THE FOUNDATIONS OF MODERN ALGEBRA

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presented to the Council for National Academic Awards

for the degree of

Master of Philosophy in January, 1972.

ABSTRACT

The objective of the thesis is to examine, in some detail the most significant contributions made by British mathematicians to the 'foundations of algebra' in the first half of the nineteenth century, and to assess the importance of these advances against the inadequacies of eighteenth century algebra and the subsequent development of modern algebra.

In order to realize this aim, it was necessary to outline the historical context in which these contributions were made. Therefore a brief account is included of problems inherited from eighteenth century algebra. Furthermore, to explain the somewhat isolated development of a school of logical algebra in Britain at this time, it was necessary to include a brief discussion of the situation in the institutions of learning and research in the first half of the nineteenth century, as a background to the work of the mathematicians considered.

The first breakthrough in algebra came in Peacock's Treatise on Algebra in 1830 and its significance is examined in some detail. In 1835, W. R. Hamilton discovered the now familiar system of number couples to describe complex numbers, this work is examined carefully since, measured against later developments, it is of considerable importance.

Another chapter is devoted to an analysis of Gregory's axiomatic system for formal algebra which appeared in 1838. His

system was closely followed by a series of important papers on the foundations of algebra by A. De Morgan. These papers have been examined in detail, since they contain a clear statement of the central problems of contemporary algebra and indicate both particular and general solutions.

The final researches considered were Hamilton's revolutionary discovery of a non-commutative algebra and De Morgan's attempt to construct a significant triple-algebra.

The concluding chapter of the thesis is an assessment of the value of these works, both in relation to the problems they overcame, and the potential for the development of new systems of algebra they created.

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INTRODUCTION

The study of the history of mathematics presents rather different problems from the study of other aspects of human development. Mathematical ideas can be regarded as more esoteric than ideas arising from changing technology or social development. Nevertheless, since mathematics has traditionally been afforded a special place in mankind's ability to rationalize developing and changing material conditions, the study of its history can reveal vital insights into the more general pattern of human social development.

Certain problems exist in studying the history of any particular development or branch of development of mathematics. While it seems that particular mathematical ideas originate from the individual mathematician, in general, no one mathematician is solely responsible for a particular discovery. The filiation of ideas is a complex process, since each branch of mathematics has its genesis in more branches than its own. The problem then becomes to abstract the most relevant pattern of development from a complex of all possible factors influential on the genesis of the new ideas.

Furthermore, one must beware, with the benefit of hindsight not to attribute to the individual mathematician, understanding of the full implication of his discovery. In general, mathematical research does not progress in the most logical, linear way: many detours and blind allies are taken before a theory is fully understood. A good example of such a detour would be the search to put the differential calculus on a rigorous basis.

Another problem for consideration, is to ascertain to what extent mathematical ideas are influenced by the prevailing ideas and conditions of the age. A cursory examination of the history of mathematics will show that the most rapid development of mathematics has been during the period of industrialization of Europe; that is the nineteenth and twentieth centuries. One can infer that the growth of ideas is strongly influenced by social factors. The problem is then to demonstrate the actual relationship between the inspiration of the individual mathematicians and the social background against which he works.

One can observe that since the mathematician is not a machine operating independently of his environment, his work may well be influenced by his social relations. In particular the state of the educational system and machinery for mathematical intercourse can severely limit or greatly assist mathematicians in their research.

While it would be mechanical to attempt to frame general laws of the manner in which the economic and political system influences the state of mathematics, it can be useful to elucidate those factors which may advance or retard the work of mathematicians during their period of activity.

The branch of mathematics to be considered in the thesis is the foundations of Algebra in the period between 1810 to 1850 in Britain. There are a number of reasons for making this particular choice. In the early nineteenth century two general trends took place in algebra. The first trend, heralded by the work of Gauss and Abel, was to construct widely inclusive theories in algebra; this trend on the continent was brought to fruition by the group theory of Galois, which was not widely publicised until the late 1840's. In Britain the trend towards abstract theories was also continued, but with an essentially British emphasis, that is, the attention was concentrated on the formal, logical basis of algebra, and major discoveries were related to that emphasis. Also the work of the British algebraists in this field preceded the major work of the continental mathematicians in that they laid down the structural basis for the major advances towards what may be called 'modern algebra'. It is for these reasons that I have concentrated on the work of the British School, and entitled the dissertation 'The Foundations of Modern Algebra'.

In presenting the subject matter of this thesis, I have attempted to take account of the problems I have outlined. The first chapter sketches the mathematical origins of the central problems taken up by the British algebraists considered. The chapter concentrates in the main on the late eighteenth century developments in algebra significant to the ideas of the British algebraists; the discussion does not attempt to outline all the details of algebraic discovery in the eighteenth century, but is confined to the genesis of the formal understanding of algebra.

Despite the fact that the actual contributions to be discussed appeared from 1830 until 1844, I have chosen to examine social climate in which they appeared from about 1810 to 1850, since certain social pressures for reforms of the Establishment took place from about 1810 onwards which I feel are relevant to developments affecting the future of the mathematics in the thesis.

I was not able to establish any very immediate relation between the actual development of algebra and the social climate. However, what I did attempt in the second chapter was to elucidate those factors which I saw as retarding the overall development of mathematics in Britain. The factors were both social, such as higher education and the Royal Society, and mathematical, such as the fluxional notation. Furthermore, I attempted to demonstrate, that by the efforts of individuals and pressure groups, which included the mathematicians to be discussed, a more favourable climate for mathematical work was being created in the period. I have also tried to show in this chapter that the situation of eighteenth century British mathematics could explain the somewhat isolated development of formal algebra in Britain in the period considered.

Having put the ideas to be discussed in a historical and social context, the next five chapters concentrate in detail on discoveries that were made constituting the Foundations of modern algebra. I mentioned that one of the problems of study is to select the most relevant material for the theme.

Since there was a reasonable quantity of algebraic work being done in the first half of the nineteenth century I have restricted myself to consider only the most original contributions; that is, the contributions that heralded the new, formal approach in algebra, and those that represented the most original advances in this field. Thus in Chapter III, I have discussed Peacock's attempt to give a formal basis to algebra, in Chapter IV, Hamilton's system of ordered couples; the first successful attempt to demystify complex numbers. In Chapter V, I have described Gregory's axiomatic approach to common algebra, in Chapter VI, De Morgan's logical expositions on the foundations of algebra and in Chapter VII, I have dealt with the revolutionary new algebra of Hamilton and De Morgan's triple algebra.

In each chapter I have suggested the outstanding significance of the contribution. However, I have had to restrict the discussion just to the central point of each discovery and could not analyse all the ramifications of it, such as Hamilton's lengthy development of Quaternions.

I have chosen to end with the discovery of non-commutative multiplication since I believe it marks the conclusion of research into the problems of the old common algebra. The discoveries which followed were principally discoveries of new systems based on the new structural approach which had been laid down.

In the eighth, concluding chapter, I have tried to show in what way this is a natural period. I have presented each contribution in relation to the mathematical situation that had preceded them; assessed their relative importance and sketched the nature of the algebraic discoveries that immediately succeeded them. Peacock, Gregory, Hamilton and De Morgan pioneered the discovery of new algebras; I have tried to show that the general theories of algebraic structure that were to follow in the second half of the eighteenth century demonstrated the permanent significance of their work.

CHAPTER I

Significant Developments in Algebra before 1800

In order to assess the mathematical significance of the contribution made by the British Algebraists to the understanding of structure and form in the first half of the nineteenth century it will be necessary to elucidate briefly the origins of the algebraic problems they took up, notably those of the late eighteenth century. It will be of particular interest to note those differences in the nature of the British and continental contributions in the eighteenth century which may have influenced the singular development of a British School of Algebra in the nineteenth century.

The tradition of European Algebra was drawn principally from the Islaamic culture of the Arabs, which flourished from the seventh to the twelfth century A.D. Islaamic algebra was, in turn, drawn from two principal sources; the Hindus (about 600 A.D.) and the Greeks of the Diophantine school. The Hindus were assisted by their aptitude in arithmetic calculation, based on a 'rational' number system which included zero. The Greeks were limited, severely, in developing algebra by the absence of such a rational arithmetic: however, there did exist a school of algebra in the later days of the Greek civilisation (about 250 A.D.). They developed methods of finding solutions in integers or rational numbers of indeterminate equations; the founder of these methods was known as Diophantus.

The algebra as inherited from the Arabs, was known as 'rhetorical'; having assimilated the developed methods of both the Greeks and the Hindus, the essence of their algebra was based on arithmetical calculation and verbal argument; they did not have a system of consistent symbolic representation. They were able to solve particular linear, quadratic and cubic equations; but without symbolic representation, the solutions of such equations had to be based on somewhat cumbersome and lengthy arguments. Clearly, given this situation, there was little possibility of developing general and all-embracing theories; results, while useful, tended to be particular and isolated. What was required to extend the domain of algebra was a notational or symbolic system, which would facilitate the processes of reasoning in algebra.

Towards the end of the sixteenth century European mathematicians began to make advances on the work of the Arabs in the direction of notational reform. In 1591, F. Vieta introduced the use of latin letters for unknown quantities, and symbols to denote the square and the cube; this system had the advantage not only of considerably abbreviating the argument, but also of depicting possible relationships between the powers.

In the following decades, further notational improvements were made; once the principle was established that new notation was facilitating the solution of equations, each symbolic system was improved upon. It then became possible to see the possibility of new generality in the use of algebra. After the improvements made by T. Harriot, A. Girard and R. Descartes, symbolic algebra emerged. However, the laws and concepts of symbolic algebra were borrowed from common arithmetic, since letters were supposed to represent unknown arithmetical quantities; clearly, it was considered they would be subject to the same laws, and likely to produce results familiar to common arithmetic. Such conceptions were, in fact, to create the problems to be considered in future chapters.

In the process of solving equations, two types of results occurred, which were unfamiliar to the results of common arithmetic. These were 'negative numbers' and what we now call 'complex numbers', and were called 'imaginaries'. The negative numbers posed the problem that a 'quantity' could be less than nothing. 'Imaginaries' posed the further problem that the product of two identical 'quantities' could be less than nothing.

In 1637 R. Descartes summarized the basic principles of symbolic algebra and the known properties of roots of polynomial equations; in this context he further referred to the prevailing mysteries of these unarithmetical quantities. Negative roots were known as 'false', complex roots as 'imaginary'; the nature of the imaginary 'quantity', unlike the negative, was that no matter by how much they were increased, reduced or multiplied, they could never be made anything but imaginary. Like that of G. Cardano in his 'Ars Magna' of 1545, the first attitudes to imaginary roots were to ignore them. However, they later became too useful for such neglect.

Descartes was also responsible for the first major extension of the domain of algebra; one offshoot of the introduction of symbolization was the extensions into the field of geometry. Descartes adopted algebra for the purpose of geometrical analysis; he used algebraic relations to represent the relations between the invariate and variable properties of geometric figures, a system known to us as analytic geometry.

As symbolic algebra lent itself more as an analytic tool than did rhetorical algebra, its applications in other branches of mathematics increased, and as this happened the problems already suggested developing within algebra took on greater significance.

In the seventeenth century, the solution of equations of higher degree by radicals generated the two central and interconnected problems in algebra; firstly, how many roots does an equation have? And secondly, can imaginary roots be included in the number? If so, what is their nature? The answer to the first question was an assertion known as the 'fundamental theorem of algebra'. In 1608 P. Roth stated the theorem, namely, that algebraic equations have the number of roots corresponding to the degree of the equation. Although attempts to prove the theorem were not until a century later, it became employed widely by many leading mathematicians. This theorem credited imaginaries with definite status, since, if the theorem was to be true in every case the number of real roots would sometimes have to be supplemented by complex ones. Further formulations of the theorem followed; one by A. Girard in 1629, another by Descartes in 1637; the first attempt at proof was not until 1742, by which time it was a well-established necessity in mathematics.

Imaginaries became well integrated into later seventeenth century mathematics; the remarkable aspect of their development was, however, that there seemed to be no regard to the legitimacy of operating with undefined entities. This disregard for the need of a formal basis or explanation of imaginaries persisted throughout the eighteenth century until their de-mystification by the British algebraists to be considered.

The only attempt to interpret complex numbers before the late eighteenth century was made by J. Wallis in 1655 in a work entitled <u>Arithmetica Infinitorum</u>. He attempted to interpret both negative numbers and imaginary numbers geometrically; the complex number $(x + iy)$ he represented in the Cartesian plane by the point (x, y). The essential detail he missed was the introduction of an axis to represent the 'imaginary' part of the number. The only other seventeenth century advances in algebra, were the generalized method of E. W. Tschirnhausen for the solution of quadratic and cubic equations, and the discovery in 1693 of a determinant method of solving simultaneous linear equations, by Leibniz. The method was not in fact published until 1850 and so had little effect on subsequent developments.

Before considering the algebraic development of the eighteenth century, it is of value to note one unfortunate event that overshadowed the mathematical exchanges of Britain and the Continent. It is now accepted that Newton and Leibniz discovered the calculus independently, the likelihood being that Newton anticipated Leibniz by a few years. However, an unpleasant feud developed between the continental analysts, supporting Leibniz, and the British followers of Newton, with mutual insinuations of plagiarism in relation to the discoverer of the calculus. The outcome of the feud was that communication of mathematical ideas between Britain and the continent was virtually at a standstill for the greater part of the eighteenth century. Also it seemed that it was British analysis which suffered. The continental analysts forged on apace with the more flexible differential notation of Leibniz; the British analysts stood at a disadvantage with the exclusive use of the fluxional notation, and at the end of the eighteenth century were well behind in the extended applications of Newton's gravitational theories.

However, returning to the objective development of algebra, in the eighteenth century one fact in particular emerged. The lack of formality in algebra tended to foster confusion between the domains of algebra and analysis. Infinite series had been studied and used extensively toward the close of the seventeenth century, but analysis, like algebra, lacked rigour. Little regard had been given to convergence of series and definition of limits; moreover infinite series were regarded as belonging to the field of algebra.

This algebraic treatment of infinite series persisted late into the eighteenth century. This could often be noted in the unqualified use of algebraic identities for series; one example was that there existed no serious doubt as to the correctness of the assertion

$$\left(\frac{1}{2} = 1 - 1 + 1 - 1 + \cdots \right)$$

One mathematician was led to conclude

$$\left(\frac{1}{2} = 0 + 0 + 0 + \cdots \right) !$$

Indeed even a mathematician of the calibre of L. Euler was content to write the proof of

$$\cdots \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = 0$$

along the following lines

$$n + n^2 + \cdots = \frac{n}{1-n}, \qquad 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1}$$

$$\frac{n}{1-n} + \frac{n}{n-1} = 0 !$$

This unfortunate confusion remained throughout the eighteenth century. Even the mighty work of Laplace on the motion of celestial bodies was based on very non-rigorous use of series. Furthermore, as late as 1797 in the Théorie des Fonctions Analytiques, Lagrange thought he had successfully obviated the problems of the use of limits in finding derivatives. He attempted a proof of Taylor's theorem with recourse to algebra alone, and derived the calculus from Taylor's theorem. However, despite the fundamental nature of some of their misconceptions, the continental mathematicians, in particular the French, made great advances in the applications of analysis and series.

The same could be said of complex numbers. Despite the serious lack of understanding of the nature of imaginaries, many formulae and applications of complex numbers were developed in the eighteenth century. The developments were along two lines. Firstly complex relations between trigonometric, logarithmic and exponential functions were discovered, and applications flowed therefrom. Secondly, towards the end of the eighteenth century an attempt to assign meaning to the notion of complex numbers was undertaken with some success.

In 1714, Roger Cotes, an Englishman and contemporary of Newton made the first breakthrough in trigonometric complex relationships; he in fact derived the formula

'$i\phi = \operatorname{Log}(\cos\phi + i\sin\phi)$' (in modern notation)

This is the first interesting departure from the mere manipulation of imaginaries in the solution of equations, towards achieving a meaningful mathematical relationship of some importance.

This discovery was closely followed by the still very useful result known as DeMoivre's theorem: namely

$$(\cos\phi + \sqrt{-1}\sin\phi)^n = \cos n\phi + \sqrt{-1}\sin n\phi$$

Although this formula bears DeMoivre's name, it is not explicitly stated by him in any of his writings. However, in many of the theorems he proved, it is clear that the relationship and its applications were well known to him from 1722 onwards. Furthermore, in certain passages it is suggested that certain eliminations be performed; on so doing, one arrives at the above formula. One such example is as follows,

> <u>Lemma 1.</u> If l and x are the cosines of two arcs A and B of a circle of radius unity, and if the first arc is to the second as the number n is to unity then
>
> $$x = \frac{1}{2}\sqrt[n]{l + \sqrt{l^2 - 1}} + \frac{\frac{1}{2}}{\sqrt[n]{l + \sqrt{l^2 - 1}}}$$
>
> (<u>Miscellanea Analytica</u>
>
> London 1830. A. DeMoivre trans. R. C. Archibald)
>
> (Quoted in D. F. Smith <u>Source Book in Mathematics</u>, p. 446).

From the above Lemma one can obtain a relationship between the two angles subtended by the arcs, and by using the theorem attributed to DeMoivre one can easily obtain the above result.

This theorem has many applications not only in trigonometry but in the eighteenth century analysis and applied mathematics.

An explicit demonstration of the formula was given in 1748 in <u>Recherches sur les Racines Imaginaires des Équations</u> by L. Euler. He demonstrated the problem as follows:

consider the product

$$(\cos\phi + \sqrt{-1}\sin\phi)(\cos\theta + \sqrt{-1}\sin\theta) = \cos(\phi+\theta) + \sqrt{-1}\sin(\theta+\phi)$$

which relationship will hold true for higher products. If $\phi = \theta$ one can obtain

$$(\cos\theta + \sqrt{-1}\sin\theta)^2 = \cos 2\theta + \sqrt{-1}\sin 2\theta$$

which will also hold true for higher products and one can write

$$(\cos\theta + \sqrt{-1}\sin\theta)^m = \cos m\theta + \sqrt{-1}\sin m\theta$$

where m is a positive integer. To prove the truth of the formula where m is any real number, Euler showed the identity remained when logs. were taken and both sides differentiated with respect to θ.

L. Euler was a prolific mathematical writer. He made many contributions to most branches of mathematics. It was on his suggestion in 1728 that the letter e be used to represent the base of natural logarithms, which was to facilitate his own contributions to complex relationships. Notably he showed that trigonometrical and exponential functions were connected by the inverse of Roger Cotes' formula, namely

$$\cos\theta + i\sin\theta = e^{i\theta}$$

At the same time he developed the familiar relationships

$$\cos v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}, \quad \sin v = \frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}$$

Strangely the imaginary numbers were giving rise to more powerful mathematical relationships. In 1746 J. D'Alembert attempted a proof that all complex numbers were of the form $a + \sqrt{-1}\, b$ where a and b were real numbers. In 1751 Euler showed that every real or imaginary number has an infinite number of logarithms, only one of which was real, and in 1777 he introduced the use of the letter i to denote the square root of -1. However, despite these developments mathematicians were still manoeuvreing in the dark.

The applications of complex numbers were becoming more numerous; significant trigonometric identities between complex numbers suggested there ought to be more of an explanation of them than algebraic accident. One mathematician sensitive to the arbitrary way in which negative and complex numbers had been assimilated into analytic proofs, was the British mathematician F. Masères. In 1758 he published a work entitled <u>A Dissertation on The Use of the Negative Sign in Algebra</u>. The work was a little more comprehensive than its title suggests. It was not a work containing new discovery, but rather an attempt to raise the problems of rigour in algebra and present some rules as regards the operations of algebra.

Masères felt there existed a need to render algebra more like geometry, to give algebra a firm logical foundation, such that its results would not only be considered useful mysteries, but very definitely mathematical fact. His point about the use of the negative sign is that it should be considered as relationally dependent, that is the so-called 'negative numbers' are not to be considered alongside the operations of algebra, the signs depend on position in relation to other numbers. He did not, like Peacock in 1830, introduce the notion that the signs could be 'signs of affection'.

The problem was, that since the laws of algebra were simply the laws of common arithmetic operating on variable quantities, it would seem that results unexplainable in arithmetic should be considered inadmissible. Indeed, in the interests of rigour this was a possible attitude. Towards the end of the eighteenth century this was the attitude of another British mathematician, W. Frend. He expounded his point of view on the need for rigour, in a text-book entitled The Principles of Algebra in 1796.

Frend takes up a very stern point of view on the hitherto accepted method in algebra.

> "The first error in teaching algebra is obvious on perusing a few pages only in the first part of Maclaurin's algebra. Numbers are there divided into two parts, positive and negative, and an attempt is made to explain the nature of negative numbers, by allusions to book-debts and other arts. Now, when

a person cannot explain the principles of a science without reference to metaphor, the probability is that he has never thought accurately upon the subject".

(The Principles of Algebra, London, 1796, Pref. p.x.)

The point Frend is making is that if one is operating with arithmetical quantities, a change in their interpretation cannot be countenanced simply when convenient.

"... though the whole world should be destroyed one will be one, and three will be three, and no art whatever can change their nature. You may put a mark before one which it will obey: it submits to be taken away from another number greater than itself, but to attempt to take it away from a number less than itself is ridiculous." (ibid p.x.)

He continues, with some amazement at the foibles of his fellow algebraists,

"... they talk of solving one equation, which requires two impossible roots to make it solvable, they can find out some impossible numbers, which, being multiplied together produce unity." (Ibid, p. XI.)

Frend's answer to such logical absurdities is to dispense with them,

"... if there should be an impossible root in the conclusion, he (the reader) will impute it to the proper cause, either to an error in his mode of reasoning, or to false premises". (Ibid, p. XI.)

An interesting point is raised here. I have said that algebra had adopted the rules of arithmetic. However, arithmetic itself had developed more or less empirically and had not the claim to rigour that geometry had; the operations of arithmetic had no axiomatic basis, no strict definitions of the nature of symbols. Hence it would have been reasonable to assume, that the logical difficulties arising in arithmetical algebra might well have their origin in the empirical basis of arithmetic.

Nevertheless, Frend did in fact attempt to set out the basis of arithmetical algebra,

"Algebraical characters are of three kinds; being either marks of numbers, or of the relations of numbers to each other, or the mode of working by numbers". (Ibid, p. 3.)

The work continued with a predominently elementary discussion of algebra, limited by the ideas stated.

What is, however, of special interest in Frend's ideas, is that it would seem that the recognition of the need for rigour in many fields was noticed by the British mathematicians. Frend could not be blamed for wanting to limit algebra conceptually to arithmetic, for at that stage, the geometrical interpretation of imaginaries was not known. But one noticeable feature of mathematical work in Britain was the tendency to try and demonstrate even the calculus geometrically; geometry being the discipline that had an axiomatic basis.

Now, essentially the difficulty of Frend's ideas was that 'impossible' numbers had become invaluable in analysis. Moreover the trigonometric identities, in particular, suggested there should be a consistent geometrical interpretation of them, in the absence of an arithmetical one.

Such an interpretation was forthcoming in 1796 by a Norwegian C. Wessel. The paper is important in two respects; it interprets complex numbers, and offers a mathematically very rigorous approach to the manipulation of 'imaginaries'. Unfortunately, this particular paper attracted less attention than later contributors on the subject; in particular the work of J. R. Argand in 1806 became better known, although it lacked Wessel's rigour.

However, it will be of some interest to contrast his approach with those later mathematicians to be considered. He opens his paper with the following statement of intent:

> "This present attempt deals with the question, how may we represent direction analytically; that is, how shall we express right lines, so that in a single equation involving one unknown line, and others known, both the length and the direction of the unknown line may be expressed". (<u>On the Analytical Representation of</u>

Direction, An Attempt, 1799) (Quoted in D. E. Smith, Source Book in Mathematics I, p. 55.)

In answer he considers firstly, one interpretation already considered of negatives, namely, the accepted means of changing the direction of a line, in an opposite sense, is by a negative product, he states,

> "To help answer this question, I base my work on two propositions which seem to me undeniable. The first one is changes in direction which can be effected by algebraic operations shall be indicated by their signs. And the second; direction is not a subject for algebra except in so far as it can be changed by algebraic operations". (Ibid, p. 55.)

Thus the way he is introducing the role of imaginaries is oblique, that is he approaches the subject from the point of view of the effect of complex numbers rather than from defining them, which seems an interesting departure from previous eighteenth century ideas. Also, it seems clear from the outset that Wessel is attempting to offer a wider interpretation to geometrical concepts than the operations of arithmetic would allow; for example he begins by extending the concept of 'addition' geometrically as follows,

> "... if one side of a triangle extends from a. to b. and the other from b. to c., the third one from a.

to c. shall be called the sum". (Ibid, p. 59.)

That is ab + bc $\neq$ ac being the shortest distance from a. to c. A line of the same magnitude as ac in the opposite sense would be denoted by -ac. The system is consistent, however many lines are summed in this way. What Wessel does in this way is to construct a system with its own definitions and rules. In inspiration the rules of the system come from arithmetic, but Wessel gave a totally new interpretation to them.

Basically what Wessel defines is a Cartesian plane with one real axis of unit 1, and one imaginary axis of unit ϵ,

"Let +1 designate the positive rectilinear unit and $+\epsilon$ a certain other unit perpendicular to the positive unit and having the same origin; then the direction angle of +1 will be 0° and that of -1 to 180°, that of ϵ to 90° and that of $-\epsilon$ to -90° or 270°. By the rule that the direction angle of the product shall equal the sum of the angles of the factors, we have;

$(+1)(+1) = +1$; $(+1)(-1) = -1$; $(-1)(-1) = +1$; $(+1)(+\epsilon) = +\epsilon$; $(+1)(-\epsilon) = -\epsilon$; $(-1)(+\epsilon) = -\epsilon$; $(-1)(-\epsilon) = +\epsilon$; $(\epsilon)(\epsilon) = -1$; $(+\epsilon)(-\epsilon) = +1$; $(-\epsilon)(-\epsilon) = -1$;

From this it is seen that ϵ is equal to $\sqrt{-1}$; and the divergence of the product is determined such that not any of the common rules of operation are contravened .... If a., b., c., denote direct lines of any length, positive or negative, and the two indirect lines $a + \epsilon b$ and $c + \epsilon d$ lie in the same plane with the absolute unit, their product can be found, even when their divergences from the absolute unit are unknown, for we need only to multiply each of the added lines that constitute one sum by each of the ones of the other and add the products; this sum is the required product in respect to the extent and direction, so that

$$(a+\epsilon b)(c+\epsilon d) = ac - bd + \epsilon(ad+bc)$$

(Ibid, p. 60 and 62)

One of the consequences of this system is that multiplication by ϵ expresses a rotation through 90° by $-\epsilon$ a rotation in the opposite sense through -90°. Certainly this discovery was a landmark in the development of complex numbers; a concrete interpretation had been given to the mystifying imaginaries. Wessel used his system to demonstrate many of the known relations between complex numbers and trigonometric functions, in the same paper; in so doing, he had only recourse to the rules he had set down for operating with them. He had discovered the use of the imaginary axis which J. Wallis had missed; for the first time an operational definition of complex numbers, as producing rotations of lines in planes, had been given.

However, the problem of finding a logical base for common algebra was not yet solved. The system of Wessel was not free either from arithmetical or geometric intuition, this was to be the essential contribution of the British school. The other significant improvements in eighteenth century algebra appeared in the methods of solution of polynomial equations and systems of linear equations. In 1750 G. Cramer, a Swiss, demonstrated the rule for the elimination of unknowns from a set of linear equations using a determinant method; a rule which still bears his name. However, the first logical exposition of the theory of determinants was not given until 1772 by A. J. Vandermonde, who is generally considered to be the founder of the theory.

There existed, in the main, only isolated results for the solution of polynomial equations; general methods of solution were available only for equations of degree less than five. No-one had successfully established a general method of solution for the quintic equation. In 1770 the eminent French mathematician J. L. Lagrange, published his results in this field in Réflections sur la résolution Algèbrique des Équations. He had studied all the methods of solution used up to that time for equations of low degree. He traced the solutions to one uniform principle. This consisted of the formation and solution of equations of lower degree, whose roots are linear functions of the roots required and the roots of unity. However, in the case of the quintic this method broke down, since the 'resolvent' turned out to be an equation of higher degree. The conclusion Lagrange did not reach, was that the quintic was insoluble by radicals; this was not proved until 1826 by the Norwegian, N. H. Abel.

However, during the course of this research Lagrange was led to consider the effects on the symmetrical root functions of different permutations on the roots of equations. Similar research was conducted by a British mathematician, E. Waring. Waring's bitter complaint was that no British mathematicians read of his researches: he had to rely for criticism and praise from the more advanced continentals such as Lagrange.

The significance of these eighteenth century researches was realized in the early decades of the nineteenth century. The solution of equations by radicals was examined in a more general way by means of Galois' group theory which did not become widely recognized until the 1840s.

Great changes took place in other branches of mathematics in the early nineteenth century. The nature of these changes was based primarily on a fresh approach to well-established mathematical practice. The first important reformation came as a result of the publication in 1821 of a series of lectures given by Cauchy to students at the École Polytechnique. The subject of the lectures was rigour in analysis. For the first time a meaningful mathematical definition was given for the limit; from this definition Cauchy was able to introduce rigour into the concepts of continuity and convergence. His work set the standard for the much needed rigour in analysis for some years.

In 1826 a Russian mathematician, N. I. Lobachewsky made public a new theory of geometry. Little notice was taken of the theory until a few years later, when its implications surprised mathematicians and philosophers alike. For 2,000 years Euclid's system of geometry was in some sense regarded as being an absolutely 'true' representation of space. Lobachewsky demonstrated the revolutionary discovery that by denying Euclids fifth (parallel) postulate, one could still retain a consistent geometry and establish new 'truths' about an unfamiliar space. Lobachewsky in abolishing the 'necessary' truth of Euclidean geometry indicated a new course for mathematicians and scientists; that of challenging other accepted 'axioms' and laws. This approach was especially fruitful in the future development of algebra.

The important changes in algebra did not take place until the 1830's and 1840's. Firstly there was the establishment by the British Algebraists of the independent logical foundations of Algebra; secondly there was the development of generalized theories of algebraic structure and algebraic systems not tied to the traditional concepts of quantity and commutativity.

One can see that by the close of the eighteenth century the development of algebra, limited by the arithmetical concept of magnitude was virtually exhausted. New and broader concepts were needed to solve those problems inherited from the eighteenth century researchers.

The somewhat isolated development of axiomatic algebra in Britain in the early decades of the 19th century can, to some extent, be traced back to these problems developing towards the close of the previous century. However, this question will be one of the subjects of the following chapter.

CHAPTER II

Background to the Development of British Algebra in the Nineteenth Century

It was suggested in the previous chapter that British mathematics had suffered a grave decline in the eighteenth century and very early nineteenth century in relation to the developments taking place on the continent. For the purposes of this thesis, it is necessary to consider in which ways this alleged decline affected the development of British mathematics; further to analyse the ways in which the position was slowly altered, particularly the way in which the mathematicians to be considered contributed to the eventual reforms. It will also be of interest to examine whether the situation of mathematics discussed in the last chapter, bore any relation to the development of a strong British School of Algebra.

The mainstream of criticism of British mathematics in the early nineteenth century was from those people who could generally be said to hold 'liberal' opinions. The reasons they offered to explain the alleged decline of science involved severe criticisms of the established institutions of learning and intercourse, namely the universities and the Royal Society, which were to a great extent responsible for prevailing scientific ideas. Thus to improve the status of British science and mathematics, reform in spheres other than the purely intellectual were deemed to be necessary.

Some of the first complaints on the nature of advanced mathematics in Britain appeared in the organ of liberal opinion, the Edinburgh Review. In a review of Laplace's Mécanique Céléste, J. Playfair, Professor of Mathematics at Edinburgh, makes the following points;

> "In the list of mathematicians and philosophers to whom the science of astronomy for the last sixty or seventy years has been indebted for its improvements, hardly a name from Great Britain falls to be mentioned .... Nothing prevented the mathematicians of England from engaging in the question of lunar theory ... but the consciousness that in the knowledge of higher geometry they were not on a footing with their brethren on the continent. We will venture to say that the number of those in this island who can read the 'Mécanique Céleste' with any tolerable facility is small indeed". (Edinburgh Review, II, 1808, p.279)

This suggests a sad state of affairs for the nation, which not a century before had boasted the prowess of Isaac Newton, whose contribution to mathematics had generated the mathematical inspiration of such men as Euler and Laplace; the Edinburgh Review of 1816 puts it thus

> "It is certainly a curious problem with respect to national genius whence it arises that the country in Europe most generally acknowledged to abound in men of strong intellect and sound judgement should for the last seventy or eighty years have been inferior to so many of its neighbours in the cultivation of that science which requires the most steady and greatest exertions of understanding, and that this relaxation should immediately follow the period when the greatest of all mathematical discoveries had been made in that same country".
> (Edinburgh Review XXVII, 1816, p. 98.)

One of the generally accepted reasons for this decline in mathematical innovation after Newton, is the aftermath of the Newton-Leibniz controversy. Much mathematical intercourse with the continent was ended; both the prolonged isolation from the new analytical methods of continental researchers, and the slavish deference to the Newtonion fluxional notation in the calculus, to the exclusion of the differential notation, had a serious and injurious result on the advance of mathematics in this country.

In 1830 Charles Babbage, who pioneered the first computer, and had earlier pioneered the notational reform at Cambridge, published Reflections on the Decline of Science in England and on Some of its Causes. The concern of this publication was not simply to show how Britain was lagging behind her continental neighbours in the development of pure science, but also, to give concrete reasons why this situation existed, thereby implying how it might be remedied. One of the reasons he put forward was the inadequacy of the Royal Society both as the central agency for communicating scientific and mathematical ideas on a national and international basis, and as an institution which should assist in the promotion of the general interests of scientific advance. He compared the nature of the Royal Society with the thriving French institution the 'Paris Academy of Sciences'. His central criticism concerned the composition of the Fellows of the Society. In similar continental institutions the membership was small and select. Only the most eminent men of science were privileged by membership; all of them were expected to have themselves made original and worthy scientific enquiries and were, therefore, recognised as a body whose pronouncements on new scientific papers and discoveries would be of the highest repute. This situation Babbage claimed, was alas, not true of the Royal Society. For example, England with a population of 22m. boasted 683 members; France with 32m. only 75, and Prussia with 12m. had 38 members of the Berlin Academy. This suggests that membership was a greater sign of scientific merit on the continent since the same seemed more competitive. Furthermore, the actual breakdown of members of the Royal Society shows that very few of them had any claim to science whatsoever. For example, in 1830 there were ten Bishop members from whom only nine papers had been contributed to the transactions; those nine all came from one Bishop, the Bishop of Cloyne. Of 63 Temporal Lords, no contributions whatsoever were made, of 74 clergymen precisely eight contributions were made.

The contribution ratios of the professional members were slightly better than these; indeed there were many distinguished contributors of whom Babbage was one. However, it is clear that a great part of the membership had, scientifically speaking no right whatsoever to membership.

The Royal Society then had, to Babbage's mind, a share in the responsibility for Britain's mathematical and scientific eclipse. His second point of grievance, is the absence of incentive in Britain for scientists to maintain scientific research and the absence of professional status;

> "The pursuit of science in England does not constitute a distinct profession, as it does in other countries .... Even men of sound sense and discernment can scarecely find means to distinguish between the possessors of knowledge merely elementary and those whose acquirements are of the highest order. This remark applies with peculiar force to all the

more difficult applications of mathematics and the fact is calculated to check the energies of those who only look to reputation in England". (Reflections on the Decline of Science, London, 1830, p. 10.)

Not only were there few professional opportunities for scientists outside the limited number of academic positions in the universities, but also, Babbage complained, little civil honour was granted to British scientists. On the continent, France in particular, he pointed out that those men of science who had honoured their country with discovery were likewise honoured by their governments. Laplace, from humble origin, became a Marquis and held public office. Monge and Fourier were personal companions of Napoleon on his voyages of conquest. Many German scientists were granted independence for their scientific labours by the patronage of princes. Babbage himself crossed swords with his own government many times over their reluctance to give him much financial assistance with his computing venture.

Despite the undoubted validity of many of his criticisms and despite support for them from eminent academics, a number of contemporaries found his remarks unjust. One such was A. B. Granville, F.R.S., who published in 1830 Science without a Head. In this work he takes to task the most virulent critics of British science, Babbage and the Edinburgh Reviewers, while setting out himself to suggest reform of the structure of the Royal Society which he recognized functioned not as it might. Another critic of Babbage's book, was a foreigner, one Dr. Moll of Utrecht. He pointed out that English scientific pursuits were still highly thought of abroad, and followed with eagerness. Also it must be said that Britain had boasted a number of important scientific discoveries, the point, however, being made by the critics was that there had been a decline in theoretical science and higher mathematics. Baden Powell, Savilian Professor of Geometry at Oxford, suggested the problem as follows:

> "It is not twenty years since we have begun to perceive that we were far behind all the rest of Europe in these (mathematical) sciences, not from want of abundance of first rate talent, but from a misapplication of that talent to unworthy objects, or at least to such as were of a nature not calculated to lead to any great advance in the state of knowledge". (History of Natural Philosophy, 1834, p. 367/8.)

Baden Powell further considered that, even when the methods and works of the continental analysts were introduced into the institutions of learning in the twenties, the spirit of the mathematics to follow was concerned more with detailed improvements and amended treatises than extensive and original researches.

Baden Powell was one of the leading critics of the nature of the mathematics taught within the Universities of Oxford and Cambridge, which until 1828 had a monopoly of academic education in England (Scotland had its own universities). Criticism of the sterile contents of the universities' syllabuses and the standards of teaching came early in the century from the Edinburgh Review and later from persons within and without the cloistered walls.

What was taught at the two great universities in the early part of the century was to a large extent governed by those subjects the students had to take for the B.A. degree. Lectures outside these syllabuses tended to be sparsely attended. For the pass degree at Cambridge, the students' knowledge of mathematics needed only to extend to the first two books of Euclid and simple and quadratic equations; for honours, the subjects examined for mathematics were arithmetic, algebra, fluxions, the doctrine of infinitesimals and increments, geometry, trigonometry, optics and astronomy. The requirements for the Oxford B.A. were considerably less than these.

However, the actual syllabus was not the only focus of criticism. Many felt that the religious tests prevented good scholars, both of scientific and artistic bents, from studying at the universities and taking degrees there, simply on grounds of dissension from the articles of the Church of England. And the clerical oligarchies controlling the universities were opposed, in general, to reform in any sphere since they felt that this would endanger religious control of the universities.

Clearly if science and mathematics were to develop more rapidly, the institutions discussed had to be reformed in many respects. The scientific academies on the continent exposed the inadequacies of the Royal Society in comparison; likewise the German universities and technical high schools and the great French scientific schools, the most famous of which was the École Polytechnique, exposed the deficiencies of the great English universities in respect of scientific education, and many British scientists became increasingly sensitive to these facts in the first half of the nineteenth century.

The first attempt to improve the situation of British mathematics came from within Cambridge itself. In 1812 a small group of undergraduates at Cambridge formed what they called the Analytical Society. Being in the habit of breakfasting together on Sunday mornings, they used the time to discuss points of common interest. The common interests included works on the calculus by the great continental mathematicians such as Lagrange and one less celebrated Cambridge mathematician, R. Woodhouse.

In 1803 Woodhouse had published The Principles of Analytical Calculation, in which he had explained the continental analysts' use of the differential notation and advocated its introduction.

However, he also included severe criticism of their analytic methods, in the sense that they operated from an intuitive rather than strictly rigorous basis. The three undergraduates were G. Peacock, C. Babbage and J. F. W. Herschel; impressed by such ideas and sensitive to the barren nature of British mathematics, their object in forming the society was to introduce continental methods into the Cambridge syllabus. Babbage coined the aphorism, that they were to advocate "the principles of pure d'ism as opposed to the dot-age of the university" (Passages from the life of a Philosopher, 1864, p. 29.) ; A dot-age was a reference to the fluxional notation of Newton.

In 1813 the Society published a volume of memoirs by the members which included work on the calculus. In 1816 they published a translation of the textbook by Lacroix entitled <u>Traité Élémentaire du Calcul Différentiel et du Calcul Intégral</u>. However, despite this useful propaganda, while the examination questions of the Senate House of Cambridge retained the fluxional notation, there was little chance of the differential notation gaining usage.

The Society's chief opportunity, therefore, came in 1817 when Peacock was made a moderator for the mathematics examinations. In the same year he introduced questions on the paper necessitating knowledge of the new methods from the student.

In 1819 the new notation was adopted at Peacock's request by

R. Gwatkin of St. John's College, and in 1820 it was adopted by Dr. W. Whewell of Trinity, who was to become himself an influential advocate of scientific reform in the university. The success of the Society was such that after 1821 the old notation appeared only at rare intervals in the Senate House examination.

In 1819 Whewell published a volume on mechanics in which the differential notation was employed, and, in the same year, the founders of the Analytical Society formed a more permanent society namely The Cambridge Philosophical Society. After their victories in the Senate House examinations, they issued, in 1820, two volumes to illustrate the new methods; the first by Peacock on differential and integral calculus, the second by Herschel on the calculus of Finite Differences. After this time all elementary works on the calculus abandoned the exclusive use of the fluxional notation. In 1826 G. B. Airy, a pupil of Peacock's, published Mathematical Tracts in which the continental works on lunar and planetary theories were elcuidated. By this time the door was open for British mathematicians to show the means and the inclination to examine the manifold discoveries of such as Euler and Lagrange, and thereby to improve on these discoveries. One thing was clear, after almost a century of isolation the British analysts were at an enormous disadvantage. The disadvantage persisted for some few years after the efforts of the Analytical Society and inhibited British mathematicians from making major contributions in the in the field. However, the full implications of this early reform will be considered shortly.

In 1828, coordinated response from liberals in politics and the academic world, religious dissenters and educational reformers enabled the establishment of a new university to take place. It was to be in London and thereby serve the interests educationally of a very large populace. It was known as the 'University of London' until 1836 when it became a college within a broader university organisation. The new college was non-residential and, therefore, more easily secular, ensuring the possibility of higher education to anyone, regardless of their religious convictions. The new university was dedicated to the idea of 'liberal education'; it was able to incorporate into its curriculum the newest branches of knowledge. Also, since it was founded independently of Church and State there existed the possibilities of using new teaching methods, and disputing new ideas, without the hindrance of centuries old laws and statutes as in the older universities.

The curriculum included classical languages, British language and literature, modern language, political economy, mathematics, physics, astronomy, chemistry, zoology, botany, English law, jurisprudence, engineering and medicine. There is a refreshing emphasis here on the sciences, but in particular the standards set in mathematics were very high. One Augustus De Morgan was made the first professor of mathematics there at the age of only twenty-two. Had he not been offered this professional appointment, it is very likely he would have taken up law and the consequences might have been to diminish his enormous contribution to mathematics; his impressive contribution to the foundations of algebra will be considered in further chapters.

From his early position of advantage as a professor of mathematics Augustus De Morgan contributed a great deal to the popularization of mathematics at various levels. He contributed many articles on the teaching and nature of elementary mathematics to Brougham's Penny Cyclopaedia; a popular series designed to inform and instruct ordinary people who would normally know little of the mainstream of contemporary political, moral and scientific thought. De Morgan also made numerous contributions at a more advanced level to the Quarterly Journal of Education, including a very important review of G. Peacock's book on the foundations of algebra, which will be discussed in detail in a forthcoming chapter. He wrote prolifically on aspects of the history of mathematical development. He was a regular contributor to the Cambridge Philosophical Society, and had in fact been a pupil of G. Peacock. In general it can be said that De Morgan was one of those who demonstrated the need for mathematical education at all levels, and the shift from leaving research to gifted amateurs and placing it in the hands of professionals.

Another undertaking by De Morgan, with a number of other committed scientists, some already mentioned, was to assist in the foundation of a society, which could fulfil what was required of the Royal Society and overcome the criticisms made by men such as Babbage. This society was known as the British Association for the Advancement of Science. Not surprisingly, it was Babbage who took the first initiative.

In 1828, whilst travelling on the coninent, Babbage attended a conference in Berlin of leading scientists and philosophers. This was the sixth of such conferences founded by the German scientist Oken. The conferences were organized to promote scientific intercourse, and Babbage was impressed by the undertaking. On his return to England he wrote accounts of the German Assembly for the Edinburgh Journal, and an appendix on it in his own book on the Decline of Science.

The contents of his article were widely discussed in scientific circles in England and a favourable review of his book appeared in the Quarterly Review in 1830 by Sir David Brewster, in which it was suggested that a similar attempt at promoting scientific exchange should be undertaken in this country.

The suggestion was taken up in 1831, the 'British Association for the Advancement of Science' was founded; the first meeting was held in York. Among its supporters were Babbage's friends of the Analytical Society, De Morgan, Whewell, Baden Powell and the various critics of the Royal Society, including some of its members. The objects for the Association laid down at its first meeting were, firstly, that the Association should bring men of science together that they might give systematic direction to scientific enquiry. Secondly, that men of high ability in each field of thought should prepare reports on the present stage of development in that sphere, This was useful from a number of standpoints, the report would advertise work that had been done and where posible research could be taken up, also it would inform people researching in other fields where useful developments contingent with their own were taking place. One of the most interesting of the early reports from the point of view of the subsequent discussion was that of G. Peacock in 1833 on the state of mathematics and prospects for development, however, it will be considered more fully in the next chapter.

One of Babbage's suggestions for the Association was that its meetings should be held in places likely to bring science into contact with the practical skills of industry such as the midland industrial towns, since the wealth of the country would ultimately depend on the degree to which the sciences assisted in accelerating technical innovation. But, perhaps, the most effective feature of the Association was the setting up of working committees which undertook to do special work where concerted action was needed.

The British Association made a tremendous difference to British science over the next decades. Its influence was felt as a pressure group in the interests of science in many spheres, and it was of great use in communication between scientists, not only British scientists, but also increased communication with the continental institutions. But most important, it was a response to the criticisms made of the Royal Society, not only did it supplement the work of the Royal Society, but instituted methods of communication not undertaken by anyone.

Having now discussed both the criticisms of the Royal Society and the universities and indicated those ways in which scientists and others chose to remedy the situation, it will be of interest to examine what relationship the institutional reforms bore to the subsequent developments in mathematics.

Certainly there seems to be a considerably body of evidence to show that the decline in mathematics, in Britain, in the eighteenth century in relation to the continental advances can be attributed to a number of interrelating factors. Firstly, and perhaps most important, the prolonged isolation from intercourse with continental institutions, and secondly the aforementioned inadequacies of the universities, Royal Society and Governmental indifference.

When the reforms had been effected, clearly in time they 'bore fruit', that is, there was a tremendous change in the quality and the quantity of mathematical research in the second half of the nineteenth century as a result of the reforming trend in the first half. However, steady advances were being made in mathematics from about 1830 before the reforms had time to become really effective. It would seem then, that those mathematicians engaged in the various aspects of reform were themselves already contributing to the improvement of mathematics.

One of the reasons for this involvement of such mathematicians is that those most actively engaged in producing new lines of research will be most aware of the greater advances made on the continent, thereby they will be most concerned to improve the social and intellectual stimuli to mathematical advance.

While the discussion so far has been able to offer some reasons for Britain's mathematical recovery in the 1830's and 1840's, the problem remains that the major original developments took place in the foundations of algebra, rather than analysis or higher arithmetic.

There were a number of factors which might have contributed to this particular trend. Firstly, Euclidean geometry had played a very important role in British mathematics and often the only formal education a mathematician received was in geometry. Newton used geometric constructions to demonstrate the calculus, and this tendency persisted in the subsequent work of British analysts. The importance of geometry was clearly in its rigorous logical foundation, its results appeared to represent infallible truths. Naturally enough if other branches of mathematics were to be as acceptable they had to approximate to its methods. The attack by G. Berkeley in 1734 on the practice of the calculus is an indication of the consciousness of British mathematicians of the importance of rigour. Even in the schools the emphasis in mathematical teaching was on Euclidean geometry.

It was mentioned earlier in the chapter that R. Woodhouse, whose work was studied by the Analytical Society, had emphasized the lack of rigour in the work of continental analysts. Peacock and Babbage must, therefore, have been aware of the importance of formalization in mathematics. Indeed, at that time on the continent Gauss and Cauchy were making attempts to put the calculus on a more formal footing.

Now it was also a feature of algebra, particularly complex algebra, in the eighteenth century that it was lacking in formal structure and, as mentioned in the last chapter, attempts to confine algebra to stricter rules had been undertaken by William Frend at the end of the eighteenth century. The attempts might have been limited in success, but they did suggest a line of enquiry for later mathematicians.

So far two main influences have been isolated; the awareness of the need for rigour in general, and in particular, the need for formalization of algebra. When the notational reform was adopted in Cambridge British mathematicians realized how far behind the continental advances in analysis they were; for some time they were therefore, unlikely to make original contributions in that field. Likewise in a number of other fields they were at a disadvantage. The German mathematician Gauss had undertaken work in higher arithmetic, analysis and the theory of equations, which again would take time to be assimilated generally in British mathematics.

Considering this combination of factors it does not seem unreasonable to suppose that the foundations of algebra should be an area in which British mathematicians might be likely to make inroads. Indeed, the first contributor was one of the Analytical Society, G. Peacock. I would not say that because the aforementioned factors existed the mathematicians took a conscious decision to pursue one line of research. In fact many lines of research outside algebra were pursued with considerable success. But mathematicians are likely to be influenced by the trends in their subject, moreover, the importance of their work can only be seen clearly in retrospect and perhaps it is for this reason the development of axiomatic algebra seems the logical consequence of the factors enumerated. Also in retrospect, the British development appears to coincide with the trends in mathematics on the continent.

The general trend was twofold; increasing rigour on the one hand and the establishment of more general theories on the other. The calculus was gradually established on a rigorous basis, the British school axiomatized algebra, group theory was developed by Galois generalizing previously isolated results in the theory equations: similarly, Gauss generalized results in the theory of numbers, Lobachewsky generalized geometry by constructing the first non-Euclidean geometry, and the results of the British school eventually led to the development of new algebras opening up whole new possibilities in algebraic systems.

The papers on the foundations of algebra to be considered appeared in a relatively short period of about fifteen years. At the end of this period British mathematics was once again flourishing. The reform movements had succeeded in most of their objects, Oxford and Cambridge after the Reform Bills in 1852 and 1854, once again became stimulating centres of research in the sciences. The culmination of the efforts of the mathematicians discussed was the establishment of the first mathematical society in London in 1865; the inspiration, in particular, of Augustus De Morgan. The society was in fact a model for many such societies in other countries established in the years that followed.

CHAPTER III

The Emergence of Formal Algebra

The first major contribution of the British school to the foundations of algebra came from George Peacock. As suggested in the previous chapter, he was personally committed to changing established attitudes in British mathematics, in education and research and in popularizing the researches of continental mathematicians. He was a talented mathematician; in 1813 he was second wrangler in mathematics at Trinity College and in 1814 took up a fellowship there. His contribution to the work of the Analytical Society has been already outlined.

His involvement in this circle was sufficient to make him aware of the critical trends in early nineteenth century mathematics. In particular, his Report to the British Association shows he was aware of the shortcomings of algebra. The alternative to these shortcomings proposed by W. Frend was clearly unsatisfactory; too many useful results with complex numbers could not be accounted for.

The work of Wessel demonstrating the consistent geometric interpretation of complex numbers was supplanted by the work of the Cambridge mathematician J. Warren. In 1828 he published a volume entitled A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities. This was an exposition in which many of the relations discovered between complex numbers were given consistent geometric interpretation. Having demonstrated the potential of complex numbers to describe real phenomena, it seemed that it was necessary to resolve the contradiction between arithmetical algebra and the existence of the, as yet, undefined 'imaginaries'.

This was precisely the task that was taken up by Peacock and continued by other British mathematicians. His ideas were set out in a work entitled A Treatise on Algebra published in 1830. The work opens with Peacock's statement of intent, in the best Euclidean tradition of the British mathematician,

> "The work which I have now the honour of presenting to the public, was written with a view of conferring upon algebra, the character of a demonstrative science, by making its first principles co-extensive with the conclusions which were founded upon them: ...."
> (A Treatise on Algebra, 1830, Pref. p. V.)

In essence this statement sums up that aspect of Peacock's work which was entirely new: he continued,

> "... and it was in consequence of the very particular examination of those principles to which I was led in the course of this inquiry, that I have felt myself compelled to depart so very widely from the form under

which they have commonly exhibited." (Ibid, Pref. p.V.)

Peacock's new departure was in fact to separate the interpretation of algebraic processes from the principles on which they were based; moreover, having asserted that the principles were independent of the interpretation, he was able to show that algebra could be given the demonstrative certainty previously ascribed exclusively to Euclidean geometry. This was Peacock's contribution to algebra; although not seemingly a very astounding discovery, it did in fact have great repercussions in the development of the next decades.

As stated his original ideas are laid out in his Treatise on Algebra of 1830, his subsequent work published in 1843-5 contains nothing fundamentally different, but his ideas are developed somewhat further.

In the preface to his Treatise of 1830, Peacock sets out substantively the process by which he came to construct his formal algebra; one of the criticisms made earlier of eighteenth century algebra was that its form had not developed beyond that of sumbolized arithmetic. Peacock himself was sensitive to precisely this problem; he states,

> "Algebra has always been considered as merely such a modification of Arithmetic as arose from the use of symbolical language, and the operations of one science have been transferred to the other without any statement of an extension of their meaning and application.

Thus symbols are assumed to be the general and unlimited representatives of every species of quantity... " (Treatise on Algebra, 1830, Pref. p. viii.)

The first problem he points out, for a system where symbols are merely general representatives of numbers and the modes of combination are arithmetical operations, is that there can be no proper limitation of their values. His example is $a-(a+b)$ represents an impossible operation for arithmetic, but if $a+b$ is replaced by c, the expression presents the same problem, but ceases to express it. Whereas, if the signs '+', '-' are allowed an independent existence, the operation - being possible in all cases, the separation of symbolical algebra and arithmetical algebra must be defined since the former, clearly, needs its own rules and definitions.

> "It is the admission of this principle, in whatever manner we are led to it, which makes it necessary to consider symbols not merely as the general representatives of numbers, but of every species of quantity, and likewise to give a form to the definitions of the operations of algebra, which must render them independent of any subordinate science". (Ibid, p. xi.)

Hence, if in framing the definitions of operations upon general symbols, the definitions are concerned only with the laws of combination, no reference to the specific nature of the symbols is necessary. Many interpretations can be given to the same symbolic system, including, with some further restrictions, arithmetic itself.

What Peacock then does is to introduce the operations of '+' and '-', and the rules by which they change the symbols and the signs attached to them, independently of what 'meaning' can be ascribed to a symbol such as '-b'. Likewise for operations of multiplication and division, there will be laws regarding the combination of the symbols operated on and secondly laws regarding the signs 'attached' to them. Furthermore, in order to construct a more powerful system it is necessary to be able to do more than simplify expressions within the system. It is necessary to be able to relate equivalent forms, that is using the parallel of arithmetic, to introduce some sign such as '=' to stand for 'algebraical result of'. Also, to be able to reduce results to other algebraic forms, it is necessary to assume the operation of $+$ to be the inverse of $-$ and the operation $\times$ inverse of $\div$. Then

$$a+b-b, \quad a-b+b, \quad a\times b\div b, \quad a\div b\times b, \quad \frac{ab}{b}$$

will be 'algebraically equivalent' to the symbol a.

Assuming such rules to be the basis for a symbolic algebra Peacock is clearly drawing on the already accepted processes of arithmetic; that is, he has a view to use the interpretation of operations on known magnitudes, to determine many of the assumptions made for symbolic algebra. Peacock calls this use of arithmetic the 'science of suggestion',

> "... that is, as the science, whose operations and general consequences of them, should serve as the guides to the assumptions which become the foundations of symbolical algebra." (Ibid, p. xii)

Hence in symbolical algebra, $a+a$ will mean the double of a and be denoted by $2a$, $a+a+a+a$ will be $4a$, $5a-3a=2a$ and so on, as one would expect from the difference of the coefficients in arithmetic.

One principle which Peacock elaborates is very important to his use of arithmetic as the 'science of suggestion'. It is known as the 'Principle of Permanence of Equivalent Forms'. When an algebraic form results from definable operations, its existence is understood as mathematically necessary. However, if an equivalent form exists, but the operations that produce it are not definable, its existence is no longer understood as necessary. Now, using Peacock's example, the law of indices is well defined for n and m as integers

$$a^n \times a^m = a^{m+n}$$

It is not defined when n and m are general symbols. The Principle asserts that if an algebraic equivalent for n and m generally exists under a suitable interpretation, it will be the one suggested by the well-defined form. We shall examine what use Peacock makes of this principle when he introduces series. The actual statement of the Principle is as follows:

> "Whatever form is algebraically equivalent to another when expressed in general symbols, must be true, whatever those symbols denote."
>
> "Conversely, if we discover an equivalent form in arithmetical algebra or any other subordinate science, when the symbols are general in form though specific in their nature, the same must be an equivalent form, when the symbols are general in their nature as well as in their form". (Ibid, p. 104.)

He states that the first proposition is necessarily true from what has been asserted in relation to symbolical algebra since the form is the necessary result of the laws independent of interpretation. The converse proposition must be true since if an equivalent form exists its symbols are general in form and nature and hence coincide with the form where the symbols have specific value as the form is unchanged from one to the other. Furthermore, since the laws of combination are assumed to coincide with laws in the subordinate science, the conclusions insofar as form is concerned are the same and hence the equivalence existing in one case exists for the other.

Having stressed the need for formality in algebra it seems strange that Peacock should regard this principle so highly. The form of an expression in symbolic algebra depends not on interpretation, but only on its own laws. The question as to whether its forms correspond to a specific subordinate science can only be examined on the consistency of interpretation, that is, whether the laws of the subordinate science correspond to that of the formal algebra, and if so, what restrictions they place on its generality. So while an expression is 'true' within its own system it may have restrictions on it under interpretation. Conversely an expression from a subordinate science can only suggest that the expression in formal terms is derivable from formal laws as Peacock laid down in his introduction to the work.

Peacock uses the principle in his chapter on series;

> "309. The law of the permanence of equivalent forms, (Art. 132) would enable us to conclude that the series which was equivalent to $(1+u)^n$, when the index was general in its form, though specific in its value, must be equivalent to it likewise, when the index is general both in form and value". (Ibid, p. 267.)

Now, at that time, Cauchy had begun to introduce rigour into the treatment of infinite series through his work on limiting processes. Mathematicians generally were beginning to feel that it was inappropriate to assign algebraic equality between $(1+u)^n$ and its corresponding power series without restriction.

Later in the twentieth century, in the sphere of 'Formal Power Series' it became possible to consider equality between power series irrespective of convergence or divergence. However, these ideas were developed after the work of Cantor in the 1870's Cantor invented the set-theoretical techniques whereby it was possible to assign meaning to the equality of infinite classes.

Such an approach was not then available to Peacock. He felt that certain structures needed to be placed on the equality between $(1+u)^n$ and its power series; he states

> "318. In the first place, if the series is divergent for any assignable number of its terms, the sign = does not indicate arithmatical equality of the quantities between which it is placed, inasmuch as the aggregation of any number of its terms, however great, will never approximate to a fixed and determinate value.
>
> "319. We must confine our attention, therefore, to those series which are convergent ..." (Ibid, p. 270)

Subject to such restrictions in the case of series the utility of the Principle is surely diminished.

The inspiration of the principle, it would seem, was the practice of eighteenth century algebraists in respect of real and complex numbers; rules of calculation known to produce consistent results for real numbers, were thereby expected to afford similar results with complex numbers.

At best, the Permanence of Equivalent Forms can be considered an heuristic guide, but I see little cause for its elevation to a principle. Nevertheless, it was generally accepted in algebra for many decades subsequently.

However, the most positive aspect of Peacock's contribution, which has been discussed generally, is the formalization of ordinary algebra. He deals with the basis of his demonstrative algebra in the first chapter of the Treatise. Significantly he opens with a definition of algebra:

> "Algebra may be defined to be, the science of general reasoning by symbolic language". (Ibid, p.1.)

The chapter sets out the properties of the elements of the system and the laws whereby the elements are combined, the following is a summary of the impotant points he makes:

> "2. The symbols of algebra may be the representatives of every species of quantity ... the operations to which they are subject are perfectly general, and are in no respect affected by the nature of the quantity which the symbols denote ..." (Ibid, p. 2.)

The symbols used are generally the early letters of the alphabet, with and without subscripts, to denote the 'known' quantities, e.g., a, b, c, d ... and for the unknown quantities v, w, x, y, z are used. He continues:

"3. All quantities of the same kind admit of being added to or subtracted from each other". (Ibid, p. 2.)

Addition is denoted by the sign + and subtraction by the sign - . Addition may have various interpretations, not just as in arithmetic.

"4. Whenever by the incorporation or combination of two symbols, two similar signs come together, whether + and +, or - and -, they are replaced by a single sign +: but if the two signs are dissimilar, whether + and - or - and + , they are replaced by the single sign -". (Ibid, p. 3.)

The rules continue on the following lines:

5. The operations commonly known as multiplication and division are denoted respectively by $\times$ and $\div$; $a \times b$ means the product of a and b and is more commonly written ab. $a \div b$ means the quotient of a divided by b.

6. The order of multiplication of two, or more, products is indifferent to the result.

7. Division is the inverse of multiplication.

Subtraction is the inverse of addition.

8. If a be multiplied by itself n times the result is written a^n where n is called the exponent.

9. Law of indices generalized.

10. Definitions of coefficient, monomial, binomial, trinomial, etc.

11. Rules pertaining to operations on symbols in brackets.

12. Definition of homogeneous terms.

13. Dimensionality of term not affected by coefficient.

14. The sign = between two expressions can mean identity, or equivalence; that is if both expressions are employed in the same operation, they will produce the same result.

15. The sign $>$ indicates the quantity preceding it is greater than the quantity succeeding it; similarly the sign $<$ indicates the succeeding quantity is less than the preceding quantity.

This chapter then provides the formal definitions for symbolic algebra; the second chapter provides the rules for the mode of operation on the symbols according to the definitions given. It consists of eight rules formalizing the processes of algebra that had been in use for many years, without adding anything new, except in the important aspect of treating the subject in a formal way. Perhaps the most interesting chapter of the Treatise is the third one. In this he considers the relationship of symbolic algebra to arithmetic, the principles of interpretation, and possible interpretations and geometry as the 'science of suggestion'.

In order to examine the role of arithmetical algebra Peacock considered the particular restrictions that the assumption of the laws of arithmetic, would place on generalized algebra. The first restriction is that the signs $+$ and $-$ in arithmetic denote only operations and not what Peacock calls 'signs of affection': that is algebraic entities such as '$+a$' and '$-b$' have no meaning in arithmetic. Furthermore, in arithmetical algebra $a-b$ can have no meaning unless a be greater than b, secondly the 'rule of signs' is proved from the rules of arithmetic, but is an assumption of general algebra. The law of indices cannot be defined in arithmetic for negative indices, but in general algebra one can define a^{-m} as that with which the product of a^{m} is unity. Peacock makes the general point:

> "In one system, all operations are limited by the possibility of interpreting the results consistently with arithmetical prototypes; in the other, the operations are perfectly unlimited, there being a symbolical result in all cases". (p. 69)

He shows that because of the new assumptions that have to be made, symbolical algebra is not derivable from arithmetical algebra, although the converse is possible, the assumptions become laws of the algebra. Symbolic algebra is then based partly on laws borrowed from arithmetical algebra and on new assumptions to circumvent the restrictions on its generality imposed by arithmetic. It is, however, once defined by its rules, independent of all other systems and its formulae are 'true' within its framework, whether they are meaningfully interpreted, or not. Peacock redefines algebra in its most general form:

> "The science which treats of the combinations of arbitrary signs and symbols by means of defined though arbitrary laws". (Ibid, p. 71)

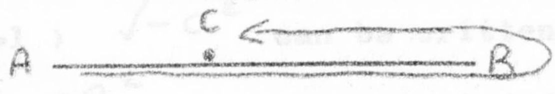
However, it is quite clear that a completely arbitrary system would not invite much interest unless it plays a positive role in terms of the relevant interpretations that can be placed on it. Whatever the interpretation may be, it must conform to the laws of algebra. For example just as + and - are inverse operations, the functions they represent must bear a similar relation. Peacock puts forward a number of possible interpretations.

1. Calculations concerning property could be represented by the algebra, the affection of the signs + and - could correctly symbolize credit and debt.

2. Within geometry the affection of the signs + and - indicate direction and the operations describe distance.

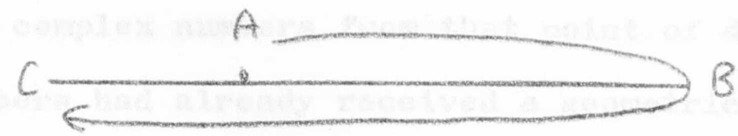
If $AB = a$ and $BC = b$

Travelling from A to B and back again to C a particle's distance from A will be the geometrical difference of AB and BC.



If $AB = a$ and $BC = b$ then $AC = a - b$

Now if $AB = BC$ the particle returns to A. If a is less than b the body will be at C then the distance AC is equal to $a - (a+c)$, if $b = a + c$



then the distance $AC = -c$

Then the signs + and - indicate in which sense the line is described.

Furthermore, the interpretation can be extended to multiplication. The product of two lines denoted by algebraic symbols indicating area, the product of three symbols would indicate volume.

3. Other examples suggested were, time past and future for the affection of signs, and temperatures above and below zero.

The final aspect of importance in the Treatise are Peacock's notions about the treatment of complex numbers. Peacock was aware of the geometrical interpretation given to complex numbers in the early nineteenth century as he mentions in his preface, in particular, that of J. Warren.

In chapter XII of the Treatise he examines the problems relating to complex numbers. He states clearly that in a system of arithmetical algebra $\sqrt{-a^2}$ can have no possible meaning. However, if in symbolized algebra the sign 'i' is introduced to represent the root of -1) $\sqrt{-a^2}$ can be written ai and the product of $ai . ai = -a^2$. If this is included in the definitions of symbolical algebra 'imaginaries' immediately become a well-defined part of the system. He introduces, then $\sqrt{-1}$ as a new 'sign of affection' and proceeds to deduce the laws associating with complex numbers from that point of departure. Since complex numbers had already received a geometric interpretation, Peacock's own geometric treatment was of no special significance. What was significant about his approach was that it was formal. The complex number system belonged to his generalized algebra which included the 'sign of affection' $\sqrt{-1}$. All the results could be deduced solely from given laws and symbols; the same general symbolic algebra could thereby represent simple arithmetic with certain restrictions, or all the known results and relations associated with complex numbers. Peacock had, therefore advanced the demystification of imaginaries a stage further than had Wessel and Warren.

The significance of this Treatise shows up against the background of eighteenth century algebra considered. While in terms of the algebra put forward there is nothing that was original, the systemization of rules and definitions was, in fact, a major breakthrough in algebra. The full significance of the breakthrough will unfold as the advances of the other British algebraists are considered.

One interesting item is that Babbage, one of the founders, with Peacock, of the Analytical Society, mentions some ideas of his own very similar, in essence, to those of Peacock. In an undated manuscript in the 'Philosophy of Analysis' (watermarks put the dates between 1812 and 1820) probably intended for the

Cambridge Philosophical Society, Babbage sketches ideas for what Peacock actually undertook. He considers the law of indices

$$'x^a \times x^b = x^{a+b}'$$

and puts forward the notion that new definitions and rules be constructed to allow the arithmetic form of the equation as a special case. He says

> ".... The definitions of the other simple operations such as addition, subtraction and multiplication must also have corresponding extension in order to enlarge their signification from a reference to mere number and their extension ought always to include the original one which was formed solely with a view to arithmetic". (Philosophy of Analysis, M.S. Brit. Mus.)

This could infer a number of possibilities; the question may have at some time been discussed by Peacock and Babbage and forgotten by one or both of them, or it is just possible that the need for reform in Algebra might have prompted both persons to develop similar ideas. However, it would seem that Peacock, in fact, developed his ideas later than Babbage.

Peacock's second publication was in two volumes appearing in 1842 and 1845. It was entitled again, <u>Treatise on Algebra</u>; the first volume was devoted to arithmetical algebra, the second symbolical algebra. The only significant development in these works is that Peacock makes a much more decisive difference between arithmetical and symbolic algebra, to the extent of bringing out separate volumes. To make a clearer distinction between the symbolic algebra and the arithmetic certainly made the symbolic more independent and thereby more flexible. However, no new contribution was made to algebra in these subsequent works.

Outside his own work, Peacock's greatest role in nineteenth century British mathematics was in popularizing the latest advances particularly continental ones. His first successes were, as discussed, with the Analytical Society. Also of great importance was his report to the newly-founded British Association for the Advancement of Science in 1833 on the Recent Progress and Present State of Certain Branches of Analysis. In this report he first outlined the problems that had existed in algebra.

> 'Algebra considered with reference to its principles has received very little attention, and consequently very little improvement during the last century'. (Reports to the British Association, 3, 1833, p. 185).

To this assertion he adds many of his own ideas on symbolic algebra with which I have dealt. However, he also discusses at some length and in some detail, the researches of continental mathematicians in several branches of mathematics. He examines Gauss' work on higher arithmetic, Abel's work on the quintic, Cauchy's work in the Cours d'Analyse and he sketches the advances until that time in the 'Theory of Equations'. He outlines Waring's work on symmetrical roots and Lagrange's general methods of solving equations up to fourth. On the subject of radicality he mentions Ruffini's work on cyclic 'groups' and Abel's contribution to the same. The paper is generally speaking a very comprehensive clear exposition of aspects of contemporary mathematical problems. As far as can be seen it was the first time such a discussion of continental work had appeared in a publication aimed at those interested and involved in the sciences in Britain.

The secondary aspect of Peacock's work for British mathematics, as has been stated, was along the lines of popularizing continental development, and the need for reform in the mathematical emphasis in Britain. Most particularly he played a leading role, not only in the reform of the mathematics syllabus at Cambridge but also in the movement to reform the structure and the statutes of the University. He published a book in 1841 on the question of the need to reform the statutes, a very significant work at that period for in the next decade sweeping changes were made in the university structure. He taught at Cambridge, in mathematics, for a number of years; in his time he was a valuable asset to the institution. In 1837 he was made Lowndean Professor of Mathematics, in 1838 he sat on the Commission for weights and measures. In 1839 he was appointed Dean of Ely, and remained in that position until his death in 1858. However, he spent the last years of his life in active service; he sat on the Cambridge Commissions of 1850 and 1855, one of the veterans who had been advocating reform of one sort or another for forty years.

CHAPTER IV

Departure from Arithmetic Intuition

The next landmark in the Foundations of Algebra, appeared three years after the publication of Peacock's Treatise in 1830. A paper was read to the Royal Irish Academy in 1833, by William Rowan Hamilton (1805-1865), a young man who had already distinguished himself in scientific circles.

Hamilton had been something of a prodigy, having mastered several difficult languages at an early age. In 1817 he was introduced to Zerah Colburn, an American boy, renowned for feats of mental calculation. He was able to communicate some of his methods to Hamilton, stimulating his interest in mathematics. By the age of seventeen Hamilton was known to have mastered the works of Newton and Lagrange; furthermore he had brought himself to the notice of Dr. Brinkley, Professor of Astronomy at Dublin, by detecting an error in Laplace's proof of the parallelogram of forces.

While studying at Trinity College Dublin, Hamilton took virtually every prize in classics and mathematics, and presented the first part of his research paper to the Royal Irish Academy on the Theory of Systems of Rays. His early success was completed, when, at the age of twenty-two, he was invited to take up the

Professorship of Astronomy vacated by Brinkley in 1827, over the head of such a distinguished applicant as G. B. Airy, later Astronomer Royal of England. Among Hamilton's early mathematical achievements was the discovery of the 'characteristic' function in the propagation of light, which was to make him internationally famous.

HamiltOn made a personal contribution to the diffusion of mathematical ideas in general: he was President of the Royal Irish Academy, a keen supporter of the British Association for the Advancement of Science, he corresponded with many of the leading algebraists and mathematicians of his day; Whewell, Baden Powell, De Morgan, Airy, Herschel, Peacock, Boole and Graves. He was especially familiar with the work of contempory algebraists notably Peacock.

HamiltOn was one of those fortunate enough to be honoured in the way Babbage felt scientists should be honoured. He was internationally acclaimed, knighted and awarded a Civil List pension from the British Government.

However, of primary interest to this thesis, are Hamilton's achievements in the field of algebra. The paper read to the Royal Irish Academy in 1833 was entitled, <u>The Theory of Conjugate Functions or Algebraic Couples</u> and appeared in the Transactions in 1835. In essence the paper transpires to be a completely fresh treatment of complex numbers. However, while the new treatment was a great improvement on anything that had gone before, the paper is of greater mathematical significance than a different approach to complex numbers would imply. It contains some very interesting general remarks on the problems of formal algebra in the Introduction to the paper.

In the introductory remarks, Hamilton states the aim of the paper as being

> "... to improve the science, not the art nor the language of Algebra. The imperfections sought to be removed, are confusions of thought, and obscurities or errors of reasoning ...." (Theory of Conjugate Functions, Trans. of R. Irish Ac. Vol. XVII, 1835, p. 104.)

Thus his object is not to extend directly the scope of algebraic application nor to prove anything new, but, like Peacock, to provide a rigorous base for existing relationships in algebra; what he calls the "science of algebra". He suggests that the tendency had been to regard algebra as a system of rules or expressions, the validity of which had no significance beyond the practical application.

This state of affairs he rightly regarded with misgivings. He felt that algebra should have some status analagous to that of geometry; as he put it, 'a system of truths', or '... a science properly so called; strict, pure, and independent, deduced by valid reasonings from its own intuitive principles'; (Ibid, p.5.)

It is on this basis that Hamilton rests his notion that algebra is the science of pure Time. As the relations of space constitute the intuition of geometry so the notion of Time, he claims, constitutes the intuition of the 'science of algebra'.

The arguments he advances in favour of this seemingly arbitrary choice, he takes from the history of algebra. The role of algebra was to consider that which 'flowed' whereas that of geometry was to consider that which was 'fixed', the notion of Time he associates with continuous progression.

The examples he cites are Newton's fluxions, Napier's logarithms based on the contemplation of continuous Progression, and Lagrange's consideration of algebra to be science of functions. Hamilton regards the essence of functions to be laws connecting change with change. Hamilton makes an interesting observation in a footnote to these examples. He states that he uses the term Algebra,

> "in the sense which is commonly but improperly given by modern mathematical writers to the name 'Analysis' and not with the narrow signification to which the unphilosophical use of the latter term (Analysis) has cause of the former term (Algebra) to be too commonly confined". (Ibid, p.6.)

I have observed before that the error of regarding such topics as 'infinite series' to be within the domain of algebra was still common in this period. It would seem that as research was being done on the axiomatic basis of algebra that the methods of algebra could not embrace the field Hamilton calls 'Analysis'.

However, regarding Hamilton's views on the need to establish algebra in a manner analogous to geometry, it was perfectly correct to assert that algebra needed its own 'truths' or axiomatic foundation. However, the strength of geometry lay not in the intuition of spatial relation that inspired its rules but rather in the rules themselves. The notion that algebra needs the intuition of time, is to an extent as irrelevant as Peacock's idea that it needs arithmetic as the 'science of suggestion'. The significant aspect of Hamilton's objective in this paper, is the desire for a system of valid reasoning based on principles independent of the mathematical systems algebra may serve. Yet, despite the strangeness of the 'intuition' chosen by Hamilton he does use this notion to the advantage of axiomatic algebra.

The immediate problems Hamilton intends to overcome are those that traditionally confused the algebraists;

> "... a greater magnitude may be subtracted from a less ... that two negative numbers ... may be multiplied the one by the other, and the product will be a positive number ... yet that numbers, called imaginary, can be found or conceived or

determined, and operated on by all the rules of positive and negative numbers ... supposed to be themselves neither positive or negative ..."

(Ibid, p. 4.)

He admits that such confused concepts had yielded practical usefulness, but the subject could hardly be developed in its own right or in extended application on such a wretched basis.

Hamilton's approach to the obviation of these difficulties, he claims, is focused on the notion of 'ORDER IN TIME'. This intuition he asserts, will yield a science of algebra as demonstrative as did the notion of 'order in space' for geometry. To remove the obstacles, the ideas of negative and imaginary quantities he proposes to substitute a theory of contrapositives and 'couples' to substitute for the operations of increasing and diminishing a magnitude, the 'more extensive' contrast between the relations of 'Before and After' or 'the directions of forward and backward.'

Hamilton's proposition is that the anomalies can be eradicated by constructing an axiomatic system based on ordered couples; he calls them 'pairs of moments'. He states that he is putting forward a concept similar to Cauchy's, in that he accepts that every 'imaginary' equation is a symbolic representation of two distinct, real equations. However, the method he adopts is different; the most important distinction is that Hamilton does not introduce the symbol $\sqrt{-1}$.

He introduces the concept of the ordered couple as a 'moment couple'. If A_1 is a primary moment and A_2 a 'secondary moment', the moment couple is denoted by $(A_1\ A_2)$. Similarly if two distinct moments B_1, and B_2 form another moment couple, $(B_1\ B_2)$, the first moment couple may be compared with the second, moment with moment, primary with primary, and secondary with secondary.

> "... examining how B_1, is ordinally related to A_1 and how B_2 is ordinally related to A_2, in the progression of time, as coincident, or subsequent, or precedent, and thus may obtain a couple of ordinal relations, which may be thus separately denoted by $B_1 - A_1$, $B_2 - A_2$, or thus collectively as a relation couple, $[B_1 - A_1,\ B_2 - A_2]$". (Ibid, p. 108)

This couple may also be thought of as the relation of one moment couple to another, and may be denoted $(B_1 B_2) - (A_1 A_2)$ whereby is established the equation,

$$(B_1\ B_2) - (A_1 A_2) = (B_1 - A_1\ ,\ B_2 - A_2)$$

In order to generate new moment couples from one, he introduces the notion of 'step couples'; if $a_1\ a_2$ are separate steps, a_1, being the transition from A_1 to B_1, a_2 the transition from A_2 to B_2, we can say $B_1 = a_1 + A_1$ $B_2 = a_2 + A_2$ or in moment notation, $B_1 = (B_1 - A_1) + A_1$ $B_2 = (B_2 - A_2) + A_2$ these are simply pairs of real equations.

Those equations in couples are $(B_1, B_2) = (a_1 + A_1 \quad a_2 + A_2)$

$$= (a_1, a_2) + (A_1, A_2)$$

$$= \{(B_1, B_2) - (A_1, A_2)\} + (A_1, A_2)$$

Hamilton uses this notion of the step couple to introduce the zero of the couple algebra. A step couple may be said to be 'effective' when it changes the couple to which it is applied, that is, it can change, one or other or both of the moments. If it changes neither, it is called the null step couple. A singly effective couple can be a pure primary step couple, $(a_1 \ 0)$ or a pure secondary step coupld $(0 \ a_2)$. Then $(0 \ 0)$ will be the null couple, and $(a_1 \ a_2)$ the doubly effective step couple.

The properties of step couples, he sets out as follows.

1. "... the sum of two step couples may be formed by coupling the two sum steps." (Ibid, p. 105.)

$$(b_1 b_2) + (a_1 a_2) = (b_1 + a_1, \ b_2 + a_2)$$

2. "...the order of any two component step couples may be changed without altering the result". (Ibid, p. 105.)

$$(b_1 b_2) + (a_1 a_2) = (a_1 a_2) + (b_1 b_2)$$

3. "... every doubly effective step couple is the sum of a pure primary and a pure secondary". (Ibid, p. 105.)

$$(a_1 a_2) = (a_1 0) + (0 \ a_2)$$

A number of consequences flow from these properties. Firstly sums of as many step couples as we choose are given by property (1): $(a_1\ a_2)+(b_1\ b_2)+(c_1\ c_2) = (a_1+b_1+c_1\ \ a_2+b_2+c_2)$

Secondly the subtraction of one step couple from another will be $(a_1\ a_2)-(b_1\ b_2)=(a_1-b_1\ a_2-b_2)$ the right hand side of the equation being that step couple which must be compounded with or added to $(b_1\ b_2)$ to produce $(a_1\ a_2)$. Furthermore we may see from (3) that every step couple can be written $(a_1\ a_2) = (0\ 0) + (a_1\ a_2)$ or

$(-a_1\ -a_2) = (0\ 0) - (a_1\ a_2)$ whence $+(a_1\ a_2)$ is another way of denoting the step couple and $-(a_1\ a_2)$ the opposite couple $(-a_1\ -a_2)$.

The next consideration is the multiplication of a step couple, by a number. Hamilton approaches this question as follows. He considers the couples generated from one moment couple $(A_1\ A_2)$ and the step coupld $(a_1\ a_2)$. By repeated application of this step couple and the opposite couple $(-a_1\ -a_2)$ the following couples can be generated: $\left[(A_1\ A_2)+(-a_1\ -a_2)+(-a_1\ -a_2)\right]$,

$\left[(A_1\ A_2)+(-a_1\ -a_2)\right]$, $\left[(A_1\ A_2)\right]$, $\left[(A_1\ A_2)+(a_1\ a_2)\right]$

$\left[(A_1\ A_2)+(a_1\ a_2)+(a_1\ a_2)\right]$ and so forth.

More concisely $\left[(A_1\ A_2)-2(a_1\ a_2)\right]$, $\left[(A_1\ A_2)-1(a_1\ a_2)\right]$

$\left[(A_1\ A_2)+0(a_1\ a_2)\right]$, $\left[(A_1\ A_2)+1(a_1\ a_2)\right]$, $\left[(A_1\ A_2)+2(a_1\ a_2)\right]$

Then one can say $-2(a_1\ a_2) = -2\times(a_1\ a_2)$

$1(a_1\ a_2) = 1\times(a_1\ a_2)$ etc.

It would then seem reasonable that the rule for multiplication of a step couple by a number n, should be

$$n \times (a_1 a_2) = n\ (a_1 a_2) = (na_1, na_2)$$

where n may be fractional.

If this relation is rewritten as $n = \frac{(na_1\ na_2)}{(a_1 a_2)}$, n expresses the ratio of one step couple to another. This may be more consistently expressed as follows:

$$\frac{(na_1\ na_2)}{(a_1 a_2)} = (n, 0)$$ where the number n becomes a pure primary number couple. It is then possible to express $\frac{(b_1 b_2)}{(a_1 0)}$ as $\left(\frac{b_1}{a_1} \frac{b_2}{a_1}\right)$.

It then follows from the addition of step couples, that we may write

$$(b_1 + a_1\ 0)(a_1 a_2) = (b_1 0)(a_1 a_2) + (a_1 0)(a_1 a_2)$$

or

$$(a_1 a_2)(b_1 + a_1\ 0) = (a_1 a_2)(b_1 0) + (a_1 a_2)(a_1 0)$$

by property (2).

This result suggests the next problem, to determine completely the concept of the couple as a ratio. It is necessary to satisfy the more general conditions;

(α_1) $(b_1 + a_1, b_2 + a_2)(n_1 n_2) = (b_1 b_2)(n_1 n_2) + (a_1 a_2)(n_1 n_2)$

and (α_2) $(n_1 n_2)(b_1 + a_1, b_2 + a_2) = (n_1 n_2)(b_1 b_2) + (n_1 n_2)(a_1 a_2)$

Now it is established that the product

$$(n_1 n_2)(a_1 a_2) = (n_1 a_1, n_1 a_2) + (0, n_2 a_1) + (0\ n_2)(0\ a_2)$$

The undetermined produce is $(0\ n_2)(0\ a_2)$ which Hamilton supposes to be another number couple,

$$(0, n_2)(0, a_2) = (c_1\ c_2)$$

For the commutative relations to hold true c_1 and c_2 must vary proportionally to the produce $n_2 a_2$ hence

$$c_1 = r_1 n_2 a_2 \ , \ r_2 n_2 a_2$$

This relationship will yield the value for the product

$$(n_1 n_2)(a_1 a_2) = \left[n_1 a_1 + r_1 n_2 a_2, \ n_1 a_2 + n_2 a_1 + r_2 n_2 a_2 \right]$$

which will satisfy $(\alpha_1)(\alpha_2)$. r_1 and r_2 may be chosen at will, the only condition should be that once chosen they are retained for the algebraic operations with the couples.

The constants chosen are $r_1 = -1, r_2 = 0$. However, while these constants are the ones which yield the algebra which provides the rigorous base for complex numbers, Hamilton continues to discuss the algebra without reference to its eventual application.

With these constants the product identity becomes

$$(n_1 n_2)(a_1 a_2) = \left[n_1 a_1 - n_2 a_2 \ , \ n_2 a_1 + n_1 a_2 \right]$$

Hamilton does, in fact, provide further evidence than intuition for the choice of constants. His argument is as follows:

If ($b_1\ b_2$) denotes the product of the step couple $(a_1\ a_2)$ and the number coupled $(A_1\ A_2)$

$$(b_1 b_2) = (A_1 A_2)(a_1 a_2)$$ we have

$$b_1 = A a_1 + \gamma_1 A_2 a_2 \ , \quad b_2 = A_1 a_2 + A_2 a_1 + \gamma_2 A_2 a_2$$

whence

$$\beta_1 = A_1 \alpha_1 + \gamma_1 A_2 \alpha_2 \ , \quad \beta_2 = A_1 \alpha_2 + A_2 \alpha_1 + \gamma_2 A_2 \alpha_2$$

where $\alpha_1 \alpha_2$ $\beta_1 \beta_2$ denote the ratios of four steps $a_1 a_2$ $b_1 b_2$ to one effective step C such that

$$a_1 = \alpha_1 C \qquad a_2 = \alpha_2 C$$
$$b_1 = \beta_1 C \qquad b_2 = \beta_2 C$$

whereby $A_1 \{\alpha_1(\alpha_1 + \gamma_2 \alpha_2) - \gamma_1 \alpha_2^2\} = \beta_1(\alpha_1 + \gamma_2 \alpha_2) - \beta_2 \gamma_1 \alpha_2$

$$A_2 \{\alpha_1(\alpha_1 + \gamma_2 \alpha_2) - \gamma_1 \alpha_2^2\} = \beta_2 \alpha_1 - \beta_1 \alpha_2$$

(from solving the two equations in β)

Then in order that $A_1 A_2$ should be determined from the product equation, when a_1 and a_2 are not null, the factor

$$a_1(\alpha_1 + \gamma_2 \alpha_2) - \gamma_1 \alpha_2^2 = (\alpha_1 + \tfrac{1}{2}\gamma_2 \alpha_2)^2 - (\gamma_1 + \tfrac{1}{4}\gamma_2^2)\alpha_2^2$$

should not become null when α_1 and α_2 are not null, it is sufficient that $\gamma_1 + \frac{1}{4}\gamma_2^2 < 0$.

We can then say whatever $\gamma_1 \ \gamma_2$ are chosen, the following will be true.

$$\frac{(C\,O)}{(C\,O)} = (1,0)\ ; \quad \frac{(O\,C)}{(C\,O)} = (0,1)\ ; \quad \frac{(O\,C)}{(O\,C)} = (1,0)$$ and

$$\frac{(C\,O)}{(O\,C)} = \left(\frac{-\gamma_2}{\gamma_1} \ , \ \frac{1}{\gamma}\right)$$ since if in the above

equation in β , if $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 0$

$$A_1 = -\frac{\gamma_2}{\gamma_1}, \qquad A_2 = \frac{1}{\gamma}$$

It can then be seen that the ratio $\frac{(c\ 0)}{(0\ c)}$ can be expressed as a pure secondary number couple if $\gamma_2 = 0$ namely $(0\ \frac{1}{\gamma_1})$. Furthermore from the condition $\gamma_1 + \frac{1}{4}\gamma_2^2 < 0$ must be contra-positive, the simplest choice for a contrapositive is clearly -1.

In general then the ratio of one step couple to another is

$$\frac{(b_1 b_2)}{(a_1 a_2)} = \frac{(\beta_1 c\ \ \beta_2 c)}{(\alpha_1 c\ \ \alpha_2 c)} = \left[\frac{\beta_1\alpha_1 + \beta_2\alpha_2}{\alpha_1^2 + \alpha_2^2}, \frac{\beta_2\alpha_1 - \beta_1\alpha_2}{\alpha_1^2 + \alpha^2}\right]$$

This simple and neat discussion has yielded a very sound algebra. The notion of ordered couples, having been defined, Hamilton has set out a series of rules governing the relationships between them. There is a significant difference between the manner in which he has presented this algebra, and the work of those before him. Not only is the system sufficient to describe the addition and multiplication of complex numbers, but also, nowhere has he referred to the intuition of previous results in that field, and introduced the mystical notation of $\sqrt{-1}$.

The discussion has yielded, then, the following definitions for the algebra of ordered couples:

(1) $(b_1\ b_2) + (a_1 a_2) = (b_1 + a_1,\ b_2 + a_2)$

(2) $(b_1\ b_2) - (a_1 a_2) = (b_1 - a_1,\ b_2 - a_2)$

(3) $(b_1 b_2)(a_1 a_2) = (a_1 a_2)(b_1 b_2) = [b_1 a_2 - b_2 a_1,\ b_2 a_1 + b_1 a_2]$

(4) $\frac{(b_1 b_2)}{(a_1 a_2)} = \left[\frac{b_1 a_2 + b_2 a_1}{a_1^2 + a_2^2}, \frac{b_2 a_1 - b_1 a_2}{a_1^2 + a_2^2}\right]$

In relation to these definitions, Hamilton makes the point that were they completely arbitrarily chosen, they would still not contradict each other, and by rigorous mathematical reasoning it would be possible to draw mathematical conclusions from them, albeit not necessarily very useful ones. However, in the light of the preamble, they are clearly not arbitrary and offer legitimate interpretation for complex numbers.

Furthermore, Hamilton shows that the definitions generate all the necessary conditions for a consistent algebra. Firstly from the definitions one can see that the addition and subtraction of number couples are mutually inverse operations; likewise are the operations of multiplication and division. Secondly the system has a unit couple; $(1, 0)$ is the primary unit and $(0, 1)$ the secondary unit. Thirdly each element or couple in the system has a reciprocal element under the operations of addition and multiplication, with the exception, of course, of the null couple.

In the remainder of the paper, Hamilton goes on to consider powers and related phenomena of ordered couples. By the introduction of a few new definitions he is able to establish all the known properties of complex numbers on a completely rigorous footing. Furthermore the method of using ordered couples renders the operations with complex numbers much more simple and the relations can be seen more clearly. The system also allows, of course, the graphical representation of complex numbers; the ordered couple represents the coordinates of a point in the complex plane.

The system of ordered couples as presented in this paper is important for two principle reasons. Firstly it provides an axiomatic base for complex algebra: secondly, the ordered system suggested extensions to three and more couples; it was on investigation along these lines that caused Hamilton to discover his next major contribution, 'quaternions'. Similar ideas to those in this paper were developed later by A. De.Morgan. However, De Morgan raised rather different problems, and it is generally accepted that Hamilton's system of ordered couples remained the most elegant and suitable system for describing complex relationships. However, his greatest contribution to algebra was still to come, and will be considered in a later chapter.

CHAPTER V

Axiomatic Algebra

In 1838, another important advance was made in the axiomatization of algebra. The mathematician responsible was Duncan Farquharson Gregory, a descendant of the celebrated seventeenth century mathematicians David and James Gregory. In 1837 he graduated from Trinity College, Cambridge, with high mathematical honours and subsequently devoted most of his research to mathematics.

His mathematical work ranged over many branches; the particular emphasis was on the laws governing the combination of symbols, not only in algebra, but also in the differential calculus. Many of his investigations appeared in the Cambridge Mathematical Journal: Gregory was, in fact, one of the interested founders of the Journal and was its editor from the time of its first appearance in 1837 until a few months before his death, seven years later. In 1840 he was elected a Fellow of Trinity College and in 1841 he became Master of Arts and moderator for the college. In the same year he published a book on the calculus, Collection of Examples of the Processes of the Differential and Integral Calculus. The book was based on the idea of bringing up to date the text book of the Analytical Society published some twenty-five years previously. It contained the more modern developments in the calculus with the emphasis on the newer applications in Physics, heat and electricity, etc.

Of special relevance to this discussion, are his two brief papers on the Foundations of Algebra; one entitled, On the Real Nature of Symbolic Algebra and On a Difficulty in the Theory of Algebra. Both papers appeared in the Cambridge Mathematical Journal, but the first paper made its first appearance in 1838 in the Transactions of the Royal Society of Edinburgh.

The professed object of this first paper was as follows:

> "The following attempt to investigate the real nature of Symbolical Algebra, as distinguished from its various branches of analysis which come under its dominion, took its rise from certain general considerations, to which I was led in following out the principle of the separation of symbols of operation from those of quantity". (On the Real Nature of Symbolical Algebra, Trans. Roy. Soc. Edinb. XIV, p. 208, 1838.)

In this attempt he was not forestalled by Peacock, in the sense that his views had not been exhibited in the same form. While Peacock had sought general principles on which to found algebra, he did not exorcize arithmetical considerations altogether. Gregory felt that what he contributed in this paper agreed in essence with the ideas of Peacock, with which he was familiar, but his own presentation was of a more general nature. This generality he sought consisted in his treatment of symbolical algebra as,

> "... The science which treats of the combination of operations defined not by their nature, that is, by what they are or what they do, but the laws of combination to which they are subject". (Ibid, p. 208.)

Instead of proceeding, like Peacock by assuming general principles inspired by known, separate systems, e.g. arithmetic and geometry Gregory reached for the abstraction that characterized a common property of all hitherto existing mathematical systems, from simple arithmetic to the calculus. That is he proceeds by,

> "... leaving out of view the nature of the operations which the symbols we use represent, we suppose the existence of classes of unknown operations subject to the same laws". (Ibid, p. 208.)

The notion which inspired Peacock to generalize the basis of symbolic algebra, was essentially practical; he wanted to eradicate the traditional difficulties of arithmetical algebra. Gregory's inspiration was more abstract; he sought to isolate the nature of algebra from its many uses in analysis, and extend it in its own right.

For example one of his objects was to define classes of operations and show that they could apply to more than one branch of mathematics not only arithmetic but also higher branches such as the differential calculus. Certain relations between different classes of operations when expressed in symbolic form will be algebraic theorems and may be equivalent to relations in geometry arithmetic calculus, etc.

Gregory, unlike Peacock gives the operations an abstract symbolic form to demonstrate relationships: he takes F and f to represent any operations whatsoever, these are prefixed to other symbols on which F or f is to be performed. Then F, f can represent sums, rotations, products, etc.

I. His first assumption is $F(a)$ and f to be connected by the following laws:

1. $FF(a) = F(a)$ 2. $ff(a) = F(a)$
3. $Ff(a) = f(a)$ 4. $fF(a) = f(a)$

This class of operations he calls the 'circulating' or 'reproductive' class of functions. Of the operations employed in arithmetic, of course, F f correspond to the operations of addition and subtraction to which the symbols '+' and '−' have been attached. The latter symbols he retains to represent the class of operations thus isolated. The important development is that Gregory has abstracted the underlying laws of combination of the operations. He points out that there exist corresponding operations in geometry, namely that F or $+$ corresponds to the transference of a point through a circumference and f or $-$ to the transference of a point through a semi-circumference and the laws of combination are still true.

In fact there is no relation between addition and rotation.

> "The relation which does exist is not due to any identity of their nature, but to the fact of their being combined by the same laws". (Ibid, p. 210)

The second group of laws Gregory isolates is that connected with index operations, 'f_m' and 'f_n' are different species of the same genus of operations. For example, if $f_1(a) = a$ and m and n are integers, the following laws represent the index operations in arithmetical algebra. The laws are as follows.

1 $f_m(a)\, f_n(a) = f_{m+n}(a)$ 2. $f_m \cdot f_n(a) = f_{mn}(a)$

e.g. $a^m a^n = a^{m+n}$ $(a^m)^n = a^{mn}$

The advantage of the abstract presentation of the laws of combination is that there is no restriction to arithmetical meaning. m, n can be negative or fractional and the laws are true, the only restriction must arise from the consistency of the interpretation.

Furthermore the traditional difficulty of the root of negative numbers can be obviated if f is '$-$' and m is fractional, the laws are true and the usual geometric interpretation will be consistent: $(+)^{\frac{1}{m}}$ is the turning of a line through $\frac{1}{m}$th of four right angles, $(-)^{\frac{1}{m}}$ is the turning through $\frac{1}{m}$th of two right angles.

> "Here we see that the geometrical family of operations admits of a more extended application

than the arithmetical ..." (Ibid, p. 211.)

But perhaps the isolation of the next class of operations is what Gregory is most famous for. He expresses it as, '... a very general class of operations, subject to the following laws:

III. 1. $f(a) + f(b) = f(a+b)$

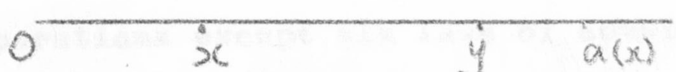
2. $f_1 f(a) = f f_1 (a)$ (Ibid. p. 211.)

The first laws he calls 'distributive', the second, 'commutative', terms which, of course, are still used for these laws in mathematics. Gregory points out that this class of operations includes several of the most important operations in mathematics, not least was Hamilton's discovery of non-commutativity, which Gregory did not, of course, foresee. One example he gives is the law where f is the operation of differentiation, another is where f is Δ; the operation of taking the difference.

The example he offers in detail is a geometrical operation subject to the above laws;

> "This is transference to a distance measured in a straight line. Thus if x represent a point, line, or any geometrical figure, $a(x)$ will represent the transference of this point or line; and it will be seen at once that
>
> $$a(x) + a(y) = a(x+y)$$
>
> or the operation a is distributive". (Ibid. p. 212.)

Thus if x is a point on an axis and y is another point



$a(x)$

represents the operation of moving x a certain distance from the origin to the point $a \cdot x$ and y to the point $a \cdot y$ then the distance $a(x) + a(y) = ax + ay$ from the origin will be the same as the $a(x+y)$ reached by moving the point $x+y$ to $a(x+y)$.

Augustus De Morgan in fact elucidates this particular example and has some interesting insights into the process; this will be discussed in the next chapter.

To continue, if x represent a point, $a(x)$ is the transference of a point to a given distance, or the tracing out of a straight line, the result of $a(x)$. Then $b[a(x)]$ will be the transferring a line to a given distance from its original position. That this may be effected, the line traced out by $a(x)$ will be moved parallel to itself by the operation b. The effect of this will be to trace out a parallelogram. Clearly the effect would be the same if a was made to act on the line traced out by $b(x)$ i.e., the same parallelogram would be traced out and $a[b(x)] = b[a(x)]$ whereby the commutative law is demonstrated.

Gregory then discusses very briefly the binomial theorem

> "The binomial theorem, the most important in symbolical algebra, is a theorem expressing a relation between distributive and commutative operations, index operations and circulating

operations. It takes cognizance of nothing in these operations except six laws of combination we have laid down, and, as we shall presently show, it holds only of functions subject to these laws". (Ibid. p. 213)

The interesting aspect of his application of all the laws to the binomial theorem is that he omits the difficulties of applying these algebraic laws to cases when the series is divergent. This seems strange in the light of the fact that his contemporaries were becoming very sensitive to the need of rigour in respect of series, and considerable advances had been made on the use of limiting processes.

The next class of operations he defines is those obeying the laws:

IV. $f(x) + f(y) = f(x+y)$

This of course corresponds to the law governing the arithmetical operation of taking logarithms if x and y are numbers.

The last class of operations he considers are those involving two operations connected by the conditions

1. $a F(x+y) = F(x) f(y) + f(x) F(y)$
2. $a f(x+y) = f(x) f(y) - c F(x) F(y)$

He states that the laws are suggested by known relations between functions of elliptic sectors; when a and c become unity, they are the laws corresponding to the combinations of sines and cosines of different angles:

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$
$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

One theorem proved from this class of functions is De Moivre's namely

$$\left(\cos x + (-1)^{\frac{1}{2}} \sin x\right)^n = \cos nx + (-1)^{\frac{1}{2}} \sin nx$$

These five classes of operations were all that Gregory considered. Quite clearly the inspiration for all of them came from known relations in arithmetic, trigonometry, geometry and analysis. However, he was unquestionably the first person to see these relations in a unified light, the first to abstract the essence of what they held in common, namely laws of combination. Peacock also did this to an extent, except that his formulations were somewhat shrouded by his dependence on arithmetic to generate the laws he laid down. Certainly Gregory's presentation stands out as more symbolic than Peacock's and his isolation of various operations opened the way for the emergence of structures in algebra. Noticeably, however, he has not considered operations as being 'inverse' to each other. This omission is to an extent considered in his subsequent paper on the Foundations of Algebra. In the paper, Gregory asserts that the commonly held view is that the symbols '+' and '-' represent in general arithmetic, addition and subtraction and that other meanings attached to them are derived from those fundamental meanings. His contention in the paper is that '+' and '-' do not in effect represent the arithmetical operations of addition and subtraction and in reality they have become representative of very different operations.

The basic argument he puts forward rests on his definition of the algebraic symbol for an operation from the last paper. That is, if the symbols + and - do not represent arithmetical addition and subtraction, the laws of combination of the symbols are not those of the operations.

The laws governing + and - he gave in the first class of operations in the last paper, namely, if F is + and f, -,

1. $F\,F(a) = F(a)$ 2. $f\,f(a) = F(a)$

3. $F\,f(a) = f(a)$ 4. $f\,F(a) = f(a)$

Now it is generally accepted that the operations of addition and subtraction are 'inverse' operations, whereas (3) and (4) are inconsistent with the inverse nature of the operations, that is, one 'undoes' what the other 'does'.

> "... so that if f and ϕ are two symbols representing inverse operations, we have
> $f\cdot\phi(a) = a$, and $\phi\, f(a) = a$". (<u>On a Difficulty in the Theory of Algebra.</u> Camb. Math. Journal, 1840, Vol. III, p. 154).

Furthermore if $a + x$ is generally held to denote x added to a and $a - x$, x subtracted from a, this is not a direct assertion that + represent addition and - subtraction. If that were the case then $+-a=-a$ or $--a=+a$ would be a contradiction. The point Gregory makes is that the error lies in expressing 'sum' and 'difference' in a way that is different from the presentation of other operations. That is, the operation is indicated after the symbol operated on, in the ordinary presentation one would prefix the operating symbol. Thus while it is reasonable to say that in $a+x$ the '+' indicates addition, it does not make it an algebraic symbol in Gregory's definition.

> "It is only when we arrive at such conclusions as $a+(x+y)=a+x+y$ involving the law $++a=+a$, that we give to + an algebraic individuality as a symbol subject to certain laws of combination, which we see at once, are not those belonging to the operation of addition". (Ibid, p. 155.)

He illustrates his observations by giving new signs for the operations, prefixing them to the subject in the usual way in order to further investigate their laws. A represents addition, B subtraction, the quantity 'added' or 'subtracted' is written as a suffix to A or B, thus $A_x(a)$ is $a+x$

$B_x(a)$ is $a-x$ The first law is

$A_x A_y(a)=A_y A_x(a)$ the commutative law.

Secondly, each of the sums is the same as if y were first added to x and that added to a, i.e.

$$A_x A_y(a) = A_{A_y(x)}(a)$$

Thirdly, it is indifferent whether x is added to a or a to x

$$A_x(a) = A_a(x)$$

Clearly the laws governing addition and subtraction are different from those governing '+' and '-' ; with regard to subtraction, as it is accepted as the operation inverse to addition,

$$A_x B_x(a) = B_x A_x(a) = a$$

Using this new notation it is eeasier to see that + is in general used as a 'separative' symbol between two others, that is, it is not permitted to write $+ax$ instead of $a+x$. Gregory gives the historical reason for this contradiction to be that the signs + and - have been called 'signs of affection' rather than accepted as 'literal symbols'. Such a distinction can exist in arithmetical, but not general, algebra. That is, when $a+b$ is written in arithmetical algebra a definite meaning is ascribed to + and no other interpretation can be given, as its laws of combination are excluded from general algebra. However, in general algebra no special meaning is ascribed to any symbol be it 'a' or '+', it is only defined in relation to combination with other symbols.

Certainly in raising this seemingly small contradiction and drawing it to its logical conclusion, Gregory has argued a very good case for treating algebra as the science of operations.

The importance of this little paper is that it is another indication of the need for rigour in algebra, and of the more abstract approach to express its results that was being put forward: an approach that was to prove eminently fruitful in the immediate years to follow, as will be demonstrated in a following chapter.

CHAPTER VI

'Technical' and 'Logical' Algebra

Perhaps some of the most penetrating analysis of the logical problems of symbolical algebra was made by Augustus De Morgan (1806-1871) between the years 1835 and 1849. However, De Morgan also distinguished himself, not only in various branches of mathematics, but also in writing histories of mathematics, teaching and popularizing new ideas.

De Morgan graduated from Trinity College Cambridge in 1827, and took the degree of fourth wrangler; among his tutors at Cambridge were W. Whewell and G. Peacock. His first intention was to read for the bar, and he entered Lincoln's Inn in 1827. However having liberal opinions on religion and the general state of society, he soon became interested in the proposals for the new 'University of London'. His interest was stimulated by W. Frend, who was mentioned for his algebraic work in Chapter I; Frend subsequently became De Morgan's father-in-law. Due to De Morgan's interest in the University, and glowing testimonials he received from various Cambridge mathematicians, he was offered the Professorship of Mathematics at the new University in February 1828, when he was only twenty-two.

This fortunate appointment committed De Morgan to a purely mathematical career, and laid the basis for his subsequent contributions in teaching and mathematical research.

Classes began in the following November; his introductory lecture, 'On the study of Mathematics', was a general statement of approach, not only to study but to the progress of knowledge and the place the reasoning processes of mathematics held in it. It was a prelude to the amazing work De Morgan was to undertake in popularizing mathematics.

De Morgan's work covered a very wide field; he was concerned with the formalization of algebra and also with reforming formal logic. His work paved the way for Boole's discovery of algebraic structure to facilitate reasoning processes in logic. De Morgan also spent much time writing articles for various popular magazines on every conceivable subject; decimal coinage, scientific and religious men, continental education, British science, among many others. As well as being a religious dissenter, he was an advocate of 'women's rights', a protagonist of the cause of the abolition of slavery, and found time to do original research in various branches of mathematics. He published text-books in algebra, logic, arithmetic, probability and the calculus.

Significantly, his first publication was a translation of the first three chapters of Bourdon's <u>Algebra</u>. This was superceded, however, in his classes by his own lectures on Arithmetic and Algebra which were published in 1831. Between the years 1831 and

1835 he published numerous articles of interest in the Quarterly Journal of Education, including reviews of certain works in algebra. The most significant of these reviews is the one on Peacock's Treatise on Algebra which appeared in 1835. Certainly Peacock's ideas profoundly influenced De Morgan's own views on structure in algebra. His own contributions began to appear only four years after this review, in the Cambridge Philosophical Transactions.

The review is worth examining in some detail as a number of De Morgan's observations shed some light, not only as to how the Treatise had been received, but also the way in which De Morgan himself was to examine the subject. It appeared in two parts, the first in No. XVII, the second in XVIII of the Quarterly Journal of Education.

The substance of the first part of the article was not so much a critique of Peacock, but more of a general discussion of the problems of algebra. The central problem he outlines, is in fact, that which principally inspired Peacock's work: De Morgan states it as follows:

> "... rejection of what we may call symbolical algebra, on account of its difficulties as opposed to its adoption without the difficulties of extension being properly placed before the student".

(Review of Peacock's Treatise on Algebra, Quarterly

Journal of Education, XVII, 1835, p. 96.)

Certain diffiuulties in algebra derive from using operations borrowed from arithmetic. This involves the use of symbols not defined in arithmetic, such as 'negatives' and 'roots of negatives'. If the difficulties are obviated by abandoning these symbols, a great many useful results must be abandoned also. De Morgan calls the problem that of 'extension '. For example, it is desirable to extend the arithmetic of '-' to a meaning which will admit 'a-b' for b greater than a. He opens his general discourse on how the problem is to be overcome by asserting that algebra should be a science of investigation, it's only rules should be those we choose to have by virtue of attaining a desired end; after all, he points out, algebra is not restricted to the province of arithmetic, it is used to investigate relations of proportion in geometry.

He spends some time explaining the need for an extended concept of algebra, to describe time, past and present; the introduction of a negative could represent 'past time ' from a given date, and positive, 'future time '. Furthermore, if a system is needed to describe the nature of relationships between lines, symbols will be needed that describe both length and direction, which implies, that simple symbols of arithmetical quantity and their accompanying rules will not be sufficient for this purpose.

Having presented the problems and various facets of them in the most general and popular way, De Morgan begins to examine the actual process of what he calls 'extension', along the same lines as Peacock in the 'Treatise'; formulae and rules can be broadened beyond what is suggested by arithmetic: for example '$a+b-b=a$' is true for the usual arithmetical meaning of $+$, $-$, but it is also true if $+$ meant $-$ and $-$ meant plus, implying that all the equation has to express is that '$+$' has an effect contrary to that of '$-$'. Thus any meaning can be assigned to $+$ and $-$ subject to the equation 'free to signify two inverse operations' which of course can include the arithmetic interpretation. Furthermore he points out that it is possible to vary the meanings of signs forming a different algebra and yet presenting theorems in the same forms as before, the theorems themselves having equivalently different meanings.

He then constructs a simple algebra in which the forms are the same as arithmetical algebra but the meanings are different and shows that the theorems have the same forms but express different truths in the new 'interpretation'. The usual symbols a, b, c, etc., represent lines, not numbers, signifying length and direction.

$a+b$ is the diagonal of a parallelogram with a, b, as sides conversely $a-b$ is a side of a parallelogram with a as diagonal, b as a side, ab is a line of length in units equivalent to $a \times b$ and inclined to an arbitrary axis at an angle equal to the sum of the angles at which a and b are inclined to that axis. On this basis every theorem of ordinary algebra will express a geometrical truth.

All of the first part of the article is concerned with introducing Peacock's innovation in a very round-a-bout, non-specialist way without actually considering the subject matter of the Treatise itself. In the second part of the article he considers the Treatise in a more detailed way.

Clearly the special emphasis of the article is on the way the ideas will influence, aid or impede the teaching of the subject; how the notion of extension should be introduced, whether in fact arithmetical algebra should be understood before the extended notions or whether Peacock's symbolical algebra should be introduced along with arithmetical algebra, avoiding later confusion.

His discussion of the treatise is bound up with the correct approach to the above problem. The first direct comments in relation to the Treatise classify it as a scholarly rather than an elementary work and thereby his comments are only relevant to the advanced student of the subject. His opening comment on the work is as follows:

> "With regard to the more advanced student, the principal difficulty which will lie in his way appears to us to arise from Mr. Peacock not having carried his own principle as he might have done". (<u>Review of a Treatise on Algebra, II</u>, Quarterly Journal of Education, XVIII, p. 300.)

The principle being, that arithmetic is rejected as the foundation of algebra, and De Morgan claims that Peacock allows a number of his definitions to be limited by arithmetical considerations. The point to which he initially draws attention is Peacock's discussion of operations on 'affected' quantities, namely the incorporation or combination of two similar signs yields '$+$' two dissimilar signs '$-$'. De Morgan maintains this should be stated.

"whichever sign it is found convenient to give to the incorporation of $+a$ and $+b$ that of $-a$ and $+b$ must have the other". (Ibid, p. 301)

He is asserting that it is only convention as to which sign is adopted, convention originating in the laws of arithmetic. He believes that for Algebra to meet Peacock's declared requirements of it, it is necessary to drop the notion that symbols are quantities, and the attempt to make arithmetic the permanent accompaniment to symbolical algebra.

De Morgan seemed to feel that while arithmetic as a 'science of suggestion' might be useful educationally, it should be kept quite separate from the definitions and rules of symbolic algebra, i.e., there is no necessary connexion between algebra and abstract number.

However, in attempting to point out what was fundamentally different about Peacock's work he compares it with that of

Warren in 1828 on the geometrical representation of 'imaginaries'. He says that Warren lays down certain definitions and proceeds to show that the equivalent forms of his algebra are the same as those in the common system. Peacock lays down definitions and shows that the interpretation of complex numbers is a necessary consequence of the relative interpretation of $+a$ and $-a$,

> "whence the geometrical interpretation of impossible quantities is a consequence of the extension which gives positive and negative quantities". (Ibid, p. 305)

Peacock's innovation was in fact to give a rigorous basis to many algebraic results based on extension of arithmetical algebra without new definitions and rules. De Morgan pointed out, that results based on arithmetical extension were only indicative of results analogous to those which could be expected if the process were based on well defined notions. It is in this context he considers Peacock's 'Permanence of Equivalent Forms'.

He raises reservations with respect to the principle on the grounds that the continental analysts doubted its generality, with respect to infinite series. However, he asserts that Peacock's usage is better founded, in that, whereas other algebraists invoke the principle without giving their underlying assumptions a necessary generality of meaning, Peacock constructs the underlying assumptions to justify the principle. However, despite reservations about assuming the principle in the definitions, De Morgan does not criticise it severely, implying that the idea must have been quite well ingrained in mathematicians at that time.

De Morgan throughout expresses general agreement with the aim of the Treatise; indeed he considers the work the most original to appear in England since Woodhouse's Analytical Calculation. He describes it as 'difficult but logical'. His own chief recommendation was as mentioned, to abandon the 'science of suggestion' except perhaps for explanation in the early stages. Certainly it suggests the lines along which De Morgan subsequently examines algebra. In fact he develops the notion of symbolical algebra away from arithmetic as is suggested by his proferred amendment.

It would appear that until the time of the Review, the Treatise had excited little notice. De Morgan puts forward the reason as being related to the novelty and extent of the new ideas contained in it; he predicts thewidespread adoption of Peacock's approach and indeed takes it up himself not four years later.

His first paper on the new approach, discussed in the article was read for the Cambridge Philosophical Society in December of 1839. His suggestion in the paper was that the attempt to separate symbols and operations of quantity from mere symbolized arithmetic, should begin the enquiry into the logic or the skeletal basis of algebra.

> "When several different hypotheses lead to results which admit of a common mode of expression, we are naturally led to look for something which the hypotheses have in common, and upon which the sameness of the method of expression depends."
>
> (On the Foundations of Algebra, Trans. Cantab. Phil. Soc. VII, 1841, p. 173)

The way in which De Morgan begins his enquiries in the paper is to examine Algebra as composed of two aspects which he calls 'technical' and 'logical'. He uses the term technical instead of 'symbolical' as the latter does not distinguish between the operations of the symbols and their interpretation. The technical aspect examines the essence of the way in which the symbols are operated on; the logical aspect examines the process by which meaning is ascribed to the symbols and the subsequent results are to be interpreted.

The definition of the symbol is the province of the technical aspect.

> "A symbol is defined when such rules are laid down for its use as will enable us to accept or reject any proposed transformation of it, or by means of it". (Ibid, p. 174)

The symbol can represent the elements of the operations of the algebra. The symbol is 'explained' when a meaning is ascribed to it consistent with the definition; a compound symbol is 'interpreted' when under the prescribed definitions, a necessay meaning can be given it from the explanation of the symbol. The latter belongs to the logical aspect of the algebra.

On the symbol itself, De Morgan makes some interesting observations suggesting the new attitude in what he terms modern algebra. He makes the point that the symbol is not an essentially objective representation of the external; the conception of the object depends on one's 'state of mind'. In the way of example, he suggests that one 'mind' may imagine the magnitude of a 'length' to be simply a given length. Another 'mind' may imagine the 'length' generated by a transition from one point moving to another, and yet a third subject would conceive the length determined by the relative position of the end points. These three ideas can, of course, be given the same kind of expression. W. R. Hamilton failed to make this point in his paper; his assertion was that algebra was the 'science of time' which De Morgan considered dogmatic, since modern algebraists were more interested in the second of the attitudes, that is, the operational concept, since it seemed more flexible.

The maxims De Morgan put forward for a symbolical algebra are as follows:

> "1. A simple symbol is the representative of one process, and of one only.
>
> 2. All processes, how many soever, may be looked at in their united effect as one process, and may be represented by one symbol.
>
> 3. Every process by which we can pass from one object of contemplation to another, involves a second by which we can re-instate the first object in its position: or every direct process has another which is its inverse. To complete the separation of these maxims from all others, I propose some considerations connected with the possible extensions of technical algebra".
>
> (Ibid, p. 176)

De Morgan makes these points as general as possible that they may be applicable to any future proposed system of algebra as well as the one studied.

His possible extensions of technical algebra are concerned with the existence of an algebra of two and three dimensions. The algebra of two dimensions requires the assignation of a symbol Ω such that

$$a+b\Omega = a_1+b_1\Omega \Rightarrow a=a_1,\ b=b_1$$

that of three requires two symbols Ω and ω such that

$$a+b\Omega+c\omega = a_1+b_1\Omega+c_1\omega \Rightarrow a=a_1,\ b=b_1,\ c=c_1$$

While no definite symbols of algebra were known to fill the second condition, for the first, the solution of the equation $\phi^2 x = -x$ (ϕ an operator) was known to fulfil the condition for a two-dimensional algebra ($\phi = \sqrt{-1}$); although not clarified as such. He does of course, expand this idea considerably in his book of 1849 and attempts the triple algebra in his paper of 1844.

To consolidate his general remarks on algebra, De Morgan considers notions of simple magnitude and analyses the operation of addition as suggested earlier in the paper. Before we arrive at the concept of a magnitude we have no object under our perception; as the symbol of this state we write 0. If the first magnitude is called 1, the transition from one state to another may be symbolized by $0+1$. The new state will then be $(0+1)$ which could be denoted by $0'$ with respect to a new magnitude, the transition again being $(0'+1)$ for the same magnitude. The result is $(0+1)+1$ which may be considered as just one operation $0+2$. This is an example of maxim 2, namely that the united effect of all processes may be viewed as the united effect yielding one process. Furthermore the first maxim is fulfilled, one process has one symbol only and '$-$' can be used to denote retracting the steps back to zero; we have an inverse fulfilling maxim 3. De Morgan summarizes this analysis of addition as follows:

> "... addition is connected with the symbol in a manner which requires us to imagine that we start from one magnitude as it were from a new 0 and renew the process by which we passed from the first 0 to that magnitude". (Ibid, p. 178)

The point he is emphasizing is the one made earlier, that the modern approach to algebra should be to consider symbols as having an operational effect on the elements of the algebra.

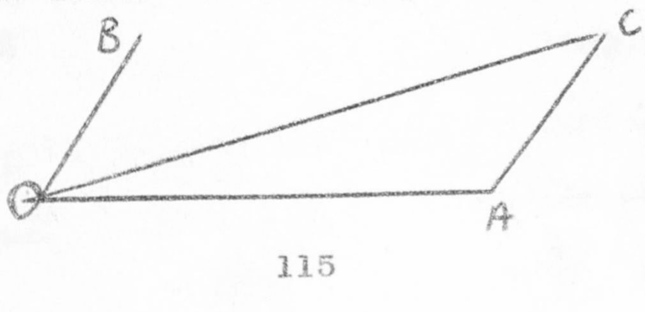
This particular analysis raises some interesting points. I mentioned in Chapter I the inadequacy of W. Frend's approach to symbolical algebra lay in the empirical status of arithmetic. It seems that here De Morgan is attempting to formulate the basic and essential processes of arithmetic. While the logical diffciulties of algebra were being obviated by rendering it more independent of arithmetic, the logical basis of arithmetic had yet to be recognized. It was not until the late nineteenth century that the mathematical logician Peano demonstrated the axiomatic basis of arithmetic. Certainly De Morgan's brief analysis of the underlying principles of addition and number contains the ingredients of this later development; principles which were not hitherto considered.

Moreover, the inductive process involved in this analysis is peculiar to De Morgan. He had in fact been the first mathematician to coin the term 'mathematical induction' and outlined its principles in 1838.

While in this analysis, De Morgan uses only arithmetical quantity, he uses it to imply that the operations could apply to any quantity; indeed his next example is a modified magnitude, namely, that of a length measured in a particular direction. The length is symbolized as magnitude, 'a' measured from a particular zero in space (of which there can be any number), the assumptions of the system are as follows:

1. Two directions measured from a line in space will be considered the same as directions measured from any line parallel to it.

2. A single symbol represents a line, two lines a and b are of the same length and direction if $a = b$. $O + a$ is the transfer of a point from O to a given length in a given direction. Thus far De Morgan has 'explained' the symbols of the technical algebra. To find the 'necessary' meaning of the compound symbol $(O + a) + b$ he proceeds as follows: let OA, OB represent the lines a and b.



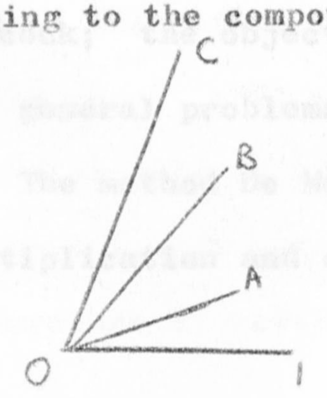
We reach A by the process $(O+a)$; taking A as a new zero and perform $(O+b)$ in the same manner as for the old zero such that $(O+a)+b$ is the line parallel to OB which is $(O+b)$ then if OC be the magnitude

$$O+C = (O+a)+b = (O+b)+a$$

So the interpretation of $(O+a)+b$ is the diagonal of the parallelogram with side lengths a and b.

As addition was dependent on the zero so multiplication is dependent on unity. It is perhaps unfortunate that as, unlike Peacock, De Morgan has not introduced arithmetic as the 'science of suggestion', he should use the familiar symbols of arithmetic, the zero and unity to demonstrate the ideas despite the fact that he had rightly introduced them via the notion of inverse operations.

However, under the present circumstance his explanation is facilitated by the arithmetical unity. The symbol b will be arrived at by the process $O+1+1+1+1$, a, is considered as a new unit, ab represents the same operation on the new unit namely $O+a+a+a+\cdots$. To give meaning to the compound symbol ab then,



'1' means a line of length 1 and direction O, OA, OB the same for a and b.

If OA is a new unit the operation whereby OI reaches OB must be performed to find ab : say it is represented by OC. Then $\angle IOB = AOC$ and the length of OC will be the result of the arithmetic operation on OA and OB, ab. Thus the multiplicative compound symbol has received its necessary meaning in the system. Clearly the division process can be explained as the inverse of multiplication by arithmetic division of lengths and subtraction of angles. This is a neat example for themaxims he had laid down although not an original system, as it had been used similarly by Peacock. What is distinct from Peacock is firstly, the absence of the 'science of suggestion' as such, while arithmetic forms are still admitted, and secondly, the emphasis in algebra is on analysing the relationships between symbols arising from various operations. That is, De Morgan has moved on from Peacock's central problem of interpreting negatives and imaginaries, to examine the logic of the operations that give rise to them. It is not until his second paper, in fact, that he actually sets out the rules governing the symbolic algebra of Peacock; the object of the present paper being to consider the general problems of the symbolic approach.

The method De Morgan uses to explain addition, subtraction multiplication and division by means of zero and unit processes clarifies what is understood by inverse processes. For example $a+x$ and $a-x$ are not inverse functions with respect to x, but with respect to a; that is, a is considered as the new zero operated on in 'reverse' manners, similarly for $a \times x$, $a \div x$.

De Morgan has thus far avoided the ambiguities of 'arithmetical algebra' by considering one geometric interpretation and certain ramifications of inverse processes. He then examines the result of extending the interpretation by also including the quantity of revolution of a line from the unit line. To denote line of magnitude a, through a revolution θ he uses the couple (a, θ); then it is true to say $(a, \theta) = (a, \theta + 2\pi) = (a, \theta + 4\pi)$ since a revolution through 2π will bring the line into the same direction. However this equality is not valid when the magnitudes considered are exponents. For example one can write

$$e^{2\pi n\sqrt{-1}} = 1, \quad \left(e^{2\pi n\sqrt{-1}}\right)^{2\pi n\sqrt{-1}} = 1^{2\pi n\sqrt{-1}}, \quad 1^{2\pi n\sqrt{-1}} = 1$$

whence $e^{-4\pi^2 n^2} = 1$ which is an absurdity. The root of the matter is that $1^{2\pi n\sqrt{-1}}$ is not necessarily $= 1$, it may have an infinite number of values of which one only is $= 1$. The equality of $1 = e^{2\pi n\sqrt{-1}}$ is valid if length and direction are concerned, but not valid for the measure of revolution. This demonstrates the importance of showing that the relationships of the technical algebra have a logically necessary existence under the interpretation.

To remove the ambiguity, De Morgan offers an interpretation of $A^{\sqrt{-1}}$ of a new kind from that already known.

Conforming with the general definitions of $\sqrt{-1}$ in the system he defines

$$\left\{(\log a, \theta)^{\sqrt{-1}}\right\}^{\sqrt{-1}} = \left\{\log a, \theta\right\}^{-1} = (-\log a, -\theta)$$

where $(\log a, \theta)$ is a line of length a, and quantity of revolution θ.

From the definitions $(\log a, \theta) \times (0, \theta) = (\log a, \theta)$ whence $(\log a, \theta)$ is the product of two functions one of a, $e^{\log a}$ and the second of θ of the form E^{θ} since

$$(0\theta) \times (0, \theta') = (0, \theta + \theta')$$

Hence aE^{θ} or $a(0\ 1)^{\theta}$ is representative of a line a inclined at an angle θ. (Where θ was an operation of rotation.)

Then $(\cos 1 + \sqrt{-1}\sin 1)^{\theta} = \cos\theta + \sqrt{-1}\sin\theta$

where cos θ and sin θ mean only the projecting factor of a length inclined at the $\angle\theta$ upon the axis of the unit line and its perpendicular.

The next point of the investigation is to connect $e^{\sqrt{-1}\theta}$ with the unit inclined at an angle θ; more generally to associate $(\ )^{\sqrt{-1}}$ with changing exponents of length into one of direction and vice versa.

What is required is an operation repeated four times on four quantities that will end in changing the sign of them all. To effect this De Morgan takes a, b, c and d as four quantities, and changes the sign of the first and makes a set of periodic interchanges, writing b for a, c for b, d for c, and a for d, thus constructing an operation which produces the desired effect. Thus

$$\phi(b,c,d,-a),\ \phi(c,d,-a,-b),\ \phi(d,-a,-b,-c),\ \phi(-a,-b,-c,-d)$$

Applying this technique to $(\log a, \theta)$ we have a method of passing from A to A^{-1} in two stages without using $\sqrt{-1}$.

thus $(\log a, \theta),\ (-\theta, \log a),\ (-\log a, -\theta)$

and $(\log a, \theta),\ (\theta, -\log a),\ (-\log a, -\theta)$

Then assuming

$$(\log a, \theta)^{\sqrt{-1}} = (-\theta, \log a)$$

$$(\log a, \theta)^{-\sqrt{-1}} = (\theta, -\log a)$$

if $A = (\log a, \theta)$

we have

$$\left(A^{\sqrt{-1}}\right)^{\sqrt{-1}} = A^{-1},\ \left(A^{-\sqrt{-1}}\right)^{-\sqrt{-1}} = A^{-1},\ \left(A^{\sqrt{-1}}\right)^{-\sqrt{-1}} = A$$

$$\left(A^{\frac{1}{\sqrt{-1}}}\right)^{\frac{1}{\sqrt{-1}}} = A^{-1},\ \left(A^{\frac{-1}{\sqrt{-1}}}\right)^{\frac{-1}{\sqrt{-1}}} = A^{-1},\ \left(A^{\frac{1}{\sqrt{-1}}}\right)^{\frac{-1}{\sqrt{-1}}} = A$$

from the assumptions, establishing the necessary relationships of $\sqrt{-1}$. Then the operation $(\ )^{\sqrt{-1}}$ clearly changes exponents of length into direction and direction into length.

Then we can write

$$\left(a, E^{\theta}\right)^{\sqrt{-1}} = E^{\theta\sqrt{-1}} \cdot e^{\log a\sqrt{-1}}$$

where $E^{\theta\sqrt{-1}}$ must be a symbol of length. Then $e^{\log a\sqrt{-1}}$ must be a unit inclined at an angle $\log a$. Then we can say $e^{\theta\sqrt{-1}}$ is a unit inclined at an angle θ, and we have

$$e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta \text{ from } E^{\theta} = \cos\theta + \sqrt{-1}\sin\theta$$

This is quite a successful 'a priori' interpretation of $(\quad)^{\sqrt{-1}}$ and concludes the paper except for a brief note on logarithms which he takes up in the second paper. It would seem, that while De Morgan had adopted Peacock's general approach, he has carried his ideas a lot further. Little use is made of arithmetical algebra, although he has not discussed the technical algebra concretely in this paper: in relation to the 'traditional difficulties' of arithmetical algebra, De Morgan has adopted a more abstract approach deriving the interpretation from a symbolic system, while the results contained in the paper are not new, the approach is quite different from any before, in that he emphasises, particularly in the interpretation of $(\quad)^{\sqrt{-1}}$ the importance of operational aspects of algebra. The very general nature of his remarks on technical algebra are extended to a particular set of rules in the next paper. He also extends the interpretation of the system begun in this paper to the discussion on logarithms.

De Morgan made his second communication on the foundations of algebra to the Cambridge Philosophical Society in 1841. It was a continuation of the first paper, its aim being to overcome an incomplete difficulty of the first one, in the transition from semi-logical to logical algebra.

The first stage in constructing his logical algebra was to separate the laws of operation from the symbols operated on. In setting out the laws, as distinct from the symbols, he had distinguished himself from Peacock in that Peacock had not separated the laws entirely from their meanings: the first rule decidedly tries to break the symbols of algebra from notions of arithmetical quantity.

> "1. The literal symbols a, b, c etc., have no necessary relation except this, that whatever any one of them may mean in any part of a process, it means the same in every other part of the same process". (On the Foundation of Algebra, No. II, Camb. Phil. Trans. VII, pt. III, 1841, p. 287.)

Thereby the symbols were completely divested of any quantitative relationship they were just entities subject to certain laws of operation to which interpretation could be given. The second law is a rigorous formulation of the meaning of equality;

"it signifies an identity of operative effect".

(Ibid, p. 288)

which is a necessary ingredient for a strictly logical formulation of the algebra. He says,

"its use implies a postulate, the only one demanded that $a=b$ gives $A=B$ whenever A is derived from a by the same operations in the same order, which produce B from b". (Ibid p. 288)

The next two rules define the nature of the two pairs of operations, the big significance of his formulation is that the pairs of operations are made to stand out as being 'inverse'.

"3. The signs $+$ and $-$ are opposite in effect; what one does the other undoes: and 0 is the symbol of a pair of such opposite operations having been performed. Thus $+a-a=0$. And such operations are convertible in their orders: thus

$$+a-b+c=+c-b+a=-b+c+a \text{ etc.}$$

"4. The signs $\times$ and $\div$ (or any substitutes for them) are opposite in effect: and 1 is the symbol of a pair of such opposite operations having been performed. Thus $\times a \div a = 1$. And these operations are also convertible in their order: thus

$$\times a \div b \times c = \times c \div b \times a = \div b \times c \times a \quad \text{etc.}$$

(Ibid. p. 288)

De Morgan's use of the notion of operations being inverse has led him to define the zero and the unit solely in terms of the operations or symbols; albeit an arithmetical zero and unit, it is an important advance in separation of symbolical algebra from arithmetic.

The remaining rules are as follows:

"5. The operations $\times$ and $\div$ are of a distributive character, when performed upon the results of the operations $+$ and $-$. Thus

$$(+a)\times(+b-c) = (+a)\times(+b) + (+a)\times(-c) \quad \text{etc.}$$

"6. Like signs ($+$ and $-$) produce $+$ in all cases, and unlike signs $-$.

And each pair of signs is, relatively to its own set, distributive.

"7. The signs 0 and 1 may themselves be considered as subjects of operation, and $1+1$ is abbreviated into 2, $1+1+1$ into 3, $1+1+1+1$ into 4 and so on.

"8. The laws by which the symbol a^b is used are $a^b \times a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$

'(Ibid, p. 288)

While all the rules would with a few restrictions be suitable for arithmetical algebra; the rules are laid down in their own right with no reference to arithmetic. Rule 7 is simply to show the operational basis of numbers as discussed in his first paper. Rule 8 is one in which difficulties arise from arithmetical algebra if b, c are extended to numbers beyond the integers. De Morgan has forestalled the issue by stating it as a rule, whereby it can be limited when subjected to arithmetic interpretation. However, he does not state the implication of his researches on $(\quad)^{\sqrt{-1}}$ namely that a^b is many-valued.

He asserts that the rules are

'neither insufficient nor redundant'. (Ibid, p. 288)

By redundant he understands that no rule can be proved from the others, by insufficient he does not make clear his meaning. Certainly systems can be constructed that are consistent with fewer rules, but the only known system at the time was the one that was sufficient for an arithmetical interpretation.

His especial concern in this paper is with the symbol a^b. He points out that while Peacock obtained the symbols of $a+b$ and ab independently of their connection in arithmetic, that is, the connection between addition and multiplication, to obtain a^b he had recourse to the multiplicative derivation resulting in insufficient notion of meaning to be attached to the symbol. De Morgan himself set out to

> "... disengage a^b from its partial dependence on a, b and having established an independent definition to examine the analogies which exist between a^b in the ancient and modern view of the subject". (Ibid, p. 291)

To establish this independent system, he proceeds at first with very general definitions.

Let $R = (r, \rho)$ be a line of r units inclined to the unit line at the angle ρ. Let $r\cos\rho = R_x$, $r\sin\rho = R_y$. Suppose the line can be given by means of another $R' = (r'\ \rho')$ such that $R'_x = \phi(r\ \rho)$, $R'_y = \psi(r\ \rho)$, ϕ, ψ being known. This line he calls the determinant of the first. (De Morgan, here has in mind to establish the logarithm as such a determinant). If the operation $+$ has been defined in its most general sense, instead of multiplying two lines, it is possible to add their determinants and the sum will be the determinant of the new line.

If $(r\ \rho)$, $(s\ \sigma)$ are the given lines and $(t\ \tau)$ the determined line

$$\phi(t\ \tau) = \phi(r\ \rho) + \phi(s\ \sigma),\ \psi(t\ \tau) = \psi(r\ \rho) + \psi(s\ \sigma)$$

For the system in which the determined line is $R \cdot S$

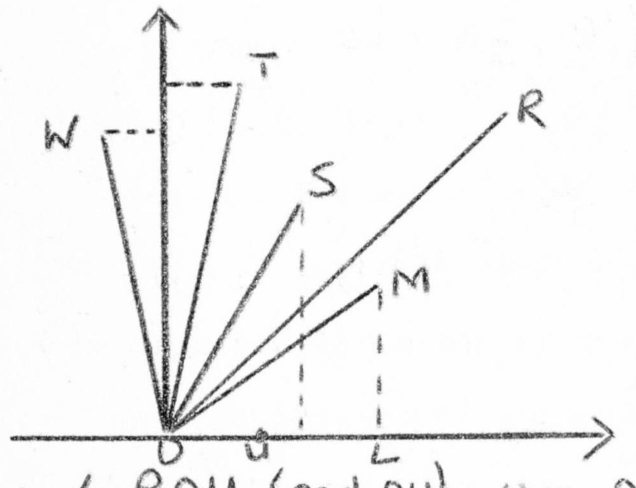
or $(r\rho)\times(s\,\sigma)$ or $(rs,\ \rho+\sigma)$ we have

$$\phi(r\rho)=\log r,\quad \psi(r\rho)=\rho$$

The log system is the arithmetical one with base e and the angle is measured by the ratio of the arc to the radius. The type of determinant suggested, De Morgan calls logometer (logarithm of $(r\ \rho)$) so the logometer of ($r\ \rho$) will be

$$\left(\sqrt{(\log r)^2+\rho^2},\ \tan^{-1}\frac{\rho}{\log r}\right)$$

This suggests the definition of R^S or $(r\rho)^{(s\,\sigma)}$ to be 'the line of which the logometer is obtained by multiplying together S and the logometer of R .'



OU being the unit line it is required to lay down OR^{OS} .

Let OL be the logarithm of OR and ML the arc of $\angle ROU$ (rad OU), then OM is the logometer of OR . Let $\angle TOU$ be $\angle MOU+\angle SOU$

Take OT a fourth proportional to OU, OM, OS . Then TO is the logometer of the required result. Place a line of which the logarithm is TV at an angle whose arc is OV; OW . Then OW is the one represented by OR^{OS} .

The laws of operation follow from this and $e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta$ is a corollary of the definition, for the logometer of E is $(1, 0)$ and $(1, 0) \times \theta\sqrt{-1}$ or $(1, 0) \times (\theta, \frac{\pi}{2})$ is $(\theta, \frac{\pi}{2})$ the logometer of a line of logarithm θ, inclined at $\frac{\pi}{2}$. Hence $e^{\theta\sqrt{-1}}$ is a unit of length, inclined at an angle θ; or $\cos\theta + \sqrt{-1}\sin\theta$.

This system will admit of an arithmetical interpretation by letting $S = (S, 0)$ where S is an integer, but it has also the ramifications required of it in that it admits of $e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta$ as a result.

Furthermore if $\Lambda(r\rho)$ represents the logometer of $(r\rho)$ then we can write $(r, \rho) = e^{\Lambda(r\rho)}$. Take $(t\,\tau)$ is $\Lambda(r, \rho)$ and $\Lambda(t, 0)$ is $(1, 0)$ then $(1, 0) \times (t, \tau)$ is $\Lambda e^{\Lambda(r\rho)}$

Hence we can say that if $\theta = \pi$, $e^{\pi\sqrt{-1}} = -1$ then $\pi = \frac{\Lambda(-1)}{\sqrt{-1}}$ which becomes a simple geometric proposition, namely that the logometer of a negative unit is a line of π units erected positively perpendicular to the unit line.

While none of the results achieved from this somewhat cumbersome system are new, De Morgan has constructed a symbolic algebra from definitions separate from arithmetic and achieved consistent unambiguous results, which has claim to originality. Unlike Peacock, De Morgan has had no recourse to to the principle of permanence of equivalent forms and he has laid down a sufficient system of rules to serve all the results of common algebra, while it is flexible enough for interpretation of different nature. Those ideas on symbolic algebra he has expressed in these papers are laid down at length in his book; also the constructions and definitions of these papers are made more elegant by the system of Double Algebra he sets out in this work.

In 1849 the book entitled Trigonometry and Double Algebra was published. In it, De Morgan discusses rather basic propositions in trigonometry in the first part of the book, as it was considered the 'science of undulating quantities'. However, in the second part of the book, he sets out systematically and with some embellishment the ideas he considered in his first three papers on algebra. The only difference is that he considers distinctly the operations of a double algebra alluded to in the first paper.

Before he begins the discussion on double algebra he summarizes his views on the nature of symbolic algebra in general. He re-emphasises that the symbols and rules of operation are independent of arithmetical notions, throughout the introductory chapter, and shows what situations can be described by a 'single algebra', that is, what kind of magnitudes can be considered as having one dimension.

In the subsequent chapter he gives fourteen fules for a symbolic calculus; the rules are not substantially different from those he put forward in the second paper on algebra, but they are rather more detailed and explicit. Having set out a complete system of rules for a single algebra without reference to any possible meaning, he devotes the next short chapter to demonstrating one interpretation of the system; the simple geometry of areas and solids.

He opens the discussion on double algebra in Chapter IV by considering the means by which meaning is assigned to the inevitable $\sqrt{-1}$. Clearly the important rule of symbolic algebra will be that governing the addition of indices, hence

$$(-1)^{\frac{1}{2}}\,(-1)^{\frac{1}{2}} = (-1)^{\frac{1}{2}+\frac{1}{2}} = (-1)^{1} = -1$$

De Morgan points out that many significant systems might admit the above as a consequence of its definitions. The one which is most interesting is that one that will also admit, the results of simple algebra, that is the 'extended' system of common algebra. What is required for the basis of significance is that $\sqrt{-1}$ must have a meaning such that successively applied to $+1$ it changes $+1$ into -1 which signify diametrically opposite units.

Now the usual systems of explantion involving the concept of opposite directions of measurement admit of no intermediate stage of 'direction'. For example the notion of time past and future, gain and loss, can be represented by positive and negative units but $\sqrt{-1}$ can represent no stage in the transition. The system of explanation of which this is not the case is the one generally admitted for the purpose in hand,

> "We can pass from a line to its opposite, not only along the line, but also by supposing the line to turn round". (Trigonometry and Double Algebra, 1849, p. 111)

that is, the usual geometric explanation of the rotational effect of $\sqrt{-1}$.

The problem then becomes to construct a symbolic algebra with a geometrical basis of significance such that the interpretation of the rotational effect of $\sqrt{-1}$ is a consequence of the interpretation of the algebra.

The object of De Morgan's double algebra is to do just this. If the symbols of single algebra denote numbers or magnitudes, the symbols of double algebra will denote lines or objects with two magnitudes as qualities like, length and direction. In his general introduction De Morgan had asserted that

> "Algebra takes cognizance only of units not of what units they are, whether of length or

time, etc ..." (Ibid, p. 113.)

Then double algebra, whether of geometrical significance or something else must admit two units; each symbol must convey a double signification each part having a different unitary base.

However, De Morgan does not begin his discussion by 'introducing' the double symbolic signification, he describes the system of his first paper already discussed where the symbols A, B etc., represent lines having both length and direction and are subject to the laws of multiplication and addition of symbolic algebra, under a particular interpretation. Having explained these laws he shows how with a double signification they can be represented. (a, α) signifies a line of length a inclined at an angle α to the unit line. Then the unit line is represented by $(1, 0)$ is $(1, \pi)$ and

$A \times B$ is $(ab, \alpha+\beta)$

$$A+B = \left\{\sqrt{(a^2+b^2 \pm 2ab\cos(\beta-\alpha)}, \tan^{-1}\frac{a\sin\alpha \mp b\sin\beta}{a\cos\alpha \pm b\cos\beta}\right\}$$

$A \div B$ is $\left(\frac{a}{b}, \alpha-\beta\right)$, $\sqrt{A} = \left(\sqrt{a}, \frac{\alpha}{2}\right)$

Then if the product of the symbol $\left(1, \frac{\pi}{2}\right)$ and $(1, 0)$ is considered twice, we have the result.

$\left(1, 0\right) \times \left(1, \frac{\pi}{2}\right) \times \left(1, \frac{\pi}{2}\right) = (1, \pi)$ or in single algebra -1. Hence the meaning of $\sqrt{-1}$ is a consequence of the geometrical interpretation of the above algebra. Similarly if $\sin \rho$ and $\cos \rho$ are the projections on the unit line and the line perpendicular to it namely $\left(1, \frac{\pi}{2}\right)$ of the line $(1, \rho)$ we have $(1, \rho) = \cos\rho + \sqrt{-1}\sin\rho$ Thus the object of double algebra has been achieved,

> "all symbols of double algebra are capable of being expressed by symbols of single algebra, combined with $\sqrt{-1}$, or $\sqrt{-1}$ is the only peculiar symbol of double algebra". (Ibid, p. 122)

The results of common algebra are all achieved from the system simply by making the directional symbol equal to zero, and the $\sqrt{-1}$ is a meaningful result of the extended system.

De Morgan demonstrates that all the rules of symbolic algebra applied to the definitions of this system are meaningful. However, he devotes a special chapter to the rule governing $(\ )^{()}$ and its interpretation. In this chapter, he considers the results concerning the exponential symbol, logs of double algebra and the rules governing them. Again the difficulties of presenting these he had considered in his second paper on algebra, and there is nothing in the chapter that is essentially new. Its presentation is more complete in that, having defined the logometer, he proves the laws of symbolic algebra related to A^B are true. In the next chapter the definition is used to embrace logs to different bases; and in the subsequent chapter he considers the roots of unity under the new algebra.

The material presented in this book is basically the same as the ideas presented in his first three papers on the foundations of algebra. The mode of presentation is more lucid in the sense that it appears in text-book form. However it must be said that, the system as he presents it in this book is, in terms of presentation, inferior to that of W. R. Hamilton's in 1835.

De Morgan's system with its inclusion of the symbol $\sqrt{-1}$ is more awkward, and algebraically not as independent as the simple and elegant presentation of Hamilton.

However, in general De Morgan's approach to the problems of symbolical algebra was very thorough. His analysis of the problems that existed was more penetrating than any of the mathematicians considered hitherto. As a logician, De Morgan was able to differentiate between the necessary relations of symbolical algebra and arithmetical interpretation. The papers discussed, suggested the line of research in algebraic logic of Boole, and contained the germs of the ideas that led to the axiomatization of arithmetic and the meta mathamatics of Peano.

His paper on triple algebra, an extension of his ideas put forward in Paper II, will be examined in the next chapter.

CHAPTER VII

New Algebras

From the time Hamilton published his paper on number couples, he had been attempting to create an algebra with a similar system of ordered triplets sufficient to describe rotations in three-dimensional space by analogy with rotations in a plane. While experimenting with these ideas, D. F. Gregory had set out an axiomatic system for common algebra, isolating different classes of operations which demonstrated the possibility of applying the laws of common algebra to different systems.

In 1843 Hamilton made the discovery which was to revolutionize the future course of algebra. According to his own account of his discovery, he was walking with his wife by a canal, when the secret of 'quaternions' flashed through his mind; he immediately carved the discovery on a stone in the bridge over the canal. The principle which he had been seeking for his new algebra was the denial of one of those laws Gregory had isolated, namely, the commutativity of multiplication.

In the same year, in a peper to the Royal Irish Academy, entitled <u>A New Species of Imaginary Quantities Connected with</u>

a Theory of Quaternions, Hamilton expounded this new principle, with which he was later to solve the aforementioned dynamical problem, and many others.

The paper opens as follows:

"It is known to all students of algebra that an imaginary quantity of the form $i^2 = 1$ has been employed so as to conduct varied and important results. Sir William Hamilton proposes to consider some of the consequences which result from the following system of imaginary equations, or equations between a system of three different imaginary quantities:

(a) $i^2 = j^2 = k^2 = -1$

(b) $ij = k, \; jk = i, \; ki = j$

(c) $ji = -k, \; kj = -i, \; ik = -j$ "

(A New Species of Imaginary Quantities, Proc. R. I. Academy, Vol. II, 1843, p. 424)

In these simple relations between imaginaries is formulated a basis for a non-commutative algebra. These quantities are used as a basis for quantities known as quaternions possessing the amazing property that $A.B \neq B.A$. Despite the fact, that the work of all the mathematicians considered has been dedicated to postulatizing algebra, freeing algebra from all intuition from other branches of mathematics, this was the first time any of the laws basically derived from other branches had been denied. This in itself opened up many new possibilities for algebra; in fact it suggested that one could construct an algebra with operations and laws entirely of one's choosing, the results may not be significant but they could be consistent.

Assuming no linear relationship between the elements, i, j, k the identity $\Theta = \Theta'$ in which

$$\Theta = w + ix + jy + kz$$

$$\Theta' = w' + ix' + jy' + kz'$$

would be equivalent to the four distinct real equations $w = w'$, $x = x'$

$y = y'$, $z = z'$ in a manner analogous to the established algebra of complex numbers. Quaternions are added or subtracted by addition or subtraction of their constituents; thus

$$\Theta + \Theta' = (w + w') + i(x + x') + j(y + y') + k(z + z')$$

Multiplication is defined by the preceeding relations, hence

$\Theta . \Theta' = \Theta'' = w'' + ix'' + jy'' + kz''$ where

$$w'' = ww' - xx' - yy' - zz'$$

$$x'' = wx' + xw' + yz' - zy'$$

$$y'' = wy' + yw' + zx' - xz'$$

$$z'' = wz' + zw' + xy' - yx'$$

These relations yield a further convenient analogy with the system of complex numbers. That is if μ, μ' be the positive quantities, $\mu = \sqrt{w^2 + x^2 + y^2 + z^2}$

$$\mu' = \sqrt{w'^2 + x'^2 + y'^2 + z'^2}$$

then $\mu\mu' = \mu''$ where $\mu'' = \sqrt{w''^2 + x''^2 + y''^2 + z''^2}$

If the quantity μ is called the modulus of Q; the modulus of the product of any two quaternions is equal to the product of the moduli.

Having thus briefly sketched the elements of the system of quaternions in this paper, Hamilton develops aspects of their significance by interpreting their properties as a calculus for proving theorems in spherical trigonometry. However, for the purpose of this discussion, the points raised in relation to spherical trigonometry are not as relevant as Hamilton's subsequent, more fundamental algebraic treatment and analysis of quaternions in the Lectures on Quaternions which were eventually published in 1853, ten years later.

The interesting feature of this publication is that in the author's preface, he submits a brief discussion of the manner in which he eventually arrived at his concept of quaternions, and of the influences on him.

As mentioned, he began by extending the idea of moments developed for couples in his paper of 1835. Instead of moment couples he generalized the notion to moment triads, and established similar ordinal relations; problems arose for multiplication, twenty-seven constants had to be assigned for the resultant coefficients of triad products. Hamilton found that with the various systems he tried,

> "There seemed to be too much room for arbitrary choice of constants, and not sufficiently decided reasons for finally preferring one triplet system to another". (Lectures on Quaternions, 1853, p.24)

For the couple system, as discussed, there was some limitation on the choice of constants, and furthermore, for the choice made, a very straight-forward and useful geometric interpretation. For the triplet system no such imperative seemed to present itself.

However Hamilton was not unaware that a system based on three moments is arbitrary, and he did discuss briefly a system based on n moments analogous to that of couples. In fact, the mathematician Grassman, was, unknown to Hamilton, working in such a direction at about the same time. Yet it was the problems with the triplets that finally led Hamilton to discover quaternions.

Just prior to his discovery in 1843, Hamilton resumed his researches on triplets with the understanding that he would retain the distributive and commutative principles. The three bases he used were $1, i, j$ so the triplet took the three rectangular coordinates, and the triplet, form $x+iy+jz$ where x, y, z were to denote a line in space. He assumed $i^2=-1$ corresponded to a rotation through π in the x, y plane, and likewise assumed $j^2=-1$ corresponded to such a rotation in the xz plane. He further assumed $ij=ji$. Then the triplet product took the form

$$(a+ib+jc)(x+iy+jz) = (ax-by-cz)+i(ay+bx)+j(az+cx) + ij(bz+cy)$$

The problem was to evaluate 'ij'.

One property Hamilton made use of was, that if the factor lines are in a common plane with the x axis whence b, c will be proportional to y, z i.e., $bz=cy$ then the coordinate projections of the product line will be $ax-by-cz, ay+bx, az+cx$ that is, it takes the form $(ax-by-cz)+(ay+bx)i+(az+cx)j$. the term $ij(bz+cy)$ reduces to zero.

Hamilton at first supposed the product ij must be zero.

> "But I saw that this fourth term (or part) of the product was more immediately given, in the calculation as the sum of the two following
>
> $ib \cdot jz, jc \cdot iy$
>
> and that this sum would vanish, under the present condition $bz=cy$ if we made what appeared to me a less harsh supposition, namely ...
>
> $ij=-ji$ or that $ij=k, ji=-k$
>
> the value of this product k being still left undetermined". (Ibid, p. 45)

Then without assuming $bz - cy = 0$ the product of the triplets becomes

$$(ax + by - cz) + i\,(ay + bx) + j\,(az + cx) + k\,(bz - cy)$$

Furthermore it is possible to establish a relation between the squares of the coefficients $(a^2 + b^2 + c^2) \cdot (x^2 + y^2 + z^2)$

$$= (ay + by - cz)^2 + (ay + bx)^2 + (az + cx)^2 + (bz - cy)^2$$

It was this that led Hamilton to believe that triplets ought to be conceived as imperfect forms of quaternions such as $a + ib + jc + kd$, where k denotes some new form of unit operator. Naturally enough k^2 was supposed to be -1 from the relations

$$k^2 = ij \cdot ij = -ii\,jj = -(-1)(-1) = -1$$

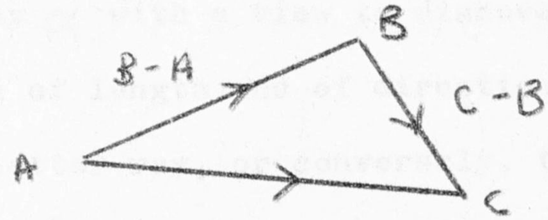
Thus all the assumptions for quaternions were made and the laws of operation flowed therefrom. Hamilton concluded that instead of representing a line by the form $x + iy + jz$ that it should be represented by the new form suggested $ix + jy + kz$. The product of two lines in space would then be expressed as a quaternion, a new instrument for applying calculation to geometry.

In the Lectures on Quaternions, Hamilton introduced the quaternions, obliquely, as part of a general calculus of geometry. The first lecture was devoted to analyzing the relative positions of points in space and the ordinal relations that can be established to describe them. He introduced some interesting terminology, which is to persist in the field of algebra. For example, lines in space he calls 'steps', or 'vectors'. A vector, AB is a directed line in space, which Hamilton defined as being the difference of two extreme points, A, B or the result of subtracting its own origin A from its own end point B. Then the following relations could be established.

(1) If a vector AB or $B-A$ be added to its own origin A, the sum is its end point B.

(2) If a 'provector' BC be added to a vector AB the sum is the 'transvector' AC,

① $(B-A)+A=B$ ② $(C-B)+(B-A)=C-A$



This notion of the end point of the vector becoming an origin for another vector and thus demonstrating the triangle law, is very similar to the presentation of De Morgan's in his first paper on the foundations of algebra, with which Hamilton was certainly familiar. However, De Morgan confined his discussion to lines in a plane, Hamilton generalized the discussion to lines in three-dimensional space.

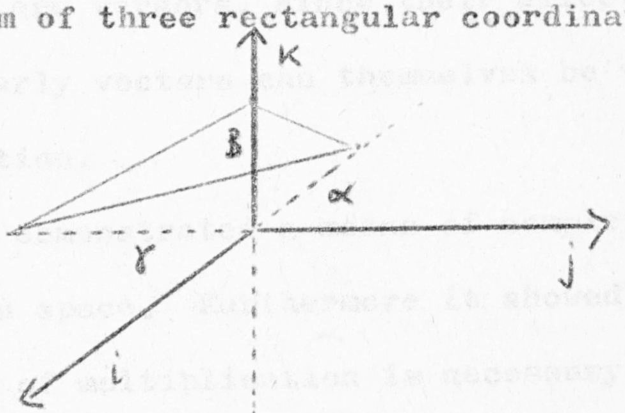
Hamilton also isolated various operators on vectors; the 'tensor', a signless number which only operates metrically on the lengths of lines, a 'sign' namely '+' and '-' which operates to preserve or reverse the direction of a vector, and is combined according to the usual rule of signs. 'Scalars' are sign-bearing numbers such as -2, +6 and can be regarded as the product of a sign and a tensor. These operators then vary lengths of vectors and can reverse direction. Hamilton also considered another kind of operation which he called version, the operators were called versors. This involved changing the directions of line vectors in space.

The problem Hamilton posed for the analysis of direction of vectors was outlined in the following proposal,

> "... to compare any one ray β, with any other ray α with a view to discover the complex relation of length and of direction of the former to the latter ray, or conversely, to construct or generate β from α by making use of such a relation". (Ibid, p. 36)

He further proposed to adopt the relation from ordinary algebra of multiplication inverse to division, thus $\beta \div \alpha \times \alpha = \beta$, $\beta \div \alpha$ would be the result of comparison of the two vectors, and denoted a 'metrographic' relation of the vector β to that of α. Its metric element would be a relation of length to length, and its graphic element, a relation of direction to direction. Now to completely determine this relationship knowledge of four elements would be necessary.

Firstly, one would need to know the relation between their lengths, secondly their mutual inclination, or the angle between them. Thirdly to specify their plane in space, it would be necessary to know the direction of the axis perpendicubar to their common place. Fourthly to specify their position in relation to this axis, it would be sufficient to know the sense of rotation relative to the axis from one vector to the other. In other words, the vector quotient was a quaternion. Hamilton showed how the situation could be described in a manner analogous to the system of plane coordinates, by a system of three rectangular coordinates.



Suppose the vectors $\alpha, \beta \gamma$ are depicted such that γ is represented in a westward direction and α a northward and β is represented in a direction perpendicular to their common plane and northward as in the figure. Furthermore their common length is assumed equal to some unit. The unit of the vertical axis is to be k and $-k$ if the direction is 'southward'. The unit of the horizontal line be j the eastward direction and the unit of the line vertically to it in the horizontal plane be i in the southward direction.

Then $\beta = k, \ \alpha = -i, \ \gamma = -j$

Consider the relation between vectors β and γ. Their relative length is unit, their mutual inclination $\frac{\pi}{2}$, the axis perpendicular to their plane is i and the direction of rotation of β to γ is in the direction of i. In fact we may write

$$\gamma \div \beta = (-j) \div (+k) = i$$

or we may find the product $i \times \beta = i \times (+k) = \gamma = -j$. The units i, j, k then are versors, since their effect is to alter direction; clearly vectors can themselves be versors as they too alter direction.

This system demonstrated a means of completely determining line products in space. Furthermore it showed that the non-commutative law of multiplication is necessary for this determination.

However, the importance of quaternions for the future development of algebra was not simply that they provided a calculus for geometry. But rather, that having realized an algebra could be consistent and fruitful by abandoning one postulate, the door was open for the development of all manner of non-commutative algebras, and further, for non-distributive algebras. In fact one can say that this particular development completely liberated algebra from dependence on other mathematical systems, and as such, it was to become a far more useful tool.

Hamilton wrote a great many papers on quaternions, applying them to geometry, astronomy, dynamics and light waves. He thought his discovery was to become as important as Newton's discovery of fluxions. There was a parallel; just as Newton's discovery was superceded by a simpler presentation so was Hamilton's. Hamilton's system was too cumbersome for use by engineers and physicists and the simpler vector algebra was invented some years after his death. Quaternions were left as a curiosity.

However, the positive repercussions of Hamilton's discovery followed very rapidly in the next few years, such as Cayley's discovery of matrices, which will be touched on in the concluding chapter. The year after Hamilton's paper was read to the Royal Irish Academy, A. De Morgan was inspired to investigate the properties of triple algebras, and in 1844 his paper was read to the Cambridge Philosophical Society.

The paper was his final one on the foundations of algebra, and in some ways the most interesting. In the paper, De

Morgan attempted to construct a significant triple algebra in the light of Hamilton's work. However, unlike Hamilton, De Morgan restricted himself to laws resembling those of common algebra for his investigation. Nevertheless, he made some very interesting observations.

The paper opens with general statements about the qualities an algebra should possess. For example an algebra of the n^{th} character, he says, should have n distinct symbols $\xi_1 \xi_2 \cdots \xi_n$ each of which is a unit such that $a_1\xi_1 + a_2\xi_2 \ldots + a_n\xi_n$ cannot be equivalent to $b_1\xi_1 + b_2\xi_2 + \ldots\ldots + b_n\xi_n$ unless $a_1 = b_1$ $a_2 = b_2 \cdots a_n = b_n$ etc. Furthermore, assuming laws of addition and multiplication requires that meanings should be assigned to $\xi_1\xi_2 , \xi_1\xi_3$ etc., such that each of them are coincident with a form of $a_1\xi_1 + a_2\xi_2 \cdots + a_n\xi_n$. The properties of the system will depend on the way in which the form is assigned. De Morgan keeps to the conservative notion of a commutative algbera for the purposes of this exercise: he is, however, aware that a perfect symbolical algebra might well exist without even his initial statement of equivalence or the ordinary laws of addition.

Nevertheless his approach is more strictly algebraic than in any previous work. The way in which the multiplication will be defined depends on the modulus of multiplication. That is if $A_1\xi_1 + A_2\xi_2 \cdots$ ... be the product of $a_1\xi_1 + a_2\xi_2 \cdots$, $a_1'\xi_1 + a_2'\xi_2 \cdots$. then $A_1\, A_2$ are definite functions of a_1, a_1' etc., and the functional equation

$$\phi(a_1 a_2 \cdots) \times \phi(a_1' a_2' \cdots) = \phi(A_1 A_2 \cdots)$$

will yield the modulus on solution.

A convenient modulus for a triple algebra would be one which will reduce to that for the double or single. Hamilton supposed that it would be $\sqrt{a^2 + b^2 + c^2}$ and therefore did not consider the possibility of constructing a triple algebra. De Morgan however, was prepared to examine the possibility of a triple system based on an a-symmetrical modulus.

He described his attempt as 'one mode of derivation' of triple algebra. The units of the system are ξ, η, ζ, they are represented on the axes of x, y, z such that $a\xi, b\eta, c\zeta$ represent lines of a, b, c units measured on those axes. It is a condition that $b = 0, c = 0$ reduces the algebra to the single system. Let η, ζ be interchangeable in the sense that they are related to ξ in the same way. Then for the action of the units on each other we have

ξ^2	means	ξ	$\eta\zeta$	means	$p\xi + q\eta + q\zeta$
η^2	"	$a\xi + b\eta + c\zeta$	$\zeta\xi$	"	$l\xi + m\eta + n\zeta$
ζ^2	"	$a\xi + c\eta + b\zeta$	$\xi\eta$	"	$l\xi + n\eta + m\zeta$

from these relations the equations $\xi^2\eta = \xi(\xi\eta)$

$$\xi\eta^2 = \eta(\xi\eta),\ \eta^2\zeta = \eta(\eta\zeta),\ \eta\zeta^2 = \zeta(\eta\zeta),\ \zeta^2\xi = \xi(\zeta\xi)$$

$$\zeta\xi^2 = \zeta(\zeta\xi),\ \xi(\eta\zeta) = \eta(\zeta\xi) = \zeta(\xi\eta)$$

are made subject to specific relations between the coefficients

of the units. These relations are arrived at as follows:

$$\xi^2\eta = \xi\eta = l\xi + n\eta + m\zeta$$

$$\xi(\xi\eta) = \xi(l\xi + n\eta + m\zeta) = l\xi + n(l\xi + n\eta + m\zeta)$$
$$+ m(l\xi + m\eta + n\zeta)$$
$$= \xi(1 + nl + ml) + \eta(n2 + m2) + \zeta(nm + nm)$$

Then by definition of the algebra:

$$l = l + nl + ml : \quad 0 = l(n+m)$$

$$n = n^2 + m^2 \quad : \quad n = n^2 + m^2$$

$$m = nm + nm \quad : \quad m = 2nm$$

a series of similar relations can be established from the

identities: twelve altogether

(1) $a(q-c) + p(q-b) = l(a-p)$ (4), (5), (6) As above

(2) $l^2 + mp + na = a + (b+c)l$ (7), (8) $ln = (q-b)m = (c-q)m$

(3) $l^2 + ma + np = p + 2ql$ (9), (10) $lm = (q-c)m = (b-q)m$

(11) $(q+c)(q-c) = am - pn$ (12) $(q+c)(q-b) = an - pm$

from (5) and (6) we have either

$$m = 0,\ n = 0 \text{ or } m = 0,\ n = 1 \text{ or } m = \tfrac{1}{2},\ n = \tfrac{1}{2} \text{ or } m = -\tfrac{1}{2},\ n = \tfrac{1}{2}$$

By analogy with double algebra the triple algebra might yield

$$\eta^3 = -\xi,\ \zeta^3 = -\xi$$

or even $\eta^2 = -\xi,\ \zeta^2 = -\xi$

The first De Morgan called the simple cubic and the second the quadratic.

Now each of the solutions of m and n corresponds to four different solutions for the relations between the units. I shall consider the most interesting of the cases considered by De Morgan, namely the one corresponding to the solution

$m = 0,\ n = 1$. In this case $\xi^2 = \xi,\ \eta\zeta = -(q^2 - c^2)\xi + q(\eta + \zeta)$
$$\zeta\xi = \zeta,\ \xi\eta = \eta$$
$$\eta^2 = (q+c)(q-b)\xi + b\eta + c\zeta,\ \zeta^2 = (q+c)(q-b)\xi + c\eta + b\zeta$$

This is the only case in which ξ has no effect in changing the other base units.

If the quadratic relation is adopted the following identities are established:

$$\xi^2 = \xi,\ \eta^2 = -\xi + \eta + \zeta,\ \zeta^2 = -\xi + \eta + \zeta,\ \eta\zeta = \xi$$
$$\zeta\xi = \zeta,\ \xi\eta = \eta$$

If the simple cubic relation is adopted the following are established:

$$\xi^2 = \xi,\ \eta^2 = -\zeta,\ \zeta^2 = -\eta,\ \eta\zeta = \xi,\ \zeta\xi = \zeta,\ \xi\eta = \eta$$

It was in fact the simple cubic De Morgan considered in greatest detail. The symbol ξ is dropped since it is inoperative and so the significant equations will be

$$\eta^2 = -\zeta,\ \zeta^2 = -\eta,\ \eta\zeta = 1$$

thereby, the product of two elements of the algebra $a + b\eta + c\zeta$, $a' + b'\eta + c'\zeta$

will be

$$bc'+cb'+aa'+(ab'+ba'-cc')\eta+(ac'+ca'-bb')\zeta$$

In order that a modulus was established for the system, De Morgan suggested that the basis might be the cube roots of -1;

$$-1, \quad \frac{1}{2}+i\frac{\sqrt{3}}{2}, \quad \frac{1}{2}-i\frac{\sqrt{3}}{2}$$

This satisfies the equations of signification, and if μ be one imaginary root and ν the other, the possibilities for the elements are $a-b-c$, $a+\mu b+\nu c$, $a+\nu b+\mu c$. Now since any product of roots of a modulus, is a modulus, by taking such roots as are required by the condition that the algebra is to become single if b and c vanish, one can have the following moduli,

(i) $a-b-c$

(ii) $\sqrt[2]{a^2+b^2+c^2+ab+ac-bc}$

(iii) $\sqrt[3]{a^3-b^3-c^3-3abc}$

The second is obtained by the product of the elements with imaginary bases, the third from the product of all three, and, as can be seen, they bear similar relation to the modulus $\sqrt{a^2+b^2}$ in double algebra. Taking the analogy further, since $a+\sqrt{-1}\,b$ is made to depend upon a length and an angle such that the modulus represents the length and the product of two elements has the product of lengths for length and the sum of angles, for an angle, it suggests that a similar dependence might be possible for the triple algebra.

De Morgan supposed that $a+b\eta+c\zeta$ should depend on the modulus and two angles such that if $a+b\eta+c\zeta$ be denoted by $[l,\theta,\phi]$

$$[l,\theta,\phi][l'\theta'\phi'] = [ll',\theta+\theta',\phi+\phi']$$

To realize this relationship it is necessary to assign

$A_{\theta\phi} = \frac{a}{l}$, $B_{\theta\phi} = \frac{b}{l}$, $C_{\theta\phi} = \frac{c}{l}$ where, by analogy $A_{\theta\phi}$ will be a species of cosine, $B_{\theta\phi}$ $C_{\theta\phi}$ species of sines. De Morgan is able to realize the relationship required with the A_s B_s and C_s as given, by means of the modulus (2) $\sqrt{a^2+b^2+c^2+ab+ac-bc}$. In this system, the equation analogous to $\sin^2\theta+\cos^2\theta=1$ of common trigonometry will be

$$A^2_{\theta\phi} + B^2_{\theta\phi} + C^2_{\theta\phi} + A_{\theta\phi}B_{\theta\phi} + A_{\theta\phi}C_{\theta\phi} - B_{\theta\phi}C_{\theta\phi} = 1$$

Thus far De Morgan had established an interesting system of what he would call the 'technical' algebra. The problem then became to make the algebra significant, that is, to give it a meaningful interpretation in terms of its operations. The interpretation he considered briefly was geometrical.

Analogy with double algebra lead him to infer that a, b, c should be lines on the axes x, y, z. Similarly l should be the absolute length of $a+b\eta+c\zeta$, but all that is necessary is that l, θ, ϕ should be sufficient determinants for the length. However, De Morgan was not able to present any striking geometrical interpretation on this basis. For if $a+b\eta+c\zeta$ represents a length $r=\sqrt{a^2+b^2+c^2}$ inclined to the axis at angles with cosines proportional to a, b, c then the modulus of multiplication has to be abandoned. Alternatively he considered the system as one in which there was a double modulus of multiplication:

"Let $l e^{\phi} = m$ and we have

$$l=\sqrt{a^2+b^2+c^2+ab+ac-bc}, \qquad m=a-b-c$$
$$a=\frac{2}{3}l\cos\theta+\frac{1}{3}m \qquad a+\frac{1}{2}(b+c)=l\cos\theta$$
$$b=\frac{2}{3}l\cos(60-\theta)-\frac{1}{3}m \qquad \frac{1}{2}\sqrt{3}\,(b-c)=l\sin\theta$$
$$c=\frac{2}{3}l\cos(60+\theta)-\frac{1}{3}m$$

The product of $[l, m, \theta]$ and $[l', m', \theta']$ is now $[ll', mm', \theta+\theta']$.

The three axes on which a, b, c are laid down, ought not to be rectangular axes, but those of y and z should be each inclined at 60° to the axis of x, so that units laid down on them may be cube roots of -1. The planes of xy and xz being at right angles, and Δ being the diagonal of the parallelepiped on a, b, c, we have $l^2=\Delta^2-\frac{2}{3}bc$". (On the Foundation of Algebra, No. IV. Cantab. Phil. Trans. XVIII, 1844)

On this basis, De Morgan pointed out that should a simple interpretation be obtained the difficulty of the 'imaginary quantity' will again occur, for the $\sqrt{m}$ in $\sqrt{[l, m, \theta]}$ may be required when m is negative. Clearly, then, the system cannot be completely explained until it is interpreted on the basis that the a_s have the form $(a + \sqrt{-1}\,a)$ etc. Then since $(a + \sqrt{-1}\,a)$ may express a line in the plane xy, it is reasonable to suppose that two new symbols will be required to express removal into the xz and yz plane and the element

$$P = (a + a\sqrt{-1}) + (b + b\sqrt{-1})\eta + (c + c\sqrt{-1})\zeta$$

formally $P = a + b\eta + c\zeta$ will then signify a line in space determined by three lines in three coordinate planes.

De Morgan continued the paper considering other cases with different unitary relations and moduli. However he encountered difficulties in the interpretation of all the cases he considered and that one just described is the one to which he gave the greatest detailed attention. He said in the conclusion of the paper that the cases could have been considered further, due to pressure of other work, he was not able to continue himself, and further hoped that it would inspire more general work to be done on the question of interpretation.

Despite the incomplete nature of the paper, some interesting new ideas were brought to light. Firstly De Morgan began the construction of triple algebra, with recourse only to logical construction no interpretation guides the actual structure, which suggests that there had been a definite shift in emphasis from his first paper to this one. Secondly he opened the discussion in the most general possible way; he set out the problems as they would exist for any proposed system of algebra, then applied them to the one under consideration. Not only does this suggest that since 'technical algebra' had been separated from 'significant algebra' it was recognised that any algebra could be constructed with any formal basis, but also what properties wer e held essentially in common for any dimension. Strangely, De Morgan did not, like Hamilton in the year before, challenge any of the rules of the common algebra, but had tried to present them all as nearly as possible. However, he might have investigated the possibilities arising from different laws if the paper had been submitted later. The last point the paper suggests is a need for generality in interpretation of a system; to examine what properties the system and the interpretation must have in common before consistent interpretation is possible. While the paper in itself did not offer any really useful results, it suggested algebraic problems which were to become central issues in the following decades.

Mathematically, the papers of Hamilton in 1843 and De Morgan in 1844 marked the end of the period when mathematicians were dealing with problems in the foundations of algebra.

Subsequent developments in algebra showed the emphasis to be on constructing new algebras and on generalizing results into all-embracing theories; none of which would have been possible without the pioneering work in the formalization of algebra of the men discussed.

Conclusion

The mathematicians that have been discussed certainly realized one aspect of the importance of their work. It was clear to them that the algebraic results of their eighteenth century forbears could only be rationalized, given a rigorous, axiomatic basis for algebra; this to a great extent, they achieved. Furthermore they realized, that having established such a formal basis for algebra, the way was opened up for wider interpretations of results. However, the full significance of their contribution, they were not able to appreciate, since it can only be assessed against later contributions in the field. For these reasons, I will not only discuss the relationships between their respective contributions, but also attempt very briefly to outline the developments made possible by their work.

While it is not possible to give a complete causal explanation of the development of algebra over the period considered one can observe certain trends in retrospect, which suggest the likelihood of developments in algebra rather than another branch of mathematics. Firstly, the work of the Analytical Society in familiarizing British mathematicians with the advances of the continental analysts, made clear the relative disadvantage of the Briton wishing to pursue new ideas in the field of analysis, and its applications in applied mathematics. Before they could take up research in analysis, it was necessary to assimilate the lengthy researches of such men as Euler, Lagrange and Laplace, and keep abreast of the work of Cauchy and others in the search for rigour in analysis. After the pioneering work of the Analytical Society in diffusing the knowledge of continental methods, many individual mathematicians took interest in analysis and mastered the major researches. However, for these ideas to permeate all mathematical circles, and to become established in University curricula, it took some years. Thus, despite isolated contributions, little research in analysis was undertaken in Britain in the early decades of the nineteenth century.

However, as we have seen, there existed numerous algebraic problems for British mathematicians to investigate. A precedent had been created for the examination of the logical foundations of algebra at the end of the eighteenth century. Firstly, there was the need to rationalize the results of arithmetical algebra; secondly, in Britain, in keeping with its traditional emphasis on rigourous demonstration, there was the work of F. Maseres and W. Frend which attempted to lay down the conditions of rigour in Algebra. Furthermore the continental mathematicians had no special advantage in the field of algebra. In Britain and on the continent, algebraic results had tended to remain isolated; continental mathematicians had made no special advances in respect of rigour either in algebra or analysis.

Certainly, the situation indicated that the subsequent development of British algebra was not unreasonable. Moreover, after Peacock made the initial advances in establishing algebra on a logical basis in 1830, there were various factors which assisted the rapid extensions that followed.

All the men considered were supporters of the body formed in 1831, the British Association for the Advancement of Science. In 1833 Peacock gave a very important report to the Association on the latest developments in mathematics, at home and abroad; of particular interest was his discussion of the situation of algebra. He outlined the outstanding advances until that time, and also stressed the central problems of algebra. Further, he discussed his own attempt to obviate the logical problems of algebra. The report was of some significance as a number of improved contributions in that field were to follow.

The British Association was one aspect of the improved communications for scientists and mathematicians. Another was the journal of the Cambridge Philosophical Society, founded with Peacock's help, contributed to by De Morgan and Gregory and widely read in scientific circles.

There was also a number of less strictly academic journals such as the "Quarterly Journal of Education", Penny Cyclopaedia, etc., in which mathematicians were able to communicate the current mathematical ideas to a wider audience than hitherto.

But perhaps the most far-reaching assistance came from the improvements in higher education; the efforts of the Analytical Society had made new demands on standards in Cambridge; A. De Morgan was in a position to initiate improved standards in teaching and mathematical methods in the University of London. While the result of their reforms had no decisive effect on the developments considered in previous chapters, they were effective in the developments of mathematics in general from the late 1840's onwards. In particular the universities helped to perpetuate the developing School of Algebra.

Having now made very general remarks about the situation from which the British School of Algebra developed, I shall outline the importance of each individual contribution as it arose and then attempt to show how these discoveries made possible subsequent important researches.

The very first work I discussed was G. Peacock's Treatise on Algebra, of 1830. As shown from Peacock's report of 1833 to the British Association, he was aware of the state of discovery, and situation of algebra at home and abroad. His book was a response to the logical difficulties of algebra rather than a new discovery, nevertheless the book was the first stage in a mathematical revolution.

The breakthrough Peacock made was really a very simple one. The way he obviated the problems of 'common' or 'arithmetical' algebra was to regard the symbols of algebra as entities controlled by specific rules of operation, and rid them of the concept of arithmetical quantity. A number of possibilities were opened up by this attitude to algebra. Firstly since the basis of the algebra is symbolic, any consistent interpretation of its symbols can be allowed. Thus one interpretation will be common algebra, limited in the way Frend conceived it. Another interpretation can be geometrical, and then complex numbers will have a perfectly intelligible interpretation. This then was the first time algebra had been ascribed its own definitions, rules and structure, independent of meaning. However, as the first attempt in the field there were still a number of shortcomings to the system.

While the system was independent of its interpretation, the definitions, rules, etc., were still dominated by arithmetical intuition. Peacock called arithmetic the 'science of suggestion'. By this he understood that the 'laws' of arithmetic had been used for symbolic algebra, but the interpretation of the laws and symbols need not be arithmetical. However, as we see later there is no need for the 'science of suggestion'. Also Peacock still uses the symbol $\sqrt{-1}$, which, due to its laws of combination has a useful interpretation; however, the accepted interpretation of the symbol is without meaning and thus a blemish in his system. He also makes use in his system of 'the Principle of Permanence of Equivalent Forms'. Roughly this principle allowed one to suppose that a result true in arithmetical algebra was true in symbolic algebra, even if the symbols were not depicting the same kind of quantities. This seemed quite respectable to Peacock's contemporaries. However, as algebra became more logically independent, this notion was abandoned as a 'principle'.

Just after Peacock published his book, W. R. Hamilton, inspired by the same problems as Peacock set out to tackle, read his paper on ordered couples in 1833 to the Royal Irish Academy. This paper represented as much of an advance as Peacock's book, and in some ways was more significant. Firstly, the system was freer from geometrical and arithmetical intrusion than previous systems, including Peacock's. The system was quite independent of all others, based entirely on its own definitions and laws of combination.

The system provided a complete account of the laws and results of complex numbers. While complex numbers were one interpretation of the system, and were indeed the object of the system, there is no reference in the system to this end. Indeed another interpretation would be equally acceptable. The approach is oblique and nowhere is there a reference to 'square roots of negative numbers'. In this sense, the paper represents a great improvement on any research that went before. Hamilton had successfully, completely, de-mystified 'imagineries', achieving all the meaningful results of complex numbers, from well-defined symbolic definitions and operations.

One small detail marred his paper; his insistence that, like Peacock, a 'science of suggestion' was needed for symbolic algebra. His suggestion was that of 'time'. Fortunately this made little difference to his excellent presentation.

Historically, the paper stands out for two principal reasons. Firstly, in using a double system of signification, he was showing that algebra could be extended to describe higher coordinate systems; his was the first major extension of algebra. Secondly Hamilton in this paper was beginning to separate the necessary ideas of an algebra from the particular details. He showed that the definitions he had chosen generate certain necessary conditions, such as the operations being mutually inverse, the existence of a unit element and reciprocal elements. It was the separating of the 'necessary' and 'particular' which was to create new algebras and make possible Hamilton's later discovery.

The separation of the necessary laws of algebra from the particular mode of expression was effected by Gregory in 1837 in his paper On the Real Nature of Symbolical Algebra in the Transactions of the Royal Society of Edinburgh. What Gregory accomplished was the final expression of what Peacock was trying to do, the axiomatization of algebra, the separation of symbols from quantitative concepts. His advance on Peacock's ideas was that, freed from arithmetic intuition, he regarded the operations of algebra as defined simply by their laws of combination.

He isolated classes of operations corresponding under interpretation to sign rules, commutative laws, etc., all drawn from analysis, geometry etc. However, Gregory was the first to see them in a uniform light, the first to see them abstracted from their context. The general theories about algebraic structure that were developed in the second half of the nineteenth century, were undoubtedly made possible by this ability to abstract the common properties of distinct, even isolated, mathematical processes.

Once it was realized that algebra was just another formal system, rather like geometry, interesting developments took place similar to those in geometry in the late 1820's at the hands of Lobachewsky.

However, continuing the development chronologically, perhaps the most detailed examination of the problems of symbolic algebra came in the series of papers by A. De Morgan On the Foundations of Algebra. De Morgan was impressed by Peacock's work of 1830, and reviewed the Treatise in some detail in the Quarterly Journal of Education in 1835. He was sufficiently interested to take up some of his own points of criticism, and make his own attempts to improve the logical status of algebra. His especial interest in this was that he was himself a logician and attempted to relate mathematical concepts to logic. His papers appeared in 1839, 1841 and 1844.

The most interesting aspect of his work was the detailed analysis of the problems in the foundations of algebra. In particular he made an analysis of the simple concept of magnitude related to the operation of addition; this contains the germs of Peano's ideas for postulates for arithmetic in the late nineteenth century. While De Morgan set out rules for operation in symbolic algebra, he actually added little that was really new. His treatment of complex numbers by double algebra in his papers and his book of 1849 was in many ways inferior to Hamilton's elegant treatment. Nevertheless, De Morgan's work highlighted the central problems of algebra for his contemporaries, offered very detailed methods of solution and suggested the course of future research.

The culmination of these ten or so years of critical examination of the logical foundations of algebra unquestionably gave rise to one of the most fruitful discoveries of the nineteenth century; that of Sir William Rowan Hamilton. Indeed it may be said that as Lobachewsky 'liberated' geometry, Hamilton 'liberated algebra'.

In 1843, Hamilton read a paper to the Royal Irish Academy in which he made his discovery. He had constructed a significant algebra in which the commutative law of multiplication was not true. Moreover, each element of the system had four coordinates. His remarkable discovery was doubtless precipitated by Gregory's isolation of the law of commutative multiplication.

The discovery was to be of great importance in the following decades, for the implication was that one could construct algebras in many new ways. One could have more coordinates, different laws of combination, fewer laws of combination and as many and more interpretations. This discovery was developed very rapidly by later British mathematicians whom I will mention briefly.

Alas, while Hamilton's discovery was of great importance, he laboured too long and too hard over the system from which it arose, the quaternions. They proved much too cumbersome for use in the sciences compared with the non-commutative algebras discovered after that time. They became in time no more than interesting antiquities.

The last contribution which was discussed was A. De Morgan's attempt at creating a significant triple algebra. While his attempt was largely unsuccessful, a number of interesting points were raised by the paper. Firstly, his approach was strictly logical and general; no potential interpretation guided the choice of definitions and laws. Secondly, he sets out the problems as they would exist for any proposed system of algebra, and shows the possible different systems that can be constructed for a triple algebra.

Unlike Hamilton in the previous year, De Morgan did not change any of the postulates for algebra, but tried, as nearly as possible, to present all the usual rules. The problems of his paper suggested the need for generality in examining the consistency of the interpretations of algebraic systems.

Having discussed the significance of these various works on the foundations of algebra, it will be of interest to outline briefly how these very rapid developments influenced the course of algebra in the latter half of the nineteenth century.

When D. F. Gregory founded his Mathematical Journal in 1837 he received a number of contributions from a young man, G. Boole. Gregory was sufficiently impressed by his work to correspond with him for some years. G. Boole's work revolutionized both algebra and logic and was perhaps the most outstanding consequence of the ideas already considered.

Boole's ideas bore some similarity to Gregory's, but were more far-reaching. Gregory abstracted the laws of combination of symbols from particular interpretations; Boole separated the symbols of operation from the symbols operated on, and investigated the operations on their own account; he invented an algebra of operations. This was a decisive break with old algebra, the interpretations of his algebra were totally divorced from any concept of magnitude, arithmetical or geometrical. He published his ideas in two books. The first was The Mathematical Analysis of Logic in 1847 and The Laws of Thought in 1854.

He created the first mathematical logic in that he invented a means of describing a chain of valid reasoning, using mathematical laws. In his first publication he gives an account of the old logic as an algebra. The basic concepts in his system are those of classes and class elements and of operations of selections of elements from their classes. He uses this system to show how the validity of a syllogistic implication may be tested, by turning the statements of the syllogism into a system of simple equations. The equations are solved to obtain the equation of the statement of the outcome.

The way in which this was done is as follows: X, Y, Z represent individual members of classes, x, y, z are 'elective' symbols such that x operating upon a subject, selects from that subject the class of all Xs which it contains.

The system also contains the 'Laws of Thought' or rules of operation on the elective symbols, and the sign of identity '=' . '1' represents the universe, or, that class of objects containing every object under consideration; '0' is the class containing none of the objects under consideration. As a consequence if x operates on the universe (symbolically $x(1)$) it selects all the Xs from that class, and x then represents the class of which every member is X.

The consequences of this system are curious. Some of the laws agree with those of common algebra, others do not: for example the distributive law $x(y+z) = xy + xz$ is common to both systems, $x^2 = x$ for all x is peculiar to Boole's system.

The interpretation of the first expression will be: the class of objects which includes X_s and either Y_s or Z_s, is the same as the class which includes both X_s and Y_s or both X_s and Z_s. $1-x$ will be those objects which are not X_s, $y(1-x)=0$ will be interpreted as the class of objects which includes both Y_s and not X_s is empty, or all Y_s are X_s. Such an equation is then interpreted as a statement about class membership.

If one constructs two such equations, they can be solved mathematically to yield another equation which can be interpreted as a new statement about class membership, e.g.

(1) All Y_s are X_s : $y(1-x)=0$

(2) All Z_s are Y_s : $z(1-y)=0$

From (1) $zy(1-x)=0 \therefore zy-zyx=0$

From (2) $z-zy=0 \therefore z=zy$

From (2) $zx-zyx=0 \therefore zx=zy, \; z=zx \therefore z(1-x)=0$

which may be interpreted 'All Z_s are X_s'.

Such a simple system for deriving the outcomes of logical propositions, was clearly a tremendous advance on the old logic. However, in addition Boole was also able to give an account of the logic of statement connections using this algebra. In this case the symbols of the algebra have different interpretations. X, Y, Z become marks of simple statements on which it is necessary to put some value in relation to truth or falsity; x becomes the period of time in which X is true, and such a variable can take values 0 and 1, true or false.

$x = 0$ will denote the proposition X is false, or there is no period of time for which it is true; $x(1-y)$ will represent the time during which X is true and Y is false.

$x(1-y) = 0$ will be interpreted, there is no period of time during which X is true and Y is false or X is true and Y is true or X is false and Y is false.

The system must have seemed extraordinary to Boole's contemporaries; by a process of reducing a series of equations, it was possible to test the consistency of a number of propositions; previously testing consistency had been a somewhat laborious logical exercise, Boole made it a simple algebraic one.

Of course, there were some difficulties in Boole's new system, but numerous logicians improved upon it in the following decades. The effect of his system was to severely shake all fixed ideas on the nature of algebra and the domain of algebra. His method had delivered the final blow to the old idea that algebra was merely 'symbolized arithmetic'.

After Hamilton's discovery of non-commutative multiplication and Boole's revolutionary logic more amazing new methods, structures and applications were found for algebra. Indeed the work of pathfinders in algebra of the first half of the century was completely vindicated by the Algebraists of the second half. Among the greatest were J. J. Sylvester, once a pupil of De Morgan and A. Cayley.

In 1858 a paper was published in the Philosophical Transactions of the Royal Society of London entitled <u>A Memoir on the Theory of Matrices</u> by A. Cayley. In this paper Cayley demonstrated the new algebra of matrices as a means of solving simultaneous linear equations. One of the radical properties of matrix algebra, was the non-commutativity under multiplication of the elements. The subject of matrices grew from observations of the manner in which linear transformations may be combined.

If one considers the following transformations

$$x = \frac{lz + r}{mz + n} \qquad y = \frac{ax + b}{px + q} \qquad z \rightarrow x,\ x \rightarrow y \text{ then}$$

$$z \rightarrow y \text{ will be } y = \frac{(al + bm)z + (ar + bn)}{(pl + qm)z + (pr + qn)}$$

Considering only the coefficients in the transformations and writing them in square arrays, we have:

$$\begin{bmatrix} a & b \\ p & q \end{bmatrix} \begin{bmatrix} l & r \\ m & n \end{bmatrix} \qquad \begin{bmatrix} al + bm & ar + bn \\ pl + qm & pr + qn \end{bmatrix}$$

Cayley's notion was that the result of performing the first two transformations could be represented by the following multiplicative rule

$$\begin{bmatrix} a & b \\ p & q \end{bmatrix} \times \begin{bmatrix} l & r \\ m & n \end{bmatrix} = \begin{bmatrix} al+bm & ar+bn \\ pl+qm & pr+qn \end{bmatrix}$$

Under this rule not only is multiplication non-commutative, but also it is not defined for every pair of matrices: unlike Hamilton's quaternions, the elements of Cayley's system could have totally different dimensions, and, according to the multiplicative rule, certain matrices of different dimensions can still be multiplied together. Moreover, unlike any algebraic system that preceded it, Cayley's algebra had divisors of zero. This system demonstrated the curious possibilities opened up by the formalization of algebra and proved very fruitful in application to the physical sciences.

Many different algebras were to follow, both non-commutative and non-associative. After all the developments sketched here, as results of attempts to free algebra from quantitative concepts, there was nothing to hinder extensive research and applications in algebra. Broader attitudes to algebra by the 1850's were adopted on the continent as well as in Britain. However, it was definitely the British School who had the advantage by mid-century. The foundations of group theory were set out by Galois in 1831. However, his work was not popularized until about 1846. From about this time the British work was becoming widely known, and the two aspects of algebraic development, the generalized theories of structure, and the logical foundations were being drawn together.

Similar developments had been taking place in other branches of mathematics. As mentioned, new geometries were discovered, rigour was being introduced in Analysis, new branches of mathematics were being developed such as topology. But perhaps the most interesting offshoots of the work in the foundations of Algebra were the various new attitudes in mathematical logic. Jevons, Pierce and Schroeder developed theories of logical relations and statement connections, influenced by the work of De Morgan and of Boole. In the 1880's Cantor developed his theory of classes which gave mathematicians a logical way of examining infinite classes.

Certainly the work of the men considered in the thesis, namely Peacock, Gregory, De Morgan and Hamilton, was more far-reaching in its implications for algebra and indeed logic, than they could have hoped at the time. However, the result of their work that they were able to see fulfilled, was the greatly improved position of British mathematics by mid-century. Each had not only advanced algebra, but had contributed to creating a situation in which mathematical ideas could be popularized and exchanged both in Britain and abroad.

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