selected. This result demonstrates that the basic reproduction number R_{01} only could not dominate the dynamics. Further, according to Lemma 4.1, R_{02} is greater or less than unity if threshold level S_c is greater or less than S_{c0}, where

$$S_{c0} = h_m + \frac{\delta+\alpha}{k} - \frac{h h_m k}{\delta(\delta+\alpha)+k(h-A)} = \frac{\delta+\alpha}{k} + \frac{h_m\Big[\delta(\delta+\alpha)-Ak\Big]}{\delta(\delta+\alpha)+k(h-A)},$$

which implies that whether a disease is eradicated or not is sensitive to the vaccination rate (h). Therefore, it is possible to eradicate a disease by increasing the vaccination rate conditional on availability of medical resources.

It follows from the formula of I_2^* given in (13) and Remark 1 (ii) that we could choose a sufficiently small value for threshold level S_c and other parameter values such that the population size of infected individuals stabilizes at some previously chosen level. In practice, it is not always possible to eradicate a disease and hence the objective changes to one of containing the disease by keeping the number of infected individuals at a low level. If it is really the case, we can determine the critical level of threshold S_c in terms of (13) such that the number of infected individuals stabilizes at a specified value. This is interesting and will help public health officers to decide when to implement the vaccination strategy.

Comparing the proposed model with the switching vaccination strategy and the one with general vaccination strategy without switching suggests that the switching vaccination strategy has many advantages. From the point view of mathematics, when disease persists, two different kinds of level of infection can be approached in the first model, whereas only a unique level of infection is possible in the second model. Theoretical analysis of the model with a switching vaccination strategy is much harder than analyzing the corresponding smooth counterpart. From the point view of biology, the first system better describes the impact of a variable vaccination strategy on the control of infectious disease, the second one represents consistent intervention; more importantly, the first one can lead the size of the infected population to stabilize at a desired level but the second one cannot do so. In conclusion, the proposed switching vaccination program is more realistic for modelling a vaccination policy with variation and is more efficacious in containing the disease.

More importantly, when we take into account some minor elements related to interventions, we derive a perturbed system version of the original one, which can better describe real world problems. Qualitative analysis demonstrates that trajectories of the system may approach a pseudo-focus of PP type or a pseudo-equilibrium, or a crossing cycle surrounding a sliding mode region, depending on the rate of acquisition of natural immunity to the virus (represented as μ_1 and μ_2). If $\mu_1 > \mu_2$, both the number of susceptibles and infecteds vary periodically and approach a dynamic equilibrium as time t is sufficiently large. If $\mu_1 = \mu_2$, a pseudo-focus of PP type is a global attractor, which implies we can choose the threshold level and other parameters appropriately such that the size of the population of infected individuals stabilizes at a previously specified level. This is a new stable level and dramatically different from the results in [1, 8, 30, 13, 12, 21, 20], by continuous switching between vaccination and no vaccination. If $\mu_1 < \mu_2$, the number of infected individuals can also stabilize at a scheduled level, but in such case, the model exhibits a rapid alternation of vaccination with no vaccination, resulting in shorter periods of both modalities [31].

By formulating and analyzing a switched epidemic model, the work presented in this paper demonstrates how such a switching vaccination policy affects disease spread. Qualitative analysis showed that some new types of global attractors are possible. In particular, the switched system possesses two novel global attractors that are generalized equilibria; while the perturbed system has three types of novel global attractors including a pseudo-focus of PP type, a pseudo-equilibrium and a crossing cycle surrounding a sliding mode region. The results demonstrate that an infectious disease can be maintained at some desired level if more realistic interventions are considered.

Acknowledgments. The authors are supported by the National Natural Science Foundation of China (NSFC, 11171268(YX)), by the Fundamental Research Funds for the Central Universities(GK 08143042 (YX)) and by the International Development Research Center, Ottawa, Canada (104519-010).

REFERENCES

[1] J. Arino, C.C. Mccluskey and P.V.D. Driessche, *Global results for an epidemic model with vaccination that exhibits backward bifurcation*, SIAM J. Appl. Math., **64** (2003), 260–276.

[2] M.di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, "Piecewise-smooth dynamical systems," Springer, New York, 2008.

[3] "China Ministry of Health and joint UN programme on HIV/AIDS WHO," Estimates for the HIV/AIDS Epidemic in China, 2009.

[4] F. Clarke, Y. Ledyaev, R. Stern and P. Wolenski, "Nonsmooth Analysis and Control Theory," Springer, New York, 1998.

[5] A.B. Claudio, P.d.S. Paulo and A.T. Marco, *A singular approach to discontinous vector fields on the plane*, J. Diff. Equa., **231** (2006), 633–655.

[6] B. Coll, A. Gasull and R. Prohens, *Degenerate hopf bifurcations in discontinuous planaer system*, J. Math. Anal. Appl., **253** (2001), 671–690.
[7] A.F. Filippov, "Differential equations with discontinuous righthand sides," Kluwer Academic, Dordrecht, 1988.
[8] D. Greenhalgh, Q.J.A. Khan and F.I. Lewis, *Recurrent epidemic cycles in an infectious disease model with a time delay in loss of vaccine immunity*, Nonl. Anal. TMA., **63** (2005), e779–e788.
[9] M.A. Han and W.N. Zhang, *On hopf bifurcation in non-smooth plannar systems*, J. Diff. Equa., **248** (2010), 2399–2416.
[10] "A/H1N1 vaccination program extends to all Beijingers", November 7, 2009. Available from: http://news.xinhuanet.com/english/2009-11/07/content_{}12404148.htm.
[11] "Guangdong starts A/ H1N1 vaccination for migrant workers", January 6, 2010. Available from: http://www.newsgd.com/news/GDNews/content/2010-01/06/content_{}7848857.htm.
[12] J. Hui and L.S. Chen, *Impulsive vaccination of SIR epidemic models with nonlinear incidence rates*, Discrete Contin. Dyn. Syst. Ser. B, **4** (2004), 595–605.
[13] G. R. Jiang and Q. G. Yang, *Bifurcation analysis in an SIR epidemic model with birth pulse and pulse vaccination*, Appl. Math. Comput., **215** (2009), 1035–1046.
[14] R.I. Leine, *Bifurcations of equilibria in non-smooth continuous systems*, Phys. D, **223** (2006), 121–137.
[15] R.I. Leine and D.H. van Campen, *Bifurcation phenomena in non-smooth dynamical systems*, Eur. J. Mech. A Solids, **25** (2006), 595–616.
[16] D. Liberzon, "Switching in systems and control," Springer-Verlag, New York, 1973.
[17] J. Melin, *Does distribution theory contain means for extending Poincare-Bendixson theory*, J. Math. Anal. Appl., **303** (2005), 81–89.
[18] M.E.M. Meza, A. Bhaya, E.K. Kaszkurewicz, D.A. Silveira and M.I. Costa, *Threshold policies control for predator-prey systems using a control Liapunov function approach*, Theor. Popul. Biol., **67** (2005), 273–284.
[19] M.E.M. Meza, M.I.S. Costa, A. Bhaya and E. Kaszkurewicz, *Threshold policies in the control of predator-prey models*, Preprints of the 15th Triennial World Congress (IFAC), Barcelona, Spain, (2002),1–6.
[20] L.F. Nie, Z.D. Teng and A. Torres, *Dynamic analysis of an SIR epidemic model with state dependent pulse vaccination*, Nonl. Anal. RWA., **13** (2012), 1621–1629.
[21] A. d'. Onofrio, *Stability properties of pulse vaccination strategy in SEIR epidemic model*, Math. Bios., **179** (2002), 57–72.
[22] L. Sanchez, *Convergence to equilibria in the Lorenz system via monotone methods*, J. Diff. Equa., **217** (2005), 341–362.
[23] S.Y. Tang and J.H. Liang, *Global qualitative analysis of a non-smooth Gause predator-prey model with a refuge*, Nonl. Anal. TMA., **76** (2013), 165–180.
[24] S.Y. Tang, J.H. Liang, Y.N. Xiao and R. A. Cheke, *Sliding bifurcation of Filippov two stage pest control models with economic thresholds*, SIAM J. Appl. Math., **72** (2012), 1061–1080.
[25] S.Y. Tang, Y.N. Xiao and et.al., *Community-based measures for mitigating the 2009 H1N1 pandemic in China*, PLoS ONE, **5** (2010), 1–11(e10911).
[26] V. I. Utkin, "Sliding Modes and Their Applications in Variable Structure Systems," Mir, Moscow, 1978.
[27] V. I. Utkin, "Sliding Modes in Control and Optimization," Springer, Berlin, 1992.
[28] A.L. Wang and Y.N. Xiao, *Sliding bifurcation and global dynamics of a Filippov epidemic model with vaccination*, Internat. J. Bifur. Chaos, **23** (2013), DOI: 10.1142/S0218127413501447.
[29] W.D. Wang, *Backward bifurcation of an epidemic model with treatment*, Math. Biosci., **201** (2006), 58–71.
[30] Y.N. Xiao and S.Y. Tang, *Dynamics of infection with nonlinear incidence in a simple vaccination model*, Nonl. Anal. RWA., **11** (2010), 4154–4163.
[31] Y.N. Xiao, X.X. Xu and S.Y. Tang, *Sliding mode control of outbreaks of emerging infectious diseases*, Bull. Math. Biol., **74** (2012), 2403–2422.
[32] Y.N. Xiao, T.T. Zhao and S.Y. Tang, *Dynamics of an infectious diseases with media/psychology induced non-smooth incidence*, Math. Biosci. Eng., **10**(2013), 445–461.
[33] T.R. Zhang and W.D. Wang, *Hopf bifurcation and bistability of a nutrient-phytoplankton-zooplankton model*, Appl. Math. Model., **36** (2012), 6225–6235.

Appendix A Local Lipschitz-continuity of system (6)

Due to the smoothness of $F(Z)$ in $G_i, i=1,2$, it is sufficient to examine that for any closed rectangle $M \subset R_+^2$ centered at (S_c, I), there is a constant $L(M) > 0$ such that
$$|F(Z_1) - F(Z_2)| \le L|Z_1 - Z_2|$$
for $Z_i \in M \cap G_i, i = 1,2$. Denote $Z_i = (S_i, I_i), \tilde{g}_1 = A - kSI/(1+\omega I) - \delta S, \tilde{g}_2 = kSI/(1+\omega I) - \delta I - \alpha I$, and one gets
$$\begin{aligned}\big|F(Z_1) - F(Z_2)\big|^2 &= \left(\tilde{g}_1(Z_1) - \tilde{g}_1(Z_2) + \frac{h(S_2 - S_c)}{h_m + (S_2 - S_c)}\right)^2 + \big(\tilde{g}_2(Z_1) - \tilde{g}_2(Z_2)\big)^2 \\ &\le \left(\Big|\tilde{g}_1(Z_1) - \tilde{g}_1(Z_2)\Big| + \frac{h}{h_m}(S_2 - S_1)\right)^2 + \big(\tilde{g}_2(Z_1) - \tilde{g}_2(Z_2)\big)^2.\end{aligned}$$
Since $\tilde{g}_i(Z), i = 1,2$, is smooth in M, there are constants $L_i > 0, i = 1,2$, such that
$$\Big|\tilde{g}_i(Z_1) - \tilde{g}_i(Z_2)\Big|^2 \le L_i |Z_1 - Z_2|^2.$$
Thus,
$$\begin{aligned}\big|F(Z_1) - F(Z_2)\big|^2 \quad &\le 2\left(L_1 + \frac{h^2}{h_m^2}\right)\left[|Z_1 - Z_2|^2 + (S_2 - S_1)^2\right] + L_2|Z_1 - Z_2|^2 \\ &\le \left[2L_1 + \frac{2h^2}{h_m^2} + L_2\right]|Z_1 - Z_2|^2.\end{aligned}$$
Let $L^2 = 2L_1 + 2h^2/h_m^2 + L_2$, and it follows that
$$\big|F(Z_1) - F(Z_2)\big| \le L|Z_1 - Z_2|.$$
Therefore, system (6) is locally Lipschitz-continuous in R_+^2.

Appendix B Existence of equilibria for system S_2

We initially examine the existence of all possible disease-free equilibria for system S_2, which satisfy equation (8) according to Section 3. It follows from
$$\begin{aligned}(A-h)S_c - Ah_m \ &> 0 \Longleftrightarrow S_c > \frac{Ah_m}{A-h}, \\ \delta S_c + (A - h - \delta h_m) \ &> 0 \Longleftrightarrow S_c > h_m - \frac{A-h}{\delta}\end{aligned}$$
and $Ah_m/(A-h) > h_m - (A-h)/\delta$ that $S_{02} > 0, S_{03} < 0$ for $S_c < Ah_m/(A-h)$; $S_{02} > 0, S_{03} > 0$ for $S_c > Ah_m/(A-h)$; $S_{02} > 0, S_{03} = 0$ for $S_c = Ah_m/(A-h)$.

Note that
$$A < h + \delta h_m \Longleftrightarrow \frac{A}{\delta} < \frac{Ah_m}{A-h}.$$
Thus, we conclude the following.

(i) When $A < h + \delta h_m$, there is a unique disease-free equilibrium (i.e. E_{02}) for $S_c < Ah_m/(A-h)$, which is real for $S_c < A/\delta$ and virtual for $S_c > A/\delta$; there exist two disease-free equilibria (i.e. E_{02} and E_{03}) for $S_c \ge Ah_m/(A-h)$ and both of them are virtual.

(ii) When $A > h + \delta h_m$, only one disease-free equilibrium (i.e. E_{02}) exists for $S_c < Ah_m/(A-h)$ and it is real; two disease-free equilibria (i.e. E_{02} and E_{03}) exist for $S_c \ge Ah_m/(A-h)$, E_{03} is always virtual while E_{02} is real for $S_c < A/\delta$ and virtual for $S_c > A/\delta$.

(iii) When $A = h + \delta h_m$, there is one disease-free equilibrium (i.e. E_{02}) for $S_c < A/\delta$, which is real; two disease-free equilibria (i.e. E_{02}, E_{03}) exist for $S_c > A/\delta$ and both of them are virtual.

For clarity, we list the results in Table 1, where ER denotes the existence of equilibrium which is real, EV represents the existence of equilibrium which is virtual and N stands for the nonexistence of equilibrium.

TABLE 1. Existence of real or virtual disease-free equilibria for system S_2

Threshold Value		E_{02}	E_{03}
$A < h + \delta h_m$	$S_c < \frac{A}{\delta}$	ER	N
	$\frac{A}{\delta} < S_c < \frac{Ah_m}{A-h}$	EV	N
	$S_c \geq \frac{Ah_m}{A-h}$	EV	EV
$A > h + \delta h_m$	$S_c < \frac{Ah_m}{A-h}$	ER	N
	$\frac{Ah_m}{A-h} \leq S_c < \frac{A}{\delta}$	ER	EV
	$S_c > \frac{A}{\delta}$	EV	EV
$A = h + \delta h_m$	$S_c < \frac{A}{\delta}$	ER	N
	$S_c > \frac{A}{\delta}$	EV	EV

It follows from Table 1 that E_{03} remains virtual if it is well defined while E_{02} is real provided $S_c < A/\delta$.

Now we consider the existence of all endemic equilibria for system S_2. To this end, it is necessary to examine the existence of positive roots for equation (12) in Section 3.

When $\delta(\delta+\alpha) + k(h-A) < 0$, one gets that

$$\Phi_2(S_c) < 0 \Longleftrightarrow S_c < S_{c0}.$$

Similarly, when $\delta(\delta+\alpha) + k(h-A) > 0$, we have

$$\Phi_2(S_c) < 0 \Longleftrightarrow S_c > S_{c0}.$$

If $\delta(\delta+\alpha) + k(h-A) = 0$, then $\Phi_2(S_c) < 0$ for all $S_c > 0$. We also get that $\Phi_1(S_c) < 0$ if and only if $S_c > S_{c1}$.

According to equation (14) in Section 3, endemic equilibrium E_3 is always virtual provided it is well defined while E_2 is real for $I_2^* > \max\{I_c, 0\}$. If $I_c \geq 0$, we conclude as following on the basis of discussion in Section 3 and inequalities (15), (16).

(i) When $R_{01} > 1, \delta(\delta+\alpha) + k(h-A) > 0$ and $S_c \geq (\delta+\alpha)/k$, there is only one endemic equilibrium (i.e. E_2), which is real for $S_c < S_1^*$ and virtual for $S_c > S_1^*$.

(ii) When $R_{01} > 1, \delta(\delta+\alpha) + k(h-A) < 0, S_{c0} < S_1^*$ and $S_c \geq (\delta+\alpha)/k$, there is a unique endemic equilibrium (i.e. E_2) for $S_c < S_{c0}$, which is real; two endemic equilibria (i.e. E_2, E_3) exist for $S_c \geq S_{c0}$, E_2 is real for $S_c < S_1^*$ and virtual for $S_c > S_1^*$ while E_3 is always virtual.

(iii) When $R_{01} > 1, \delta(\delta+\alpha) + k(h-A) < 0, S_{c0} > S_1^*$ and $S_c \geq (\delta+\alpha)/k$, there is a unique endemic equilibrium (i.e. E_2) for $S_c < S_{c0}$, which is real for $S_c < S_1^*$ and virtual for $S_c > S_1^*$; there exist two endemic equilibria (i.e. E_2, E_3) for $S_c \geq S_{c0}$ and both of them are virtual.

(iv) When $R_{01} > 1, \delta(\delta+\alpha)+k(h-A) < 0, S_{c0} = S_1^*$ and $S_c \geq (\delta+\alpha)/k$, only one endemic equilibrium (i.e. E_2) exists for $S_c < S_{c0}$ which is real; two endemic equilibria (i.e. E_2, E_3) exist for $S_c > S_{c0}$, both of which are virtual.

(v) When $R_{01} > 1, \delta(\delta+\alpha)+k(h-A) = 0$ and $S_c \geq (\delta+\alpha)/k$, a unique endemic equilibrium (i.e. E_2) does exist, which is real for $S_c < S_1^*$ and virtual for $S_c > S_1^*$.

When $I_c < 0$, one gets no endemic equilibrium existing for

$$R_{01} > 1, \quad \delta(\delta+\alpha)+k(h-A) > 0, \quad S_c \leq S_{c0},$$

based on the discussion in Section 3.

For simplification and convenience, we list all results related to the existence of endemic equilibria for system S_2 in Table 2.

TABLE 2. Existence of real or virtual endemic equilibria of system S_2 for $R_{01} > 1$

Threshold Value		E_2	E_3
$\delta(\delta+\alpha)+k(h-A) > 0$	$S_c \leq S_{c0}$	N	N
	$S_{c0} < S_c < S_1^*$	ER	N
	$S_c > S_1^*$	EV	N
$\delta(\delta+\alpha)+k(h-A) < 0$ $Sc0 < S_1^*$	$S_c < S_{c0}$	ER	N
	$S_{c0} \leq S_c < S_1^*$	ER	EV
	$S_c > S_1^*$	EV	EV
$\delta(\delta+\alpha)+k(h-A) < 0$ $Sc0 > S_1^*$	$S_c < S_1^*$	ER	N
	$S_1^* < S_c < S_{c0}$	EV	N
	$S_c \geq S_{c0}$	EV	EV
$\delta(\delta+\alpha)+k(h-A) < 0$ $Sc0 = S_1^*$	$S_c < S_1^*$	ER	N
	$S_c \geq S_{c0}$	EV	EV
$\delta(\delta+\alpha)+k(h-A) = 0$	$S_c < S_1^*$	ER	N
	$S_c > S_1^*$	EV	N

It follows from Table 2 that endemic equilibrium E_3 remains virtual if it is well defined while endemic equilibrium E_2 is real for $S_c < S_1^*$ provided it is well defined.

Received xxxx 20xx; revised xxxx 20xx.

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