RESPONSE OF A SLENDER STRUCTURE SUBJECT TO STOCHASTIC GROUND MOTION

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Abstract. The stochastic analysis of the deflection behaviour of an idealised slender structure subject to stochastic disturbance is studied. In a previous work by the authors, the response of an Euler-Bernoulli beam subject to stochastic disturbance was studied. The current work extends the same techniques to a modified Euler-Bernoulli beam with both flexural beam and shear properties. The beam is subjected to a stochastic ground motion in the form of periodic motion with disturbance in the amplitude of the motion. The disturbance is in the form of Gaussian white noise. This results in a Stochastic Partial Differential Equation (SPDE) version of the modified Euler-Bernoulli beam equation. The stochastic analysis was then conducted by numerical methods using a combination of a finite difference scheme and Monte-Carlo Simulation. Given that the input force is Gaussian, it is also observed that the response of the system is a Gaussian process.

Key words: *Euler-Bernoulli beam; Stochastic PDE; White noise; Uncertainty quantification; Implicit numerical scheme; Monte carlo method.*

1 Introduction

Stochastic modelling is important for problems with inputs containing uncertainties as is it helps in capturing the uncertainties in the system. For example, one can describe the system response when one is able to obtain the response distribution function either numerically or in closed form where possible. As a follow up to the work of the author (see [1]), in this work we examine the effect of both stochastic body forces and ground motion by examining such forces as the input of a modified Euler-Bernoulli beam equation [2]. For a uniform mass, uniform stiffness and undamped vibration, the modified Partial Differential Equation (PDE) is given by

$$EI\frac{\partial^4 w}{\partial y^4} + \mu\frac{\partial^2 w}{\partial t^2} - GA\frac{\partial^2 w}{\partial y^2} = Q(y,t) \tag{1}$$

where w is the lateral deflection, y is the longitudinal spatial variable, t is the time variable, Q is the input load, μ is the mass per unit length, E is the Young's modulus, I is the second moment of the area of the beam's cross section. The product EI is known as the flexural rigidity that measures the force required to bend the beam. GA is the shear rigidity of the beam, where G is the shear modulus and A is the cross-section area of the beam. As in the related paper [1], in this study, we consider a stochastic force of the form

$$Q(y,t) \propto \xi(y,t)\,, \tag{2}$$

where $\xi(y,t)$ is a Gaussian white noise giving rise to a Stochastic Partial Differential Equation (SPDE).

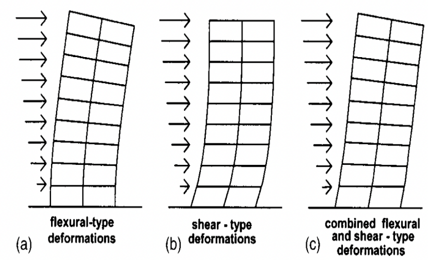

Figure 1: Lateral deformations of the beam [2]

2 Formulation

A beam of length h with uniform density, flexural rigidity and shear rigidity subject to undamped vibration caused by ground acceleration is considered. Its lower end is assumed fixed at the foundation forming a cantilever structure. The beam is assumed to have both flexural and shear properties, thus, able to deform in both bending and shear configurations. The response of such a beam, illustrated in Figure 1, subjected to a sinusoidal ground motion is governed by a modified Euler-Bernoulli equation as

$$\frac{\partial^4 w}{\partial y^4} + \frac{\mu}{EI}\frac{\partial^2 w}{\partial t^2} - \frac{GA}{EI}\frac{\partial^2 w}{\partial y^2} = -\frac{\mu}{EI}\frac{\partial^2}{\partial t^2}\big(\alpha \sin(\omega t)\big). \tag{3}$$

The problem is formulated in a two-dimensional Cartesian coordinate system with origin at the fixed end of the beam. The pressure load acts along the x-axis thus the displacement of the beam is denoted by $x = w(y;t)$. By the process of non-dimensionalisation we can re-scale equation (1) as

$$w'''' + \ddot{w} - \beta^2 w'' = \alpha \sin \omega t\,, \tag{4}$$

by choosing h, $\sqrt{\mu h^4/EI}, EI/h^4, and \sqrt{EI/\mu h^4}$ as the reference length, time, pressure respectively and angular frequency respectively. For ease of notations, a prime $'$ is used to denote the partial derivative of the displacement with respect to y, and a dot $\cdot$ is used to denote the partial derivative of the displacement with respect to t; w and α are the scaled deflection and amplitude; y and t are the spatial and temporal variable respectively and $\beta = h\sqrt{\frac{GA}{EI}}$ is the parameter that controls the overall shear and flexural behaviour of the beam. The boundary conditions for the cantilever beam are

$$w(0,t) = w'(0,t) = w''(1,t) = w'''(1,t) - \beta^2 w'(1,t) = 0\,. \tag{5}$$

For an initial value problem (IVP), the beam is assumed to be initially at rest, so

$$w(y,0) = \dot{w}(y,0) = 0\,. \tag{6}$$

By considering a disturbance in the amplitude of the period ground motion we obtain the stochastic PDE of the form

$$w'''' + \ddot{w} - \beta^2 w'' = (\alpha + \sigma\xi)\sin \omega t \tag{7}$$

where σ^2 is the variance of the stochastic process. The white noise (ξ), see [3], is defined as the time derivative of standard Wiener process W

$$\xi(y,t) = \dot{W}(y,t). \tag{8}$$

2.1 Numerical Scheme

In this study, we will analyse the stochastic process of the system via a numerical approach. To this end we employ an implicit finite difference scheme to ensure the stability of the computation. The domain of the problem is uniformly discretised into N grid points $y_j = \frac{j}{N}$, for $j = 1, 2, ..., N$, with step size $\Delta y = 1/N$ and $w_j \approx w(y_j)$ for $j = 1, 2, ..., N$. By imposing a second order finite difference scheme on the boundary conditions in (5) a finite difference formula for the ghost points can be derived then a matrix of the finite difference scheme for w over $\{y_j\}_{j=1,2,...,N}$ can be written.

If one introduces an artificial variable $v = \dot{w}$ which is second order in time, equation (7) can be rewritten as a system of two coupled PDEs of the first order

$$\frac{\partial U}{\partial t} = \left(\mathcal{M}_1 + \beta^2 \mathcal{M}_2\right) U + F, \tag{9}$$

where $U = \begin{pmatrix} w \\ v \end{pmatrix}$, $\mathcal{M}_1 = \begin{pmatrix} 0 & 1 \\ -D^4 & 0 \end{pmatrix}$, $\mathcal{M}_2 = \begin{pmatrix} 0 & 0 \\ D^2 & 0 \end{pmatrix}$, $F = \begin{pmatrix} 0 \\ q \end{pmatrix}$, D^4 and D^2 is fourth-order and second-order finite difference scheme respectively. The time domain $[0, T]$ is divided into n steps with $\Delta t = T/n$, where T is the final time. By discretising spatially in y and temporally in t the discretised variables are

$$U_j^k = \begin{pmatrix} w_j \\ v_j \end{pmatrix} \text{ at } t = t_k = k\Delta t. \tag{10}$$

The backward Euler scheme can then be written in the matrix form

$$\begin{bmatrix} I_N & -\Delta t I_N \\ \Delta t\left(\mathbf{M}_1 + \beta^2 \mathbf{M}_2\right) & I_N \end{bmatrix} \begin{bmatrix} W \\ V \end{bmatrix}^{k+1} = \begin{bmatrix} W \\ V \end{bmatrix}^{k} + \begin{bmatrix} 0 \\ \Delta t Q \end{bmatrix}, \tag{11}$$

where I_N is the identity matrix and $\mathbf{M}_1$ and $\mathbf{M}_2$ are the matrix replacement of the fourth-order and second-order finite difference scheme for D^4 and D^2 respectively, $W = [w_1, w_2, \ldots, w_N]^T$, $V = [v_1, v_2, \ldots, v_N]^T$ and the discretised load, $Q = [q_1, q_2, \ldots, q_N]^T$, where $q_i = (1 + \sigma\zeta)\sin\omega t_k$ and $\zeta = dW/dt \approx N(0,1)/\sqrt{\Delta t}$, where $N(0,1)$ is a normally distributed random variable with zero mean and unit variance. Equation (11) is used to conduct Monte Carlo (MC) simulations to obtain data for analysis. For good accuracy, $\Delta y = 0.002$ and $\Delta t = 0.1s$ were chosen for 10,000 simulations and the standard deviation of the stochastic process is chosen as $\sigma = 0.01$.

3 Results

The snapshot of the deflection profile of the MC simulation at time T is shown in Figure 2a where we see a cluster of the deflection about a region. The Expectation $\mathbb{E}[w]$ shown in Figure 2b seemed to be the region about which the cluster is formed which is what is expected due to the fact that the input force is Gaussian. The resulting output is also a Gaussian process as can be confirmed by the distribution of the deflection at the top of the beam shown in Figure 3.

4 Conclusions

The stochastic analysis of a beam that can deformed by a combination of flexural and shear deformation is conducted. The beam is subjected to stochastic ground motion in the form of periodic motion with disturbance in the amplitude of the motion. The deformation of the beam is governed by a modified form

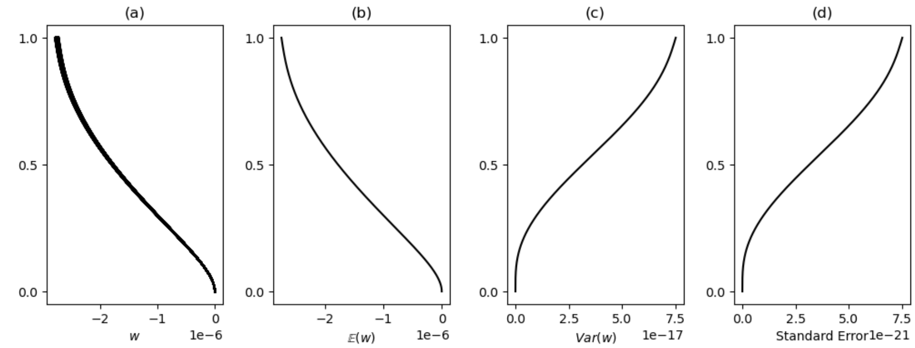

Figure 2: (a) Snapshots of w due to stochastic load in (7) using Monte-Carlo method at time T (b) Expectation $\mathbb{E}(w)$ of the stochastic process at T along the beam (c) Variance of the stochastic process at T along the beam (d) Standard error of the Monte Carlo method at T along the beam

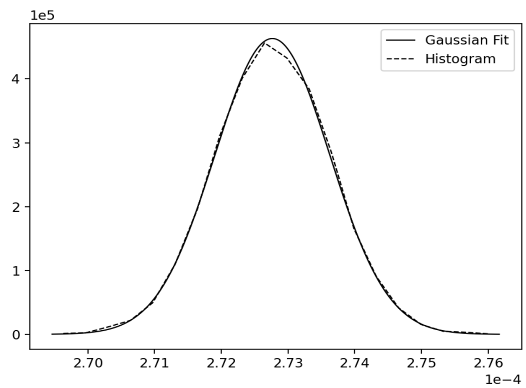

Figure 3: Histogram of the deflection data, w_1, from time T and distribution fitted to the data.

of Euler-Bernoulli beam equation. The stochastic analysis was conducted by numerical methods using a combination of a finite difference scheme and Monte-Carlo Simulation. Given that the input force is Gaussian, it is also observed that the response of the system is a Gaussian process.

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