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Derivations of a family of quantum second Weyl algebras

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A B S T R A C T

In view of a well-known theorem of Dixmier, its is natural to consider primitive quotients of $U_q^+(\mathfrak{g})$ as quantum analogues of Weyl algebras. In this work, we study primitive quotients of $U_q^+(G_2)$ and compute their Lie algebra of derivations. In particular, we show that, in some cases, all derivations are inner showing that the corresponding primitive quotients of $U_q^+(G_2)$ should be considered as deformations of Weyl algebras.

1. Introduction

Weyl algebras have been extensively studied in the last 60 years due to their link to Lie theory, differential operators, quantum mechanics, etc. One of the main questions remaining is the famous Dixmier Conjecture that asserts that every endomorphism of a complex Weyl algebra is an automorphism.

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Let $\mathbb{K}$ be a field and q be an element of $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ that is not a root of unity. The aim of this article is to produce quantum analogues of the second Weyl algebra and to compare their properties to those of the second Weyl algebra. There exist in the literature various families of "quantum Weyl algebras", e.g. the so-called quantum Weyl algebras and generalized Weyl algebras (GWA for short). Most of the time, they are obtained by generators and relations through a deformation of the classical defining relation of the first Weyl algebra: $xy - yx = 1$.

To produce potential quantizations, we take a different approach in this article. Our inspiration comes from a Theorem of Dixmier (see, for instance, [7, Théorème 4.7.9]) that asserts that primitive quotients of enveloping algebras of complex nilpotent Lie algebras are isomorphic to Weyl algebras.

We have at hand a quantum analogue of at least some enveloping algebras of complex nilpotent Lie algebras, namely the positive part $U_q^+(\mathfrak{g})$ of a quantized enveloping algebra $U_q(\mathfrak{g})$ of a complex simple Lie algebra $\mathfrak{g}$. As a consequence, it is natural to consider primitive quotients of $U_q^+(\mathfrak{g})$ as quantum analogues of Weyl algebras. In the A_2 and B_2 cases, primitive ideals of $U_q^+(\mathfrak{g})$ have been classified and it turns out that in the B_2 case, some of the resulting primitive quotients provide 'nice' quantum analogues of the first Weyl algebra. For instance, they are simple—this is not the case of quantum Weyl algebras—and do not possess non-trivial units—this is not the case of a quantum GWA over a Laurent polynomial ring (see [13] for details). It is also worth mentioning that Lopes [16] has studied the primitive ideals in the 0-stratum of $U_q^+(A_n)$ $(n \geq 2)$, and in particular, describes fully all the primitive spectra in all strata of $U_q^+(A_3)$. Furthermore, he concluded that the primitive quotients of $U_q^+(A_n)$, although they have even Gelfand-Kirillov dimension, are not isomorphic to quantum Weyl algebras (see [16, Corollary 3.6]).

The present article is concerned with the G_2 case. More precisely, we identify a family of primitive ideals of $U_q^+(G_2)$ and then proceed in proving that the corresponding primitive quotients have (at least for some choices of the parameters) properties similar to those of the second Weyl algebra. More precisely, the centre of $U_q^+(G_2)$ is a polynomial algebra $\mathbb{K}[\Omega_1, \Omega_2]$ in two variables, and we prove that the quotient algebra

$$A_{\alpha,\beta} := U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$$

is simple for all $(\alpha, \beta) \neq (0, 0)$. We then proceed and study these quotient algebras. In particular, we show that $A_{\alpha,\beta}$ has the same Gelfand-Kirillov dimension as the second Weyl algebra $A_2(\mathbb{K})$. We also establish that for certain choices of the parameters α and β, the algebra $A_{\alpha,\beta}$ is a deformation of a quadratic extension of $A_2(\mathbb{K})$ at $q = 1$.

In the final section, we compute the derivations of $A_{\alpha,\beta}$. Our results show that when α and β are both non-zero, all derivations of $A_{\alpha,\beta}$ are inner, a property that is well known to hold in $A_2(\mathbb{K})$ when the characteristics of $\mathbb{K}$ is zero.

In view of the celebrated Dixmier Conjecture, it would be interesting to describe automorphisms and endomorphisms of $A_{\alpha,\beta}$ when α and β are both non-zero. We intend to come back to these questions in the future.

This article is organized as follows. In Section 2, we recall the presentation of $U_q^+(G_2)$ as a so-called quantum nilpotent algebra (QNA for short). This allows the use of two different tools to study the prime and primitive spectra of $U_q^+(G_2)$: the $\mathcal{H}$-stratification theory of Goodearl and Letzter, and the deleting derivation theory of Cauchon. We recall both theories in the context of $U_q^+(G_2)$ in Section 2. In Section 3, we use these two theories to establish that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is a maximal ideal of $U_q^+(G_2)$ when $(\alpha, \beta) \neq (0, 0)$.

In Section 4, we focus on comparing $A_{\alpha,\beta}$ with the second Weyl algebra $A_2(\mathbb{K})$. In particular, we show that both have Gelfand-Kirillov dimension equal to 4. Through a direct computation, we also establish that $A_{1,\frac{1}{9(q^6-1)}}$ is a quadratic extension of $A_2(\mathbb{K})$ at $q = 1$. In this section, we also compute a linear basis for $A_{\alpha,\beta}$.

In the final section, we compute the derivations of $A_{\alpha,\beta}$. Our strategy here is to make use of the following tower of algebras arising from the deleting derivations algorithm (DDA for short):

$$\mathcal{R}_7 = A_{\alpha,\beta} \subset \mathcal{R}_6 = \mathcal{R}_7\Sigma_6^{-1} \subset \mathcal{R}_5 = \mathcal{R}_6\Sigma_5^{-1} \subset \mathcal{R}_4 = \mathcal{R}_5\Sigma_4^{-1} \subset \mathcal{R}_3.$$

The later algebra $\mathcal{R}_3$ is a simple quantum torus whose derivations have been described by Osborn and Passman in [19]. We pull back their description to obtain a description of the derivations of $A_{\alpha,\beta}$ through a step-by-step process consisting in "reverting" the DDA. Our results show that when α or β is equal to zero, then the first Hochschild cohomology group of $A_{\alpha,\beta}$ is a 1-dimensional vector space, whereas when both α and β are non-zero, all derivations are inner.

2. The quantum nilpotent algebra $U_q^+(G_2)$ and its primitive ideals

2.1. The quantum nilpotent algebra $U_q^+(G_2)$

Let $\mathbb{K}$ be a field and q be a non-zero element of $\mathbb{K}$ that is not a root of unity.

The algebra of $U_q^+(G_2)$ is the so-called positive part of the quantum enveloping algebra $U_q(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ of type G_2. It is well known, see for instance [2], that this algebra is generated over $\mathbb{K}$ by two indeterminates E_α and E_β subject to the following quantum Serre relations:

$$(S1) \quad E_\alpha^4 E_\beta - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q E_\alpha^3 E_\beta E_\alpha + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q E_\alpha^2 E_\beta E_\alpha^2 - \begin{bmatrix} 4 \\ 1 \end{bmatrix}_q E_\alpha E_\beta E_\alpha^3 + E_\beta E_\alpha^4 = 0,$$

$$(S2) \quad E_\beta^2 E_\alpha - \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{q^3} E_\beta E_\alpha E_\beta + E_\alpha E_\beta^2 = 0,$$

where $\begin{bmatrix} n \\ i \end{bmatrix}_z$ denotes the quantum binomial coefficients (see [2, I.6.1]).

One can construct a PBW-basis of $U_q^+(G_2)$ using the so-called Lusztig automorphisms of $U_q(\mathfrak{g})$, see for instance [2, I.6.8]. In the present case, such a basis was computed by De Graaf in [5]. We will use the convention of that paper, but with $E_1 := E_\alpha,\ E_2 := E_{3\alpha+\beta},\ E_3 := E_{2\alpha+\beta},\ E_4 := E_{3\alpha+2\beta},\ E_5 := E_{\alpha+\beta}$ and $E_6 := E_\beta$.

With these notations, the defining relations of $U_q^+(G_2)$ are as follows:

$$
\begin{aligned}
E_2E_1 &= q^{-3}E_1E_2 & E_3E_1 &= q^{-1}E_1E_3 - (q+q^{-1}+q^{-3})E_2\\
E_3E_2 &= q^{-3}E_2E_3 & E_4E_1 &= E_1E_4 + (1-q^2)E_3^2\\
E_4E_2 &= q^{-3}E_2E_4 - \frac{q^4-2q^2+1}{q^4+q^2+1}E_3^3 & E_4E_3 &= q^{-3}E_3E_4\\
E_5E_1 &= qE_1E_5 - (1+q^2)E_3 & E_5E_2 &= E_2E_5 + (1-q^2)E_3^2\\
E_5E_3 &= q^{-1}E_3E_5 - (q+q^{-1}+q^{-3})E_4 & E_5E_4 &= q^{-3}E_4E_5\\
E_6E_1 &= q^3E_1E_6 - q^3E_5 & E_6E_2 &= q^3E_2E_6 + (q^4+q^2-1)E_4\\
& & &\quad + (q^2-q^4)E_3E_5\\
E_6E_3 &= E_3E_6 + (1-q^2)E_5^2 & E_6E_5 &= q^{-3}E_5E_6\\
E_6E_4 &= q^{-3}E_4E_6 - \frac{q^4-2q^2+1}{q^4+q^2+1}E_5^3,
\end{aligned}
$$

and the monomials $E_1^{k_1}\dots E_6^{k_6}$ $(k_1,\dots,k_6 \in \mathbb{N})$ form a basis of $U_q^+(G_2)$ over $\mathbb{K}$.

Even better, one may write $U_q^+(G_2)$ as a QNA (short for quantum nilpotent algebra) or Cauchon-Goodearl-Letzter (CGL) extension in the sense of [14, Definition 3.1], by adjoining the generators E_i in order. This means in particular that $U_q^+(G_2)$ can be presented as an iterated Ore extension:

$$U_q^+(G_2) = \mathbb{K}[E_1][E_2;\sigma_2,\delta_2]\cdots[E_6;\sigma_6,\delta_6],$$

where the σ_i are automorphisms and the δ_i are left σ_i-derivations of the appropriate subalgebras. We would not need the precise definition of a QNA for what follows, but it is worth reminding the reader of the algebraic torus action involved in writing $U_q^+(G_2)$ as a QNA.

The algebraic torus $\mathcal{H} = (\mathbb{K}^\times)^2$ acts by automorphisms on $U_q^+(G_2)$ as follows:

$$h \cdot E_i = h_iE_i \text{ for all } i \in \{1,6\} \text{ and } h = (h_1,h_6) \in \mathcal{H}.$$

Note that the action of the automorphism h on the generators $E_2,\dots,E_5$ follows from the above defining relations.

By [2, Theorem II.2.7], the action of $\mathcal{H}$ on $U_q^+(G_2)$ is rational in the sense of [2, Definition II.2.6].

A consequence of the QNA condition is that important tools such as Cauchon's deleting derivations procedure and the Goodearl-Letzter stratification theory (this is the origin of the CGL extension terminology, see [14]) are available to study prime and primitive ideals. These ideas will be introduced in the following sections. At the moment, we merely note that it is immediate that $U_q^+(G_2)$ is a noetherian domain and that all prime ideals are completely prime (in the case of $U_q^+(G_2)$, it was proved in [20, Section 5]). We denote by F_q its skew-field of fractions, i.e. $F_q := \mathrm{Frac}(U_q^+(G_2))$.

2.2. *Prime ideals in $U_q^+(G_2)$ and $\mathcal{H}$-stratification*

A two-sided ideal I of $U_q^+(G_2)$ is said to be *$\mathcal{H}$-invariant* if $h \cdot I = I$ for all $h \in \mathcal{H}$. An *$\mathcal{H}$-prime ideal* of $U_q^+(G_2)$ is a proper $\mathcal{H}$-invariant ideal J of $U_q^+(G_2)$ such that if J contains the product of two $\mathcal{H}$-invariant ideals of $U_q^+(G_2)$ then J contains at least one of them. We denote by $\mathcal{H}\text{-Spec}(U_q^+(G_2))$ the set of all $\mathcal{H}$-prime ideals of $U_q^+(G_2)$. Observe that if P is a prime ideal of $U_q^+(G_2)$ then

$$(P : \mathcal{H}) \;:=\; \bigcap_{h \in \mathcal{H}} h \cdot P \tag{1}$$

is an $\mathcal{H}$-prime ideal of $U_q^+(G_2)$. Indeed, let J be an $\mathcal{H}$-prime ideal of $U_q^+(G_2)$. We denote by $\mathrm{Spec}_J(U_q^+(G_2))$ the *$\mathcal{H}$-stratum* associated to J; that is,

$$\mathrm{Spec}_J(U_q^+(G_2)) = \{P \in \mathrm{Spec}(U_q^+(G_2)) \mid (P : \mathcal{H}) = J\}. \tag{2}$$

Then the $\mathcal{H}$-strata of $\mathrm{Spec}(U_q^+(G_2))$ form a partition of $\mathrm{Spec}(U_q^+(G_2))$ [2, Chapter II.2]; that is,

$$\mathrm{Spec}(U_q^+(G_2)) = \bigsqcup_{J \in \mathcal{H}\text{-Spec}(U_q^+(G_2))} \mathrm{Spec}_J(U_q^+(G_2)). \tag{3}$$

This partition is the so-called *$\mathcal{H}$-stratification* of $\mathrm{Spec}(U_q^+(G_2))$.

It follows from the work of Goodearl and Letzter [9] that every $\mathcal{H}$-prime ideal of $U_q^+(G_2)$ is completely prime, so $\mathcal{H}\text{-Spec}(U_q^+(G_2))$ coincides with the set of $\mathcal{H}$-invariant completely prime ideals of $U_q^+(G_2)$. Moreover there are precisely $|W|$ $\mathcal{H}$-prime ideals in $U_q^+(G_2)$, where W denotes the Weyl group of type G_2 (see [17, Remark 6.2.2]). As a consequence, the $\mathcal{H}$-stratification of $\mathrm{Spec}(U_q^+(G_2))$ is finite and so the full strength of the $\mathcal{H}$-stratification theory of Goodearl and Letzter is available to study $\mathrm{Spec}(U_q^+(G_2))$.

For each $\mathcal{H}$-prime ideal J of $U_q^+(G_2)$, the space $\mathrm{Spec}_J(U_q^+(G_2))$ is homeomorphic to the prime spectrum $\mathrm{Spec}(\mathbb{K}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}])$ of a commutative Laurent polynomial ring whose Krull dimension depends on J, [2, Theorems II.2.13 and II.6.4]. These Krull dimensions were computed in [1,22]. Finally, let us mention that the primitive ideals of $U_q^+(G_2)$ are precisely the prime ideals that are maximal in their $\mathcal{H}$-strata [2, Theorem II.8.4].

In this article, we will mainly focus on one specific $\mathcal{H}$-stratum. Since $U_q^+(G_2)$ is a domain, 0 (technically, $\langle 0\rangle$) is clearly an $\mathcal{H}$-invariant completely prime ideal of $U_q^+(G_2)$, and so an $\mathcal{H}$-prime. We will focus on computing its stratum, the so-called 0-stratum. The motivation here is twofold: first, in the B_2 case, we obtain "new" quantum deformations of the first Weyl algebra as $U_q^+(B_2)/P$, where P is a primitive ideal from the 0-stratum of $\operatorname{Spec}(U_q^+(B_2))$ [13]. Next, in the present case, we would like to construct algebras of Gelfand-Kirillov dimension 4 as explained in the introduction. Since Tauvel's height formula holds in $U_q^+(G_2)$ [8], we need to quotient $U_q^+(G_2)$ by a primitive ideal of height 2. Given that the $\mathcal{H}$-spectrum of $U_q^+(G_2)$ (viewed as a poset under set inclusion) is isomorphic as a poset to the Weyl group of type G_2 (viewed as a poset under the Bruhat order), such primitive ideals can only be found in the 0-stratum and the strata associated to one of the two height 1 $\mathcal{H}$-primes. In this article, we mainly present results for the 0-stratum, but we will also indicate results obtained for the primitive quotients coming from the height 1 $\mathcal{H}$-prime strata.

2.3. Deleting derivations algorithm (DDA) in $U_q^+(G_2)$

As $U_q^+(G_2)$ is a QNA, we can apply Cauchon's DDA to study its prime spectrum. Recall first that $U_q^+(G_2)$ is an iterated Ore extension of the form:

$$U_q^+(G_2) = \mathbb{K}[E_1][E_2;\sigma_2][E_3;\sigma_3,\delta_3][E_4;\sigma_4,\delta_4][E_5;\sigma_5,\delta_5][E_6;\sigma_6,\delta_6];$$

where, σ_2 denotes the automorphism of $\mathbb{K}[E_1]$ defined by:

$$\sigma_2(E_1) = q^{-3}E_1,$$

σ_3 denotes the automorphism of $\mathbb{K}[E_1][E_2;\sigma_2]$ defined by:

$$\sigma_3(E_1) = q^{-1}E_1 \qquad \sigma_3(E_2) = q^{-3}E_2,$$

δ_3 denotes the σ_3-derivation of $\mathbb{K}[E_1][E_2;\sigma_2]$ defined by:

$$\delta_3(E_1) = -(q+q^{-1}+q^{-3})E_2 \qquad \delta_3(E_2) = 0,$$

σ_4 denotes the automorphism of $\mathbb{K}[E_1]\cdots[E_3;\sigma_3,\delta_3]$ defined by:

$$\sigma_4(E_1) = E_1 \qquad \sigma_4(E_2) = q^{-3}E_2 \qquad \sigma_4(E_3) = q^{-3}E_3,$$

δ_4 denotes the σ_4-derivation of $\mathbb{K}[E_1]\cdots[E_3;\sigma_3,\delta_3]$ defined by:

$$\delta_4(E_1) = (1-q^2)E_3^2 \qquad \delta_4(E_2) = \frac{-q^4+2q^2-1}{q^4+q^2+1}E_3^3 \qquad \delta_4(E_3) = 0,$$

σ_5 denotes the automorphism of $\mathbb{K}[E_1]\cdots[E_4;\sigma_4,\delta_4]$ defined by:

$$\sigma_5(E_1) = qE_1 \qquad \sigma_5(E_2) = E_2 \qquad \sigma_5(E_3) = q^{-1}E_3 \qquad \sigma_5(E_4) = q^{-3}E_4,$$

δ_5 denotes the σ_5-derivation of $\mathbb{K}[E_1]\cdots[E_4;\sigma_4,\delta_4]$ defined by:

$$\delta_5(E_1)=-(1+q^2)E_3 \quad \delta_5(E_2)=(1-q^2)E_3^2 \quad \delta_5(E_3)=-(q+q^{-1}+q^{-3})E_4 \quad \delta_5(E_4)=0,$$

σ_6 denotes the automorphism of $\mathbb{K}[E_1]\cdots[E_5;\sigma_5,\delta_5]$ defined by:

$$\sigma_6(E_1)=q^3E_1 \qquad \sigma_6(E_2)=q^3E_2 \quad \sigma_6(E_3)=E_3 \quad \sigma_6(E_4)=q^{-3}E_4 \quad \sigma_6(E_5)=q^{-3}E_5,$$

and δ_6 denotes the σ_6-derivation of $\mathbb{K}[E_1]\cdots[E_5;\sigma_5,\delta_5]$ defined by:

$$\delta_6(E_1)=-q^3E_5 \qquad \delta_6(E_2)=(q^2-q^4)E_3E_5+(q^4+q^2-1)E_4 \qquad \delta_6(E_3)=(1-q^2)E_5^2$$
$$\delta_6(E_4)=\frac{-q^4+2q^2-1}{q^4+q^2+1}E_5^3 \qquad \delta_6(E_5)=0.$$

The DDA constructs by a decreasing induction a family $\{E_{1,j},\ldots,E_{6,j}\}$ of elements of the division ring of fractions $F_q=\mathrm{Fract}(U_q^+(G_2))$ of $U_q^+(G_2)$ for each $2\le j\le 7$. The precise definition of these elements in the general context of QNAs can be found in [4].

In the present case, a direct computation leads to:

$$\begin{aligned}
E_{1,6}&=E_1+rE_5E_6^{-1}\\
E_{2,6}&=E_2+tE_3E_5E_6^{-1}+uE_4E_6^{-1}+nE_5^3E_6^{-2}\\
E_{3,6}&=E_3+sE_5^2E_6^{-1}\\
E_{4,6}&=E_4+bE_5^3E_6^{-1}\\
E_{1,5}&=E_{1,6}+hE_{3,6}E_{5,6}^{-1}+gE_{4,6}E_{5,6}^{-2}\\
E_{2,5}&=E_{2,6}+fE_{3,6}^2E_{5,6}^{-1}+pE_{3,6}E_{4,6}E_{5,6}^{-2}+eE_{4,6}^2E_{5,6}^{-3}\\
E_{3,5}&=E_{3,6}+aE_{4,6}E_{5,6}^{-1}\\
E_{1,4}&=E_{1,5}+sE_{3,5}^2E_{4,5}^{-1}\\
E_{2,4}&=E_{2,5}+bE_{3,5}^3E_{4,5}^{-1}\\
E_{1,3}&=E_{1,4}+aE_{2,4}E_{3,4}^{-1}\\
T_1&:=E_{1,2}=E_{1,3}\\
T_2&:=E_{2,2}=E_{2,3}=E_{2,4}\\
T_3&:=E_{3,2}=E_{3,3}=E_{3,4}=E_{3,5}\\
T_4&:=E_{4,2}=E_{4,3}=E_{4,4}=E_{4,5}=E_{4,6}\\
T_5&:=E_{5,2}=E_{5,3}=E_{5,4}=E_{5,5}=E_{5,6}=E_5\\
T_6&:=E_{6,2}=E_{6,3}=E_{6,4}=E_{6,5}=E_{6,6}=E_6,
\end{aligned}$$

where the parameters a,b,e,f,g,h,n,p,r,s,t,u are all defined in Appendix A.2.

In the following, we set $A := U_q^+(G_2)$ and we denote by $A^{(j)}$ the subalgebra of F_q generated by $E_{1,j}, \ldots, E_{6,j}$. The following results were proved by Cauchon [4, Théorème 3.2.1 and Lemme 4.2.1]. For $2 \leq j \leq 7$, we have:

1. When $j = 7$, $(E_{1,7}, \ldots, E_{6,7}) = (E_1, \ldots, E_6)$, so that $A^{(7)} = A = U_q^+(G_2)$;
2. $A^{(j)}$ is isomorphic to an iterated Ore extension of the form

$$\mathbb{K}[y_1] \ldots [y_{j-1}; \sigma_{j-1}, \delta_{j-1}][y_j; \tau_j] \cdots [y_6; \tau_6]$$

   by an isomorphism that sends $E_{i,j}$ to y_i $(1 \leq i \leq 6)$, where $\tau_j, \ldots, \tau_6$ denote the $\mathbb{K}$-linear automorphisms such that $\tau_\ell(y_i) = \lambda_{\ell,i} y_i$ $(1 \leq i \leq \ell - 1)$ and the $\lambda_{\ell,i}$ are defined by $\sigma_\ell(E_i) = \lambda_{\ell,i} E_i$.
3. Assume that $j \neq 7$. The set $\Sigma_j := \{E_{j,j+1}^n \mid n \in \mathbb{N}\} = \{E_{j,j}^n \mid n \in \mathbb{N}\}$ is a multiplicative system of regular elements of $A^{(j)}$ and $A^{(j+1)}$, and satisfies the Ore condition in both $A^{(j)}$ and $A^{(j+1)}$. Moreover we have

$$A^{(j)} \Sigma_j^{-1} = A^{(j+1)} \Sigma_j^{-1}.$$

It follows from these results that $A^{(j)}$ is a noetherian domain, for all $2 \leq j \leq 7$.

As in [4], we use the following notation.

Notation 2.1. We set $\overline{A} := A^{(2)}$ and $T_i := E_{i,2}$ for all $1 \leq i \leq 6$.

It follows from [4, Proposition 3.2.1] that $\overline{A}$ is a quantum affine space in the indeterminates $T_1, \ldots, T_6$ and so can be presented as an iterated Ore extension in the T_is with no skew-derivations. It is for this reason that Cauchon used the expression "effacement des dérivations". More precisely, let $M = (\mu_{i,j}) \in M_6(\mathbb{Z})$ be a skew-symmetric matrix defined as follows:

$$M := \begin{bmatrix} 0 & 3 & 1 & 0 & -1 & -3 \\ -3 & 0 & 3 & 3 & 0 & -3 \\ -1 & -3 & 0 & 3 & 1 & 0 \\ 0 & -3 & -3 & 0 & 3 & 3 \\ 1 & 0 & -1 & -3 & 0 & 3 \\ 3 & 3 & 0 & -3 & -3 & 0 \end{bmatrix}. \tag{4}$$

Then, we have:

$$\overline{A} = \mathbb{K}_{q^M}[T_1, \ldots, T_6], \tag{5}$$

where $\mathbb{K}_{q^M}[T_1, \ldots, T_6]$ denotes the $\mathbb{K}$-algebra generated by $T_1, \ldots, T_6$ with relations $T_i T_j = q^{\mu_{i,j}} T_j T_i$ for all i, j.

2.4. Canonical embedding

Since $A = U_q^+(G_2)$ is a QNA, one can use Cauchon's DDA to relate the prime spectrum of A to the prime spectrum of the associated quantum affine space $\overline{A}$. More precisely, the DDA allows the construction of embeddings

$$\psi_j : \mathrm{Spec}(A^{(j+1)}) \longrightarrow \mathrm{Spec}(A^{(j)}) \qquad (2 \leq j \leq 6). \tag{6}$$

Recall from [4, Section 4.3] that these embeddings are defined as follows.

Let $P \in \mathrm{Spec}(A^{(j+1)})$. Then

$$\psi_j(P) = \begin{cases} P\Sigma_j^{-1} \cap A^{(j)} & \text{if } E_{j,j+1} = T_j \notin P \\ g_j^{-1}(P/\langle E_{j,j+1}\rangle) & \text{if } E_{j,j+1} \in P \end{cases}$$

where g_j denotes the surjective homomorphism

$$g_j : A^{(j)} \to A^{(j+1)}/\langle E_{j,j+1}\rangle$$

defined by

$$g_j(E_{i,j}) := E_{i,j+1} + \langle E_{j,j+1}\rangle$$

(for more details, see [4, Lemme 4.3.2]). It was proved by Cauchon [4, Proposition 4.3.1] that ψ_j induces an increasing homeomorphism from the topological space

$$\{P \in \mathrm{Spec}(A^{(j+1)}) \mid E_{j,j+1} \notin P\}$$

onto

$$\{Q \in \mathrm{Spec}(A^{(j)}) \mid E_{j,j} \notin Q\}$$

whose inverse is also an increasing homeomorphism. Also, ψ_j induces an increasing homeomorphism from

$$\{P \in \mathrm{Spec}(A^{(j+1)}) \mid E_{j,j+1} \in P\}$$

onto its image by ψ_j whose inverse similarly is an increasing homeomorphism. Note however that, in general, ψ_j is not a homeomorphism from $\mathrm{Spec}(A^{(j+1)})$ onto its image.

Composing these embeddings, we get an embedding

$$\psi := \psi_2 \circ \cdots \circ \psi_6 : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(\overline{A}), \tag{7}$$

which is called the *canonical embedding* from $\mathrm{Spec}(A)$ into $\mathrm{Spec}(\overline{A})$.

The canonical embedding ψ is $\mathcal{H}$-equivariant so that $\varphi(\mathcal{H}-\mathrm{Spec}(A)) \subseteq \mathcal{H}-\mathrm{Spec}(\overline{A})$. Interestingly, the set $\mathcal{H}-\mathrm{Spec}(\overline{A})$ has been described by Cauchon as follows. For any subset C of $\{1,\dots,6\}$, let K_C denote the $\mathcal{H}$-prime ideal of $\overline{A}$ generated by the T_i with $i \in C$, that is

$$K_C = \langle T_i \mid i \in C \rangle.$$

It follows from [4, Proposition 5.5.1] that

$$\mathcal{H}-\mathrm{Spec}(\overline{A}) = \{K_C \mid C \subseteq \{1,\dots,6\}\},$$

so that

$$\psi(\mathcal{H}-\mathrm{Spec}(A)) \subseteq \{K_C \mid C \subseteq \{1,\dots,6\}\}.$$

3. Primitive ideals of $U_q^+(G_2)$ in the 0-stratum

The aim of this section is to give explicit generating sets for the primitive ideals of $U_q^+(G_2)$ that belong to the 0-stratum. They are intimately related to the centre of $U_q^+(G_2)$ and so we start this section by making explicit the centre of $U_q^+(G_2)$ and of related algebras.

Throughout the rest of this paper, the notation $q^\bullet$ will mean any arbitrary integer power of q. We will often use this notation whenever the exponent of q is of no interest.

3.1. Centre of $U_q^+(G_2)$

Recall that $\overline{A} = A^{(2)} = \mathbb{K}_\Lambda[T_1,\dots,T_6]$ is a quantum affine space. Set $\Omega_1 := T_1T_3T_5$ and $\Omega_2 := T_2T_4T_6$. One can easily verify that Ω_1 and Ω_2 are central elements of $\overline{A}$ by checking they commute with all the T_is.

We now want to successively pull Ω_1 and Ω_2 from the quantum affine space $\overline{A}$ into the algebra A using the data of the DDA of A discussed above. A direct computation shows that

$$\begin{aligned}
\Omega_1 &:= T_1T_3T_5 \\
&= E_{1,4}E_{3,4}E_{5,4} + aE_{2,4}E_{5,4} \\
&= E_{1,5}E_{3,5}E_{5,5} + aE_{2,5}E_{5,5} \\
&= E_{1,6}E_{3,6}E_{5,6} + aE_{1,6}E_{4,6} + aE_{2,6}E_{5,6} + a'E_{3,6}^2 \\
&= E_1E_3E_5 + aE_1E_4 + aE_2E_5 + a'E_3^2,
\end{aligned}$$

and

$$\begin{aligned}\Omega_2 &:= T_2T_4T_6\\ &= E_{2,5}E_{4,5}E_{6,5} + bE_{3,5}^3E_{6,5}\\ &= E_{2,6}E_{4,6}E_{6,6} + bE_{3,6}^3E_{6,6}\\ &= E_2E_4E_6 + bE_2E_5^3 + bE_3^3E_6 + b'E_3^2E_5^2 + c'E_3E_4E_5 + d'E_4^2,\end{aligned}$$

where the parameters a, b, a', b', c', d' can be found in Appendix A.2. Note that Ω_1 and Ω_2 are central elements of $A^{(j)}$ for each $2 \leq j \leq 7$, since $\Omega_i \in A^{(7)} \subseteq A^{(j)} \subseteq \overline{A}$.

We now want to show that the centre $Z(A)$ of A and of other related algebras is a polynomial ring generated by Ω_1 and Ω_2 over $\mathbb{K}$. The following discussions will lead us to the proof.

Set $S_j := \{\lambda T_j^{i_j}T_{j+1}^{i_{j+1}} \dots T_6^{i_6} \mid i_j, \dots, i_6 \in \mathbb{N} \text{ and } \lambda \in \mathbb{K}^*\}$ for each $2 \leq j \leq 6$. One can observe that S_j is a multiplicative system of non-zero divisors of $A^{(j)} = \mathbb{K}\langle E_{i,j} \mid \text{for all } i = 1, \dots, 6\rangle$. Furthermore, the elements $T_j, \dots, T_6$ are all normal in $A^{(j)}$. Hence, S_j is an Ore set in $A^{(j)}$. We can therefore localize $A^{(j)}$ at S_j as follows:

$$R_j := A^{(j)}S_j^{-1}.$$

Recall that $\Sigma_j := \{T_j^n \mid n \in \mathbb{N}\}$ is an Ore set in both $A^{(j)}$ and $A^{(j+1)}$ for each $2 \leq j \leq 6$, and that

$$A^{(j)}\Sigma_j^{-1} = A^{(j+1)}\Sigma_j^{-1}.$$

For all $2 \leq j \leq 6$, we have that:

$$\begin{aligned} R_j = A^{(j)}S_j^{-1} &= (A^{(j)}\Sigma_j^{-1})S_{1+j}^{-1} = (A^{(j+1)}\Sigma_j^{-1})S_{1+j}^{-1}\\ &= (A^{(j+1)}S_{j+1}^{-1})\Sigma_j^{-1} = R_{j+1}\Sigma_j^{-1}. \end{aligned} \tag{8}$$

Note that $R_7 := A$.

Again, one can also observe that T_1 is normal in R_2. As a result, we can form the localization $R_1 := R_2[T_1^{-1}]$. The algebra R_1 is the quantum torus associated to the quantum affine space $\overline{A}$. As a result, $R_1 = \mathbb{K}_{q^M}[T_1^{\pm 1}, \dots, T_6^{\pm 1}]$, where $T_iT_j = q^{\mu_{ij}}T_jT_i$ for all $1 \leq i, j \leq 6$ and μ_{ij} are the entries of M. Similarly to [15, Section 3.1], we construct the following tower of algebras:

$$\begin{aligned} A = R_7 \subset R_6 = R_7\Sigma_6^{-1} \subset R_5 = R_6\Sigma_5^{-1} \subset R_4 = R_5\Sigma_4^{-1}\\ \subset R_3 = R_4\Sigma_3^{-1} \subset R_2 = R_3\Sigma_2^{-1} \subset R_1. \end{aligned} \tag{9}$$

Note that the family $(E_{1,j}^{k_1} \dots E_{6,j}^{k_6})$, where $k_i \in \mathbb{N}$ if $i < j$ and $k_i \in \mathbb{Z}$ otherwise is a PBW-basis of R_j for all $1 \leq j \leq 7$. In particular, the family $(T_1^{k_1} \dots T_6^{k_6})_{k_1, \dots, k_6 \in \mathbb{Z}}$ is a PBW-basis of R_1.

Lemma 3.1.

1. $Z(R_1) = \mathbb{K}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$.
2. $Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$.
3. $Z(\overline{A}) = \mathbb{K}[\Omega_1, \Omega_2]$.
4. $Z(A) = \mathbb{K}[\Omega_1, \Omega_2]$.

Proof. 1. It follows from [9, 1.3] that $Z(R_1)$ is a commutative Laurent polynomial ring generated by certain monomials in the T_is. A direct computation proves the result.

2. Clearly, $\mathbb{K}[\Omega_1, \Omega_2] \subseteq Z(R_3)$. For the reverse inclusion, let $y \in Z(R_3)$. Then, y can be written in terms of the basis of R_3 (recall that $T_i = E_{i,3}$) as:

$$y = \sum_{(i,\dots,n)\in\mathbb{N}^2\times\mathbb{Z}^4} a_{(i,\dots,n)} T_1^i T_2^j T_3^k T_4^l T_5^m T_6^n.$$

   Using the fact that $T_1, \dots, T_6$ are all normal elements in R_3 and $yT_i = T_i y$ for all i, one easily concludes that $i = k = m$ and $j = l = n$ for all monomials appearing in y. Since $i, j \geq 0$, we have that $y = \sum_{(i,j)\in\mathbb{N}^2} q^\bullet a_{(i,j)} T_1^i T_3^i T_5^i T_2^j T_4^j T_6^j = \sum_{(i,j)\in\mathbb{N}^2} q^\bullet a_{(i,j)} \Omega_1^i \Omega_2^j$. This implies that $y \in \mathbb{K}[\Omega_1, \Omega_2]$ as expected.

3. Observe that $\mathbb{K}[\Omega_1, \Omega_2] \subseteq Z(\overline{A}) \subseteq Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$. Hence, $Z(\overline{A}) = \mathbb{K}[\Omega_1, \Omega_2]$.
4. Since R_i is a localization of R_{i+1}, it follows that $Z(R_{i+1}) \subseteq Z(R_i)$. From (9), we have that $Z(A) \subseteq Z(R_3)$. Observe that $\mathbb{K}[\Omega_1, \Omega_2] \subseteq Z(A) \subseteq Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$. Hence, $Z(A) = \mathbb{K}[\Omega_1, \Omega_2]$. □

Remark 3.2. Since $Z(A) = Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$ and $Z(R_{i+1}) \subseteq Z(R_i)$, it follows from (9) that $Z(A) = Z(R_6) = Z(R_5) = Z(R_4) = Z(R_3) = \mathbb{K}[\Omega_1, \Omega_2]$. One can also deduce from the proof of Lemma 3.1 that $Z(R_2) = \mathbb{K}[\Omega_1, \Omega_2^{\pm 1}]$.

Remark 3.3. The centre of the positive part of a quantized enveloping algebra of a simple Lie algebra has been described by Caldero in [3] but we will need Remark 3.2 later on.

3.2. Ω_1 and Ω_2 generate completely prime ideals of $U_q^+(G_2)$

The aim of this subsection is to show that $\langle\Omega_1\rangle$ and $\langle\Omega_2\rangle$ are (completely) prime. We will make use of the DDA to establish these facts. Note that we could also have used the results of [10] to obtain these results. However, we will need some of the intermediate steps obtained here to compute the derivations of certain primitive quotients of $U_q^+(G_2)$ in the final section.

From Subsection 2.4 we know that there is a bijection between $\{P \in \operatorname{Spec}(A^{(j+1)}) \mid P \cap \Sigma_j = \emptyset\}$ and $\{Q \in \operatorname{Spec}(A^{(j)}) \mid Q \cap \Sigma_j = \emptyset\}$ via $P = Q\Sigma_j^{-1} \cap A^{(j+1)}$. Note that $\langle T_1\rangle$ and $\langle T_2\rangle$ are prime ideals of the quantum affine space $\overline{A}$, since each of the factor algebras $\overline{A}/\langle T_1\rangle$ and $\overline{A}/\langle T_2\rangle$ is isomorphic to a quantum affine space of rank 5 which is well known to be a domain.

The following result and its proof show that $\langle T_1\rangle$ belongs to the image $\mathrm{Im}(\psi)$ of the canonical embedding ψ and that $\langle \Omega_1\rangle$ is the completely prime ideal of A such that $\psi(\langle \Omega_1\rangle) = \langle T_1\rangle$.

Lemma 3.4. $\langle \Omega_1\rangle \in \mathrm{Spec}(A)$.

Proof. We will prove this result in several steps by showing that:

1. $\langle T_1\rangle_{A^{(3)}} \in \mathrm{Spec}(A^{(3)})$.
2. $\langle E_{1,4}T_3 + aT_2\rangle = \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$, hence $Q_1 := \langle E_{1,4}T_3 + aT_2\rangle \in \mathrm{Spec}(A^{(4)})$.
3. $\langle E_{1,5}T_3 + aE_{2,5}\rangle = Q_1[T_4^{-1}] \cap A^{(5)}$, hence $Q_2 := \langle E_{1,5}T_3 + aE_{2,5}\rangle \in \mathrm{Spec}(A^{(5)})$.
4. $\langle \Omega_1\rangle_{A^{(6)}} = Q_2[T_5^{-1}] \cap A^{(6)}$, hence $\langle \Omega_1\rangle_{A^{(6)}} \in \mathrm{Spec}(A^{(6)})$.
5. $\langle \Omega_1\rangle_A = \langle \Omega_1\rangle_{A^{(6)}}[T_6^{-1}] \cap A$, hence $\langle \Omega_1\rangle_A \in \mathrm{Spec}(A)$.

We now proceed to prove the above claims.

1. One can easily verify that $A^{(3)}/\langle T_1\rangle$ is isomorphic to a quantum affine space of rank 5, which is a domain, hence $\langle T_1\rangle$ is a prime ideal in $A^{(3)}$.

2. Note that $T_1 = E_{1,4} + aT_2T_3^{-1}$. We want to show that $\langle E_{1,4}T_3 + aT_2\rangle = \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$. Observe that $\langle E_{1,4}T_3 + aT_2\rangle \subseteq \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$. We established the reverse inclusion. Let $y \in \langle T_1\rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)}$. Then, $y \in \langle T_1\rangle_{A^{(3)}}[T_3^{-1}]$. Therefore, there exists $i \in \mathbb{N}$ such that $yT_3^i \in \langle T_1\rangle_{A^{(3)}}$. This implies that $yT_3^i = T_1 v$, for some $v \in A^{(3)}$. Since $A^{(3)}[T_3^{-1}] = A^{(4)}[T_3^{-1}]$, there exists $j \in \mathbb{N}$ such that $vT_3^j = v'$, for some $v' \in A^{(4)}$. It follows that $yT_3^{i+j} = T_1 v T_3^j = T_1 v' = (E_{1,4} + aT_2T_3^{-1})v' = (E_{1,4}T_3 + aT_2)T_3^{-1}v'$. The multiplicative system generated by T_3 satisfies the Ore condition in $A^{(4)}$, hence, there exists $k \in \mathbb{N}$ and $v'' \in A^{(4)}$ such that $T_3^{-1}v' = v''T_3^{-k}$. One can therefore write $yT_3^{i+j} = (E_{1,4}T_3 + aT_2)v''T_3^{-k}$. This implies that $yT_3^{\delta} = \Omega_1' v''$, where $\Omega_1' := E_{1,4}T_3 + aT_2$ and $\delta = i + j + k$. Set $S := \{s \in \mathbb{N} \mid \exists v'' \in A^{(4)} : yT_3^s = \Omega_1' v''\}$. Note that $S \neq \emptyset$, since $\delta \in S$. Let $s = s_0$ be the minimum element of S such that $yT_3^{s_0} = \Omega_1' v''$. We want to show that $s_0 = 0$. Remember: $\Omega_1' T_5 = \Omega_1$ in $A^{(4)}$. Since Ω_1 is central in $A^{(4)}$, and T_5 is normal in $A^{(4)}$, we must have Ω_1' to be a normal element in $A^{(4)}$, otherwise, there will be a contradiction. Therefore, there exists $w \in A^{(4)}$ such that $yT_3^{s_0} = \Omega_1' v'' = w\Omega_1'$. Now, $A^{(4)}$ can be viewed as a free left $\mathbb{K}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle$-module with basis $\left(T_3^{\xi}\right)_{\xi\in\mathbb{N}}$. One can therefore write $y = \sum_{\xi=0}^n \alpha_\xi T_3^{\xi}$ and $w = \sum_{\xi=0}^n \beta_\xi T_3^{\xi}$, where $\alpha_\xi, \beta_\xi \in \mathbb{K}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle$. This implies that $\sum_{\xi=0}^n \alpha_\xi T_3^{\xi+s_0} = \sum_{\xi=0}^n \beta_\xi T_3^{\xi}\Omega_1' = \sum_{\xi=0}^n q^{\bullet}\beta_\xi \Omega_1' T_3^{\xi}$ (note that $T_3\Omega_1' = q^{-1}\Omega_1' T_3$). Given that $\Omega_1' = E_{1,4}T_3 + aT_2$, we have that $\sum_{\xi=0}^n \alpha_\xi T_3^{\xi+s_0} = \sum_{\xi=0}^n q^{\bullet}\beta_\xi E_{1,4}T_3^{1+\xi} + \sum_{\xi=0}^n q^{\bullet} a\beta_\xi T_2 T_3^{\xi}$. Suppose that $s_0 > 0$. Then, identifying the constant coefficients, we have $q^{\bullet}a\beta_0 T_2 = 0$. As a result, $\beta_0 = 0$, since $q^{\bullet}aT_2 \neq 0$. Hence, w can be written as $w = \sum_{\xi=1}^n \beta_\xi T_3^{\xi}$. Returning to $yT_3^{s_0} = w\Omega_1'$, we have that $yT_3^{s_0} = \sum_{\xi=1}^n \beta_\xi T_3^{\xi}\Omega_1' = \sum_{\xi=1}^n q^{\bullet}\beta_\xi \Omega_1' T_3^{\xi} = \Omega_1' \sum_{\xi=1}^n q^{\bullet}\beta_\xi' T_3^{\xi}$. This implies that $yT_3^{s_0-1} = \Omega_1' w'$, where $w' = \sum_{\xi=1}^n q^{\bullet}\beta_\xi' T_3^{\xi-1} \in A^{(4)}$, with $\beta_\xi' \in \mathbb{K}\langle E_{1,4}, T_2, T_4, T_5, T_6\rangle$. Consequently, $s_0 - 1 \in S$, a contradiction! Therefore, $s_0 = 0$

and $y = \Omega_1' v'' \in \langle \Omega_1' \rangle = \langle E_{1,4}T_3 + aT_2 \rangle$. Hence, $\langle T_1 \rangle_{A^{(3)}}[T_3^{-1}] \cap A^{(4)} \subseteq \langle E_{1,4}T_3 + aT_2 \rangle$ as desired.

The following steps are proved in a similar manner to Step 2. They are left to the reader who might want to check details in [18, Section 2.3]. $\square$

Using similar techniques, one can prove that $\langle T_2 \rangle \in \mathrm{Im}(\psi)$ and that $\langle \Omega_2 \rangle$ is the completely prime ideal of A such that $\psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle$. Again, we refer the interested reader to [18, Section 2.3] for details. We record these facts in the following lemma.

Lemma 3.5. *$\langle \Omega_2 \rangle$ is a completely prime ideal of A and $\psi(\langle \Omega_2 \rangle) = \langle T_2 \rangle$.*

Since $\mathcal{H}$-Spec$(U_q^+(G_2))$ is isomorphic as a poset to the Weyl group W of type G_2 by [21], there are only 2 $\mathcal{H}$-primes in $U_q^+(G_2)$ of height 1. Since Ω_1 and Ω_2 are central, the prime ideals that they generate have height less than or equal to 1, and so equal to 1. As an immediate consequence, we get the following result.

Lemma 3.6.

1. *$\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$ are the only height one $\mathcal{H}$-invariant prime ideals of A.*
2. *Every non-zero $\mathcal{H}$-invariant prime ideal of A contains either $\langle \Omega_1 \rangle$ or $\langle \Omega_2 \rangle$.*

Remark 3.7. The first part of Lemma 3.6 can be deduced from [10, Theorem 4.3]. However, we needed the argument to establish the second part of the lemma.

3.3. Description of the 0-stratum and beyond

In this subsection, we will often assume that our base field $\mathbb{K}$ is algebraically closed. This assumption is actually not necessary for the main result of this subsection, Theorem 3.12, but makes the description of the 0-stratum easier to present.

This subsection focuses on finding the height two maximal ideals of $A = U_q^+(G_2)$. Note first that such ideals can only belong to the $\mathcal{H}$-stratum of an $\mathcal{H}$-prime of height less than or equal to 1 (since $\mathcal{H}$-Spec(A) is isomorphic as a poset to W). It follows from the previous subsections that we need to compute the $\mathcal{H}$-strata of 3 $\mathcal{H}$-primes: 0, $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$. We start with the 0-stratum.

The strategy is similar to [13, Propositions 2.3 and 2.4]. Note that in this subsection, all ideals in A will simply be written as $\langle \Theta \rangle$, where $\Theta \in A$. However, if we want to refer to an ideal in any other algebra, say R, then that ideal will be written as $\langle \Theta \rangle_R$, where in this case, $\Theta \in R$.

Proposition 3.8. *Assume $\mathbb{K}$ is algebraically closed. Let $\mathcal{P}$ be the set of those monic irreducible polynomials $P(\Omega_1, \Omega_2) \in \mathbb{K}[\Omega_1, \Omega_2]$ with $P(\Omega_1, \Omega_2) \neq \Omega_1$ and $P(\Omega_1, \Omega_2) \neq \Omega_2$. Then,* $\mathrm{Spec}_{\langle 0 \rangle}(A) = \{\langle 0 \rangle\} \cup \{\langle P(\Omega_1, \Omega_2) \rangle \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \mid \alpha, \beta \in \mathbb{K}^*\}$.

Proof. We claim that $\mathrm{Spec}_{\langle 0\rangle}(A) = \{Q \in \mathrm{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\}$. To establish this claim, let us assume that this is not the case. Suppose that there exists $Q \in \mathrm{Spec}_{\langle 0\rangle}(A)$ such that Ω_1 or Ω_2 belongs to Q; then the product $\Omega_1\Omega_2$ which is an $\mathcal{H}$-eigenvector belongs to Q. Consequently, $\Omega_1\Omega_2 \in \bigcap_{h\in\mathcal{H}} h \cdot Q = \langle 0\rangle$, a contradiction. Hence, $\mathrm{Spec}_{\langle 0\rangle}(A) \subseteq \{Q \in \mathrm{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\}$. Conversely, suppose that $Q \in \mathrm{Spec}(A)$ such that $\Omega_1, \Omega_2 \notin Q$, then $\bigcap_{h\in\mathcal{H}} h \cdot Q$ is an $\mathcal{H}$-invariant prime ideal of A, which contains neither Ω_1 nor Ω_2. Obviously, the only possibility for $\bigcap_{h\in\mathcal{H}} h \cdot Q$ is $\langle 0\rangle$ since every non-zero $\mathcal{H}$-invariant prime ideal contains at least Ω_1 or Ω_2. Thus, $\bigcap_{h\in\mathcal{H}} h \cdot Q = \langle 0\rangle$. Hence, $Q \in \mathrm{Spec}_{\langle 0\rangle}(A)$. Therefore, $\{Q \in \mathrm{Spec}(A) \mid \Omega_1, \Omega_2 \notin Q\} \subseteq \mathrm{Spec}_{\langle 0\rangle}(A)$. This confirms our claim.

Since $\Omega_1, \Omega_2 \in Z(A)$, we have that the set $\{\Omega_1^i\Omega_2^j \mid i, j \in \mathbb{N}\}$ is a right denominator set in the noetherian domain A. One can now localize A as $R := A[\Omega_1^{-1}, \Omega_2^{-1}]$. Let $Q \in \mathrm{Spec}_{\langle 0\rangle}(A)$, the map $\phi : Q \mapsto Q[\Omega_1^{-1}, \Omega_2^{-1}]$ is an increasing bijection from $\mathrm{Spec}_{\langle 0\rangle}(A)$ onto $\mathrm{Spec}(R)$.

Since Ω_1 and Ω_2 are $\mathcal{H}$-eigenvectors, and $\mathcal{H}$ acts on A, we have that $\mathcal{H}$ also acts on R. Since every non-zero $\mathcal{H}$-prime ideal of A contains Ω_1 or Ω_2, one can easily check that R is $\mathcal{H}$-simple (in the sense that the only $\mathcal{H}$-invariant proper ideal of R is the 0 ideal).

We proceed to describe $\mathrm{Spec}(R)$ and $\mathrm{Spec}_{\langle 0\rangle}(A)$. We deduce from [2, Exercise II.3.A] that the action of $\mathcal{H}$ on R is rational. This rational action coupled with R being $\mathcal{H}$-simple implies that the extension and contraction maps provide mutually inverse bijections between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(Z(R))$ [2, Corollary II.3.9]. From Lemma 3.1, $Z(A) = \mathbb{K}[\Omega_1, \Omega_2]$, and so $Z(R) = \mathbb{K}[\Omega_1^{\pm 1}, \Omega_2^{\pm 1}]$. Since $\mathbb{K}$ is algebraically closed, we have that $\mathrm{Spec}(Z(R)) = \{\langle 0\rangle_{Z(R)}\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_{Z(R)} \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_{Z(R)} \mid \alpha, \beta \in \mathbb{K}^*\}$. Since there is an inverse bijection between $\mathrm{Spec}(R)$ and $\mathrm{Spec}(Z(R))$, and also R is $\mathcal{H}$-simple, one can recover $\mathrm{Spec}(R)$ from $\mathrm{Spec}(Z(R))$ as follows: $\mathrm{Spec}(R) = \{\langle 0\rangle_R\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_R \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \mid \alpha, \beta \in \mathbb{K}^*\}$. It follows that $\mathrm{Spec}_{\langle 0\rangle}(A) = \{\langle 0\rangle_R \cap A\} \cup \{\langle P(\Omega_1, \Omega_2)\rangle_R \cap A \mid P(\Omega_1, \Omega_2) \in \mathcal{P}\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap A \mid \alpha, \beta \in \mathbb{K}^*\}$.

Undoubtedly, $\langle 0\rangle_R \cap A = \langle 0\rangle$. We now have to show that $\langle P(\Omega_1, \Omega_2)\rangle_R \cap A = \langle P(\Omega_1, \Omega_2)\rangle$, $\forall P(\Omega_1, \Omega_2) \in \mathcal{P}$, and $\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle_R \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle$, $\forall \alpha, \beta \in \mathbb{K}^*$ to complete the proof.

Fix $P(\Omega_1, \Omega_2) \in \mathcal{P}$. Observe that $\langle P(\Omega_1, \Omega_2)\rangle \subseteq \langle P(\Omega_1, \Omega_2)\rangle_R \cap A$. To show the reverse inclusion, let $y \in \langle P(\Omega_1, \Omega_2)\rangle_R \cap A$. This implies that $y = dP(\Omega_1, \Omega_2)$, where $d \in R$, since $y \in \langle P(\Omega_1, \Omega_2)\rangle_R$. Also, $d \in R$ implies that there exist $i, j \in \mathbb{N}$ such that $d = a\Omega_1^{-i}\Omega_2^{-j}$, where $a \in A$. Therefore, $y = a\Omega_1^{-i}\Omega_2^{-j}P(\Omega_1, \Omega_2)$, which implies that $y\Omega_1^i\Omega_2^j = aP(\Omega_1, \Omega_2)$. Choose $(i, j) \in \mathbb{N}^2$ minimal (in the lexicographic order on $\mathbb{N}^2$) such that the equality holds. Without loss of generality, suppose that $i > 0$, then $aP(\Omega_1, \Omega_2) \in \langle \Omega_1\rangle$. Given that $\langle \Omega_1\rangle$ is a completely prime ideal, this implies that $a \in \langle \Omega_1\rangle$ or $P(\Omega_1, \Omega_2) \in \langle \Omega_1\rangle$. Since $P(\Omega_1, \Omega_2) \in \mathcal{P}$, it follows that $P(\Omega_1, \Omega_2) \notin \langle \Omega_1\rangle$, hence $a \in \langle \Omega_1\rangle$. This further implies that $a = t\Omega_1$, where $t \in A$. Returning to $y\Omega_1^i\Omega_2^j = aP(\Omega_1, \Omega_2)$, we have that $y\Omega_1^i\Omega_2^j = t\Omega_1 P(\Omega_1, \Omega_2)$. Therefore, $y\Omega_1^{i-1}\Omega_2^j = tP(\Omega_1, \Omega_2)$. This clearly contradicts the minimality of (i, j), hence $(i, j) = (0, 0)$, and $y = aP(\Omega_1, \Omega_2) \in \langle P(\Omega_1, \Omega_2)\rangle$. Consequently, $\langle P(\Omega_1, \Omega_2)\rangle_R \cap A = \langle P(\Omega_1, \Omega_2)\rangle$ for all $P(\Omega_1, \Omega_2) \in \mathcal{P}$ as desired.

Similarly, one can also verify that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle_R \cap A = \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$; $\forall \alpha, \beta \in \mathbb{K}^*$. $\square$

Using similar techniques, we obtain the following description for the $\mathcal{H}$-strata of $\langle \Omega_1 \rangle$ and $\langle \Omega_2 \rangle$.

Proposition 3.9. *Assume $\mathbb{K}$ is algebraically closed.*

1. $\mathrm{Spec}_{\langle \Omega_1 \rangle}(A) = \{\langle \Omega_1 \rangle\} \cup \{\langle \Omega_1, \Omega_2 - \beta \rangle \mid \beta \in \mathbb{K}^*\}$.
2. $\mathrm{Spec}_{\langle \Omega_2 \rangle}(A) = \{\langle \Omega_2 \rangle\} \cup \{\langle \Omega_1 - \alpha, \Omega_2 \rangle \mid \alpha \in \mathbb{K}^*\}$.

Since maximal ideals in their strata are primitive for a QNA, we obtain the following result.

Corollary 3.10. *Assume $\mathbb{K}$ is algebraically closed and let $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. The ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ of A is primitive.*

Remark 3.11. The statement of the above corollary is still valid without the assumption that $\mathbb{K}$ is algebraically closed. The proof is actually similar as we only use this assumption to get a full description of the strata we were interested in.

We can actually prove a stronger result.

Theorem 3.12. *Let $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. The prime ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ of A is maximal.*

Proof. Let $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Suppose that there exists a maximal ideal I of A such that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subsetneq I \subsetneq A$. Let J be the $\mathcal{H}$-invariant prime ideal in A such that $I \in \mathrm{Spec}_J(A)$.

We claim that J cannot be $\langle 0 \rangle$, $\langle \Omega_1 \rangle$ or $\langle \Omega_2 \rangle$. For instance, if $\alpha, \beta \neq 0$, then J cannot be equal to $\langle 0 \rangle$ since in this case $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is maximal in the 0-stratum. Moreover, $J \neq \langle \Omega_1 \rangle$ as otherwise I would contain $\alpha = \Omega_1 - (\Omega_1 - \alpha)$, a contradiction. The other cases are similar and left to the reader.

This means that J is an $\mathcal{H}$-prime of height at least equal to 2. As the poset of $\mathcal{H}$-primes is isomorphic to W, this forces J to contain both Ω_1 and Ω_2. Moreover, since $J \subseteq I$, it follows that $\Omega_1, \Omega_2 \in I$. Given that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle \subset I$, we have that $\Omega_1 - \alpha, \Omega_2 - \beta \in I$. It follows that $\alpha, \beta \in I$, hence $I = A$, a contradiction! This confirms that $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is a maximal ideal in A. $\square$

4. Simple quotients of $U_q^+(G_2)$ and their relation to the second Weyl algebra

We refer the reader to [12] for background on the Gelfand-Kirillov dimension. We denote by $\mathrm{GKdim}(R)$ the Gelfand-Kirillov dimension of an algebra R.

Now that we have found maximal ideals of $A = U_q^+(G_2)$, we are going to study their corresponding simple quotient algebras. In view of Dixmier's theorem, we consider these simple quotients as deformations of a Weyl algebra (of appropriate Gelfand-Kirillov dimension), and so we compare their properties with some known properties of the Weyl algebras. In this section, we prove that the Gelfand-Kirillov dimension of $A_{\alpha,\beta}$ is 4 and consequently prove that the height of the maximal ideal $\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ is 2 as expected. Then we focus on describing a linear basis of $A_{\alpha,\beta}$; we use this basis in the following section to study the derivations of $A_{\alpha,\beta}$. Finally, we show that with appropriate choices of α and β, the algebra $A_{\alpha,\beta}$ is a quadratic extension of the second Weyl algebra $A_2(\mathbb{K})$ at $q = 1$.

Recall from Theorem 3.12 that $\Omega_1 - \alpha$ and $\Omega_2 - \beta$, where $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$, generate a maximal ideal of A. As a result, the corresponding quotient

$$A_{\alpha,\beta} := \frac{A}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle}$$

is a simple noetherian domain. Denote the canonical images of E_i in $A_{\alpha,\beta}$ by $e_i := E_i + \langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$ for all $1 \le i \le 6$. The algebra $A_{\alpha,\beta}$ satisfies the following relations:

$$\begin{aligned}
e_2e_1 &= q^{-3}e_1e_2 & e_3e_1 &= q^{-1}e_1e_3 - (q + q^{-1} + q^{-3})e_2 \\
e_3e_2 &= q^{-3}e_2e_3 & e_4e_1 &= e_1e_4 + (1 - q^2)e_3^2 \\
e_4e_2 &= q^{-3}e_2e_4 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}e_3^3 & e_4e_3 &= q^{-3}e_3e_4 \\
e_5e_1 &= qe_1e_5 - (1 + q^2)e_3 & e_5e_2 &= e_2e_5 + (1 - q^2)e_3^2 \\
e_5e_3 &= q^{-1}e_3e_5 - (q + q^{-1} + q^{-3})e_4 & e_5e_4 &= q^{-3}e_4e_5 \\
e_6e_1 &= q^3e_1e_6 - q^3e_5 & e_6e_2 &= q^3e_2e_6 + (q^4 + q^2 - 1)e_4 + (q^2 - q^4)e_3e_5 \\
e_6e_3 &= e_3e_6 + (1 - q^2)e_5^2 & e_6e_4 &= q^{-3}e_4e_6 - \frac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}e_5^3 \\
e_6e_5 &= q^{-3}e_5e_6,
\end{aligned}$$

and

$$e_1e_3e_5 + ae_1e_4 + ae_2e_5 + a'e_3^2 = \alpha, \tag{10}$$

$$e_2e_4e_6 + be_2e_5^3 + be_3^3e_6 + b'e_3^2e_5^2 + c'e_3e_4e_5 + d'e_4^2 = \beta. \tag{11}$$

Note that the constants a, b, a' and b' are defined in Appendix A.2.

4.1. Gelfand-Kirillov dimension (GKdim) of $A_{\alpha,\beta}$

Assume first that $\alpha, \beta \neq 0$. Recall from Section 3.1 that $R_1 = \mathbb{K}_{q^M}[T_1^{\pm 1}, \ldots, T_6^{\pm 1}]$ is the quantum torus associated to the quantum affine space $\overline{A} = A^{(2)}$. Also, $\Omega_1 = T_1T_3T_5$

and $\Omega_2 = T_2T_4T_6$ in $\overline{A}$. It follows from [4, Theorem 5.4.1] that there exists an Ore set $S_{\alpha,\beta}$ in $A_{\alpha,\beta}$ such that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong R_1/\langle T_1T_3T_5-\alpha, T_2T_4T_6-\beta\rangle$.

Now, set

$$\mathscr{A}_{\alpha,\beta} := \frac{R_1}{\langle T_1T_3T_5-\alpha, T_2T_4T_6-\beta\rangle}.$$

Let $t_i := T_i + \langle T_1T_3T_5-\alpha, T_2T_4T_6-\beta\rangle$ denote the canonical images of the generators T_i of R_1 in $\mathscr{A}_{\alpha,\beta}$. The algebra $\mathscr{A}_{\alpha,\beta}$ is generated by $t_1^{\pm1},\dots,t_6^{\pm1}$ subject to the following relations:

$$t_it_j = q^{\mu_{ij}}t_jt_i \qquad t_1 = \alpha t_5^{-1}t_3^{-1} \qquad t_2 = \beta t_6^{-1}t_4^{-1},$$

for all $1 \le i,j \le 6$; and μ_{ij} are the entries of the skew-symmetric matrix M (see (4)). Observe that $\mathscr{A}_{\alpha,\beta} \cong \mathbb{K}_{q^N}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$, where the skew-symmetric matrix N can easily be deduced from M (by deleting the first two rows and columns) as follows:

$$N := \begin{bmatrix} 0 & 3 & 1 & 0 \\ -3 & 0 & 3 & 3 \\ -1 & -3 & 0 & 3 \\ 0 & -3 & -3 & 0 \end{bmatrix}.$$

Secondly, suppose that $\alpha = 0$ and $\beta \neq 0$.
Then, $A_{0,\beta}S_{0,\beta}^{-1} \cong \mathscr{A}_{0,\beta} = \mathbb{K}_{q^{M'}}[T_2^{\pm1},\dots,T_6^{\pm1}]/\langle T_2T_4T_6-\beta\rangle$, where M' is the skew-symmetric matrix obtained by deleting the first row and column of M. The algebra $\mathscr{A}_{0,\beta}$ is generated by $t_2^{\pm1},\dots,t_6^{\pm1}$ subject to the relations

$$t_it_j = q^{\mu_{ij}}t_jt_i \quad \text{and} \quad t_2 = \beta t_6^{-1}t_4^{-1},$$

for all $1 \le i,j \le 6$ and $\mu_{ij} \in M$. We also have that $\mathscr{A}_{0,\beta} \cong \mathbb{K}_{q^N}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$.

Finally, when $\alpha \neq 0$ and $\beta = 0$, one can also verify that $\mathscr{A}_{\alpha,0} \cong \mathbb{K}_{q^N}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$.

From the above discussion, in all cases, we have that $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} \cong \mathscr{A}_{\alpha,\beta} \cong \mathbb{K}_{q^N}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$. With a slight abuse of notation, we write $A_{\alpha,\beta}S_{\alpha,\beta}^{-1} = \mathscr{A}_{\alpha,\beta} = \mathbb{K}_{q^N}[t_3^{\pm1}, t_4^{\pm1}, t_5^{\pm1}, t_6^{\pm1}]$ for all $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. It follows from [8, Theorem 6.3] that $\text{GKdim}(A_{\alpha,\beta}) = \text{GKdim}(A_{\alpha,\beta}S_{\alpha,\beta}^{-1}) = \text{GKdim}(\mathscr{A}_{\alpha,\beta}) = 4$. Since Tauvel's height formula holds in $A = U_q^+(G_2)$ [8], we have that $\text{GKdim}(A) = \text{ht}(\langle \Omega_1-\alpha, \Omega_2-\beta\rangle) + \text{GKdim}(A_{\alpha,\beta})$. Since $\text{GKdim}(A) = 6$, we conclude that $\text{ht}(\langle \Omega_1-\alpha, \Omega_2-\beta\rangle) = 2$ for all $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$.

Proposition 4.1. $\text{GKdim}(A_{\alpha,\beta}) = 4$ *for all* $(\alpha,\beta) \neq (0,0)$.

4.2. Linear basis for $A_{\alpha,\beta}$

Set $A_\beta := A/\langle \Omega_2-\beta\rangle$, where $\beta \in \mathbb{K}$. Now, denote the canonical images of E_i by $\widehat{e}_i := E_i + \langle \Omega_2-\beta\rangle$ in A_β. Clearly, $A_{\alpha,\beta} \cong A_\beta/\langle \widehat{\Omega}_1-\alpha\rangle$. As a result, one can identify

$A_{\alpha,\beta}$ with $A_\beta/\langle\widehat{\Omega}_1-\alpha\rangle$. Moreover, the algebra A_β satisfies the relations of $A=U_q^+(G_2)$ and

$$\widehat{e_2}\widehat{e_4}\widehat{e_6}+b\widehat{e_2}\widehat{e_5}^3+b\widehat{e_3}^3\widehat{e_6}+b'\widehat{e_3}^2\widehat{e_5}^2+c'\widehat{e_3}\widehat{e_4}\widehat{e_5}+d'\widehat{e_4}^2=\beta. \tag{12}$$

From Propositions 3.8 and 3.9, one can conclude that $\langle\Omega_2-\beta\rangle$ is a completely prime ideal (since it is a prime ideal) of A for all $\beta\in\mathbb{K}$. Hence, the algebra A_β is a noetherian domain.

We are now going to find a linear basis for $A_{\alpha,\beta}$, where $(\alpha,\beta)\in\mathbb{K}^2\setminus\{(0,0)\}$. Since $A_{\alpha,\beta}$ is identified with $A_\beta/\langle\widehat{\Omega}_1-\alpha\rangle$, we will first and foremost find a basis for A_β, and then proceed to find a basis for $A_{\alpha,\beta}$. Note that the relations in Lemma A.1 are also valid in A_β and $A_{\alpha,\beta}$, and are going to be very useful in this section.

Proposition 4.2. *The set* $\mathfrak{S}=\{\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}^\xi\widehat{e_5}^l\widehat{e_6}^m \mid i,j,k,l,m\in\mathbb{N}$ *and* $\xi=0,1\}$ *is a* $\mathbb{K}$*-basis of* A_β*.*

Proof. Since the family $(\Pi_{s=1}^6 E_s^{i_s})_{i_s\in\mathbb{N}}$ is a PBW-basis of A over $\mathbb{K}$, it follows that the family $(\Pi_{s=1}^6\widehat{e_s}^{i_s})_{i_s\in\mathbb{N}}$ is a spanning set of A_β over $\mathbb{K}$. We want to show that $\mathfrak{S}$ spans A_β. We do this by showing that $\Pi_{s=1}^6\widehat{e_s}^{i_s}$ can be written as a finite linear combination of the elements of $\mathfrak{S}$ for all $i_1,\dots,i_6\in\mathbb{N}$ by an induction on i_4. The result is obvious when $i_4=0$ or 1. For $i_4\geq 1$, assume that

$$\prod_{s=1}^6\widehat{e_s}^{i_s}=\sum_{(\xi,\underline{v})\in I}a_{(\xi,\underline{v})}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}^\xi\widehat{e_5}^l\widehat{e_6}^m,$$

where $\underline{v}:=(i,j,k,l,m)\in\mathbb{N}^5$ and the $a_{(\xi,\underline{v})}$ are all scalars. Note that I is a finite subset of $\{0,1\}\times\mathbb{N}^5$. It follows from the commutation relations of A_β (see Lemma A.1) that

$$\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4+1}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6}=q^\bullet\prod_{s=1}^6\widehat{e_s}^{i_s}\widehat{e_4}-q^\bullet d_1[i_6]\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4}\widehat{e_5}^{i_5+3}\widehat{e_6}^{i_6-1}.$$

From the inductive hypothesis, $\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4}\widehat{e_5}^{i_5+3}\widehat{e_6}^{i_6-1}\in\mathrm{Span}(\mathfrak{S})$. Hence, we proceed to show that $\Pi_{s=1}^6\widehat{e_s}^{i_s}\widehat{e_4}$ is also in the span of $\mathfrak{S}$. From the inductive hypothesis, we have

$$\prod_{s=1}^6\widehat{e_s}^{i_s}\widehat{e_4}=\sum_{(\xi,\underline{v})\in I}a_{(\xi,\underline{v})}\widehat{e_1}^i\widehat{e_2}^j\widehat{e_3}^k\widehat{e_4}^\xi\widehat{e_5}^l\widehat{e_6}^m\widehat{e_4}.$$

Using the commutation relations in Lemma A.1, we have that

$$\prod_{s=1}^{6} \widehat{e_s}^{i_s}\widehat{e_4} = \sum_{(\xi,\underline{v})\in I} q^{\bullet} a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi+1}\widehat{e_5}^{l}\widehat{e_6}^{m}$$
$$+ \sum_{(\xi,\underline{v})\in I} q^{\bullet} d_1[m] a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi}\widehat{e_5}^{l+3}\widehat{e_6}^{m-1}.$$

All the terms in the above expression belong to the span of $\mathfrak{S}$ except $\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m}$. From (12), we have that

$$\widehat{e_4}^{2} = \beta_0\widehat{e_2}\widehat{e_4}\widehat{e_6} + b\beta_0\widehat{e_2}\widehat{e_5}^{3} + b\beta_0\widehat{e_3}^{3}\widehat{e_6} + b'\beta_0\widehat{e_3}^{2}\widehat{e_5}^{2} + c'\beta_0\widehat{e_3}\widehat{e_4}\widehat{e_5} - \beta\beta_0, \tag{13}$$

where $\beta_0 = -1/d'$. Substituting (13) into $\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m}$, one can easily verify that

$$\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{2}\widehat{e_5}^{l}\widehat{e_6}^{m} \in \mathrm{Span}(\mathfrak{S}).$$

Therefore, $\widehat{e_1}^{i_1}\widehat{e_2}^{i_2}\widehat{e_3}^{i_3}\widehat{e_4}^{i_4+1}\widehat{e_5}^{i_5}\widehat{e_6}^{i_6}$ can be written as a finite linear combination of the elements of $\mathfrak{S}$ over $\mathbb{K}$ for all $i_1, \ldots, i_6 \in \mathbb{N}$. By the principle of mathematical induction, $\mathfrak{S}$ is a spanning set of A_β over $\mathbb{K}$.

Next we show that $\mathfrak{S}$ is a linearly independent set. Suppose that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}\widehat{e_1}^{i}\widehat{e_2}^{j}\widehat{e_3}^{k}\widehat{e_4}^{\xi}\widehat{e_5}^{l}\widehat{e_6}^{m} = 0.$$

Since $A_\beta = A/\langle \Omega_2 - \beta\rangle$, it follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} E_1^i E_2^j E_3^k E_4^\xi E_5^l E_6^m = (\Omega_2 - \beta)\nu,$$

with $\nu \in A$. Write $\nu = \sum_{(i,\ldots,n)\in J} b_{(i,\ldots,n)} E_1^i E_2^j E_3^k E_4^l E_5^m E_6^n$, where J is a finite subset of $\mathbb{N}^6$ and $b_{(i,\ldots,n)}$ are all scalars. It follows that

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} E_1^i E_2^j E_3^k E_4^\xi E_5^l E_6^m = \sum_{(i,\ldots,n)\in J} b_{(i,\ldots,n)} E_1^i E_2^j E_3^k (\Omega_2 - \beta) E_4^l E_5^m E_6^n. \tag{14}$$

Before we continue the proof, the following needs to be noted.

♣ Let $(i', j', k', l', m', n'), (i, j, k, l, m, n) \in \mathbb{N}^6$. We say that $(i, j, k, l, m, n) <_4 (i', j', k', l', m', n')$ if $[l < l']$ or $[l = l'$ and $i < i']$ or $[l = l',\ i = i'$ and $j < j']$ or $[l = l',\ i = i',\ j = j'$ and $k < k']$ or $[l = l',\ i = i',\ j = j',\ k = k'$ and $m < m']$ or $[l = l',\ i = i',\ j = j',\ k = k',\ m = m'$ and $n \le n']$. Note that the purpose of the square bracket [ ] is to differentiate the options.

From Section 3.1, we have that $\Omega_2 = E_2E_4E_6 + bE_2E_5^3 + bE_3^3E_6 + b'E_3^2E_5^2 + c'E_3E_4E_5 + d'E_4^2$ in $A = U_q^+(G_2)$. Now,

$$\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}E_1^iE_2^jE_3^kE_4^{\xi}E_5^lE_6^m = \sum_{(i,\dots,n)\in J} b_{(i,\dots,n)}E_1^iE_2^jE_3^k(\Omega_2-\beta)E_4^lE_5^mE_6^n$$
$$= \sum_{(i,\dots,n)\in J} d'b_{(i,\dots,m)}E_1^iE_2^jE_3^kE_4^{l+2}E_5^mE_6^n + \mathrm{LT}_{<_4},$$

where $\mathrm{LT}_{<_4}$ contains lower order terms with respect to $<_4$ (as in ♣). Moreover, $\mathrm{LT}_{<_4}$ vanishes when $b_{(i,\dots,n)} = 0$ for all $(i,\dots,n) \in J$ (one can easily confirm this by fully expanding the right hand side of (14)).

Now, suppose that there exists $(i,j,k,l,m,n) \in J$ such that $b_{(i,j,k,l,m,n)} \neq 0$. Let (i',j',k',l',m',n') be the greatest element of J with respect to $<_4$ (defined in ♣ above) such that $b_{(i',j',k',l',m',n')} \neq 0$. Note that the family $(E_1^iE_2^jE_3^kE_4^lE_5^mE_6^n)_{(i,\dots,n)\in\mathbb{N}^6}$ is a basis of A. Since $\mathrm{LT}_{<_4}$ contains lower order terms, identifying the coefficients of $E_1^{i'}E_2^{j'}E_3^{k'}E_4^{l'+2}E_5^{m'}E_6^{n'}$ in the above equality, we have that $d'b_{(i',\dots,n')} = 0$. Since $b_{(i',j',k',l',m',n')} \neq 0$, it follows that $d' = q^{12}/(q^6-1) = 0$, a contradiction (see Appendix A.2 for the expression of d'). As a result, $b_{(i,j,k,l,m,n)} = 0$ for all $(i,j,k,l,m,n) \in J$. Therefore, $\sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})}E_1^iE_2^jE_3^kE_4^{\xi}E_5^lE_6^m = 0$. Since $(E_1^iE_2^jE_3^kE_4^lE_5^mE_6^n)_{(i,\dots,n)\in\mathbb{N}^6}$ is a basis of A, it follows that $a_{(\xi,\underline{v})} = 0$ for all $(\xi,\underline{v}) \in I$. In conclusion, $\mathfrak{S}$ is a linearly independent set and hence forms a basis of A_β as desired. □

Proposition 4.3. *Let $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. The set $\mathcal{B} = \{e_1^ie_2^je_3^{\epsilon_1}e_4^{\epsilon_2}e_5^ke_6^l \mid i,j,k,l \in \mathbb{N}$ and $\epsilon_1,\epsilon_2 \in \{0,1\}\}$ is a $\mathbb{K}$-basis of $A_{\alpha,\beta}$.*

Proof. The proof is analogous to the proof of Proposition 4.2. We refer the interested reader to [18, Section 3.2] for the details of this proof. □

Remark 4.4. Given the basis of $A_{\alpha,\beta}$, we have computed the group of units of $A_{\alpha,\beta}$. However, we do not include the details in this manuscript due to the voluminous computations involved. We only summarize our findings below. Set

$$h_1 := e_3e_5 + ae_4 \quad \text{and} \quad h_2 := (q^{-3}-q^{-9})e_2e_4 - (q^4-2q^2+1)/(q^4+q^2+1)e_3^3.$$

Theorem 4.5. *Let $(\alpha,\beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$ and $\mathcal{U}(A_{\alpha,\beta})$ denote the group of units of $A_{\alpha,\beta}$. We have that:*

$$\mathcal{U}(A_{\alpha,\beta}) = \begin{cases} \{\lambda h_1^i \mid \lambda \in \mathbb{K}^*,\ i \in \mathbb{Z}\} & \text{if } \alpha = 0 \\ \{\lambda h_2^i \mid \lambda \in \mathbb{K}^*,\ i \in \mathbb{Z}\} & \text{if } \beta = 0 \\ \mathbb{K}^* & \text{otherwise.} \end{cases}$$

4.3. $A_{\alpha,\beta}$ as a q-deformation of a quadratic extension of $A_2(\mathbb{K})$

In this subsection, we assume that the characteristic of $\mathbb{K}$, denoted $\mathrm{char}(\mathbb{K})$, not equal to 3.

Recall that $\mathrm{GKdim}(A_{\alpha,\beta}) = 4$ and so we should compare $A_{\alpha,\beta}$ to the second Weyl algebra. We prove that, for a suitable choice of α and β, the simple algebra $A_{\alpha,\beta}$ is a q-deformation of a quadratic extension of $A_2(\mathbb{K})$.

Recall that $A_2(\mathbb{K})$ is generated by x_1, x_2, y_1 and y_2 subject to the relations:

$$\begin{array}{llll} y_1y_2 = y_2y_1 & x_2y_1 = y_1x_2 & x_1x_2 = x_2x_1 & x_1y_1 - y_1x_1 = 1 \\ y_1y_2 = y_2y_1 & x_1y_2 = y_2x_1 & x_2y_1 = y_1x_2 & x_2y_2 - y_2x_2 = 1. \end{array}$$

Given the relations of $A_{\alpha,\beta}$ at the onset of this section, we have that $A_{1,\frac{1}{9(q^6-1)}}$ satisfies the following relations:

$$\begin{array}{ll} e_2e_1 = q^{-3}e_1e_2 & e_3e_1 = q^{-1}e_1e_3 - (q + q^{-1} + q^{-3})e_2 \\ e_3e_2 = q^{-3}e_2e_3 & e_4e_1 = e_1e_4 + (1 - q^2)e_3^2 \\ e_4e_2 = q^{-3}e_2e_4 - \dfrac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}e_3^3 & e_4e_3 = q^{-3}e_3e_4 \\ e_5e_1 = qe_1e_5 - (1 + q^2)e_3 & e_5e_2 = e_2e_5 + (1 - q^2)e_3^2 \\ e_5e_3 = q^{-1}e_3e_5 - (q + q^{-1} + q^{-3})e_4 & e_5e_4 = q^{-3}e_4e_5 \\ e_6e_1 = q^3e_1e_6 - q^3e_5 & e_6e_2 = q^3e_2e_6 + (q^4 + q^2 - 1)e_4 + (q^2 - q^4)e_3e_5 \\ e_6e_3 = e_3e_6 + (1 - q^2)e_5^2 & e_6e_4 = q^{-3}e_4e_6 - \dfrac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}e_5^3 \\ e_6e_5 = q^{-3}e_5e_6, & \end{array}$$

and

$$(q^{-2} - 1)e_1e_3e_5 + (q^2 + 1 + q^{-2})e_1e_4 + (q^2 + 1 + q^{-2})e_2e_5 - q^4e_3^2 = q^{-2} - 1,$$
$$(q^6 - 1)e_2e_4e_6 + \frac{2q^{-1} - q^{-3} - q}{q^4 + q^2 + 1}e_2e_5^3 + \frac{2q^{-1} - q^{-3} - q}{q^4 + q^2 + 1}e_3^3e_6$$
$$+\frac{(q^6 - 1)(q^{13} - q^{11})}{(q^4 + q^2 + 1)^2}e_3^2e_5^2 - \frac{q^9(q^6 - 1)}{q^4 + q^2 + 1}e_3e_4e_5 + q^{12}e_4^2 = \frac{1}{9}.$$

Note that we have made the necessary substitutions for a, a', b, b', c' and d' from Appendix A.2.

Set $F := \mathbb{K}[z^{\pm 1}]$. One can define a $F[(z^4+z^2+1)^{-1}]$-algebra A_z generated by $e_1, \ldots, e_6$ subject to the following relations:

$$\begin{aligned}
&e_2e_1 = z^{-3}e_1e_2 && e_3e_1 = z^{-1}e_1e_3 - (z + z^{-1} + z^{-3})e_2 \\
&e_3e_2 = z^{-3}e_2e_3 && e_4e_1 = e_1e_4 + (1 - z^2)e_3^2 \\
&e_4e_2 = z^{-3}e_2e_4 - \frac{z^4 - 2z^2 + 1}{z^4 + z^2 + 1}e_3^3 && e_4e_3 = z^{-3}e_3e_4 \\
&e_5e_1 = ze_1e_5 - (1 + z^2)e_3 && e_5e_2 = e_2e_5 + (1 - z^2)e_3^2 \\
&e_5e_3 = z^{-1}e_3e_5 - (z + z^{-1} + z^{-3})e_4 && e_5e_4 = z^{-3}e_4e_5 \\
&e_6e_1 = z^3e_1e_6 - z^3e_5 && e_6e_2 = z^3e_2e_6 + (z^4 + z^2 - 1)e_4 + (z^2 - z^4)e_3e_5 \\
&e_6e_3 = e_3e_6 + (1 - z^2)e_5^2 && e_6e_4 = z^{-3}e_4e_6 - \frac{z^4 - 2z^2 + 1}{z^4 + z^2 + 1}e_5^3 \\
&e_6e_5 = z^{-3}e_5e_6,
\end{aligned}$$

$$(z^{-2} - 1)e_1e_3e_5 + (z^2 + 1 + z^{-2})e_1e_4 + (z^2 + 1 + z^{-2})e_2e_5 - z^4e_3^2 = z^{-2} - 1, \quad \text{and}$$

$$\begin{aligned}
(z^6 - 1)e_2e_4e_6 &+ \frac{2z^{-1} - z^{-3} - z}{z^4 + z^2 + 1}e_2e_5^3 + \frac{2z^{-1} - z^{-3} - z}{z^4 + z^2 + 1}e_3^3e_6 \\
&+ \frac{(z^6 - 1)(z^{13} - z^{11})}{(z^4 + z^2 + 1)^2}e_3^2e_5^2 - \frac{z^9(z^6 - 1)}{z^4 + z^2 + 1}e_3e_4e_5 + z^{12}e_4^2 = \frac{1}{9}.
\end{aligned}$$

Set $A_1 := A_z/\langle z - 1\rangle$ and observe that A_1 satisfies the following relations:

$$\begin{aligned}
&e_2e_1 = e_1e_2 && e_3e_1 = e_1e_3 - 3e_2 && e_3e_2 = e_2e_3 \\
&e_4e_1 = e_1e_4 && e_4e_2 = e_2e_4 && e_4e_3 = e_3e_4 \\
&e_5e_1 = e_1e_5 - 2e_3 && e_5e_2 = e_2e_5 && e_5e_3 = e_3e_5 - 3e_4 \\
&e_5e_4 = e_4e_5 && e_6e_1 = e_1e_6 - e_5 && e_6e_2 = e_2e_6 + e_4 \\
&e_6e_3 = e_3e_6 && e_6e_4 = e_4e_6 && e_6e_5 = e_5e_6 \\
&e_4^2 = 1/9 && && 0 = e_3^2 - 3e_1e_4 - 3e_2e_5.
\end{aligned}$$

Lemma 4.6. *$e_4 \in Z(A_1)$ and also it is invertible.*

Proof. Since $e_4e_i = e_ie_4$ for all $1 \leq i \leq 6$, we have that $e_4 \in Z(A_1)$. Again, from $e_4^2 = 1/9$, we have that $e_4(9e_4) = (9e_4)e_4 = 1$. Hence e_4 is invertible with $e_4^{-1} = 9e_4$. □

Given that $e_4^{-1} = 9e_4$ and $e_4 \in Z(A_1)$, it follows from the relation $e_3^2 - 3e_1e_4 - 3e_2e_5 = 0$ that $e_1 = 3e_3^2e_4 - 9e_2e_4e_5$. Therefore, A_1 can be generated by only $e_2, \dots, e_6$. All these generators commute except that

$$e_6e_2 = e_2e_6 + e_4 \quad \text{and} \quad e_5e_3 = e_3e_5 - 3e_4.$$

Since e_4 is invertible, one can also verify that $9e_2e_4, 3e_3e_4, e_4, e_5$ and e_6 generate A_1.

Let R be an algebra generated by f_2, f_3, f_4, f_5, f_6 subject to the following defining relations:

$$\begin{array}{lll} f_3f_2 = f_2f_3 & f_4f_2 = f_2f_4 & f_4f_3 = f_3f_4 \\ f_5f_2 = f_2f_5 & f_5f_4 = f_4f_5 & f_6f_3 = f_3f_6 \\ f_6f_4 = f_4f_6 & f_6f_5 = f_5f_6 & f_4^2 = 1/9 \\ f_6f_2 = f_2f_6 + 1 & f_5f_3 = f_3f_5 - 1. & \end{array}$$

Proposition 4.7. $R \cong A_1$.

Proof. One can easily check that we define a homomorphism $\phi : R \longrightarrow A_1$ via

$$\phi(f_2) = 9e_2e_4 \qquad \phi(f_3) = 3e_3e_4 \qquad \phi(f_4) = e_4 \qquad \phi(f_5) = e_5 \qquad \phi(f_6) = e_6.$$

Recall that $e_4^2 = 1/9$. To check that ϕ is indeed a homomorphism, we just need to check its compatibility with the defining relations of R. We check this on the relation $f_6f_2 - f_2f_6 = 1$, and leave the remaining ones for the reader to verify. We do that as follows: $\phi(f_6)\phi(f_2) - \phi(f_2)\phi(f_6) = 9e_6e_2e_4 - 9e_2e_4e_6 = 9e_4(e_6e_2 - e_2e_6) = 9e_4^2 = 9(1/9) = 1$ as required.

Conversely, one can check that a homomorphism $\varphi : A_1 \longrightarrow R$ can be defined via

$$\begin{array}{lll} \varphi(e_1) = 3f_3^2f_4 - f_2f_5 & \varphi(e_2) = f_2f_4 & \varphi(e_3) = 3f_3f_4 \\ \varphi(e_4) = f_4 & \varphi(e_5) = f_5 & \varphi(e_6) = f_6. \end{array}$$

We check this on the relation $e_3^2 - 3e_1e_4 - 3e_2e_5 = 0$, and leave the remaining ones for the reader to verify. We do that as follows: $\varphi(e_3)^2 - 3\varphi(e_1)\varphi(e_4) - 3\varphi(e_2)\varphi(e_5) = (3f_3f_4)^2 - 3(3f_3^2f_4 - f_2f_5)f_4 - 3f_2f_4f_5 = 9f_3^2f_4^2 - 9f_3^2f_4^2 + 3f_2f_4f_5 - 3f_2f_4f_5 = 0$ as expected.

To conclude we just observe that ϕ and φ are inverses of each other. □

The corollary below can easily be deduced from the above proposition.

Corollary 4.8. *Set $\mathbb{F} := \mathbb{K}[f_4]/\langle f_4^2 - 1/9\rangle$, we have that $R \cong A_2(\mathbb{F})$, where $A_2(\mathbb{F})$ is the second Weyl algebra over the ring $\mathbb{F}$.*

Remark 4.9. We have the following isomorphisms:

$$\mathbb{F} \cong \mathbb{K}[f_4]/\langle f_4 - 1/3\rangle \oplus \mathbb{K}[f_4]/\langle f_4 + 1/3\rangle \cong \mathbb{K} \oplus \mathbb{K}.$$

These induce an isomorphism $A_2(\mathbb{F}) \cong A_2(\mathbb{K}) \oplus A_2(\mathbb{K})$.

Remark 4.10. Observe that the subalgebra B of R generated by f_2, f_3, f_5, f_6 is isomorphic to $A_2(\mathbb{K})$ and $R \cong B[f_4]/\langle f_4^2 - 1/9\rangle \cong A_2(\mathbb{K})[f_4]/\langle f_4^2 - 1/9\rangle$. Thus R is a *quadratic extension* of $A_2(\mathbb{K})$. Note also that $A_{1,\frac{1}{9(q^6-1)}}$ is a q-deformation of $A_1 \cong R \cong A_2(\mathbb{F}) \cong A_2(\mathbb{K})[f_4]/\langle f_4^2 - 1/9\rangle \cong A_2(\mathbb{K}) \oplus A_2(\mathbb{K})$.

5. Derivations of the simple quotients of $U_q^+(G_2)$

In this section, we compute the derivations of the algebra $A_{\alpha,\beta}$ using the DDA that allows to embed $A_{\alpha,\beta}$ into a suitable quantum torus. Derivations of quantum tori are known, thanks to the work of Osborn and Passman [19]. In our cases, such derivations are always the sum of an inner derivation and a scalar derivation (of the quantum torus). Since $A_{\alpha,\beta}$ can be embedded into a quantum torus, we first extend every derivation of $A_{\alpha,\beta}$ to a derivation of such quantum torus, and then pull back their description as a derivation of the quantum torus to a description of their action on the generators of $A_{\alpha,\beta}$ by "reverting" the DDA process. We conclude that every derivation of $A_{\alpha,\beta}$ is inner when α and β are both non-zero. However, when either α or β is zero, we conclude that every derivation of $A_{\alpha,\beta}$ is the sum of an inner and a scalar derivation. In fact, the first Hochschild cohomology group of $A_{\alpha,\beta}$ is of dimension 0 when α and β are both non-zero and 1 when either α or β is zero.

5.1. Preliminaries and strategy

Let $2 \leq j \leq 7$ and $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. Set

$$A_{\alpha,\beta}^{(j)} := \frac{A^{(j)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta\rangle},$$

where $A^{(j)}$ is defined in Section 3.1 and, Ω_1 and Ω_2 are the generators of the centre of $A^{(j)}$, see Remark 3.2. Note in particular that $A_{\alpha,\beta}^{(7)} = A_{\alpha,\beta}$. For each $2 \leq j \leq 7$, denote the canonical images of the generators $E_{i,j}$ of $A^{(j)}$ in $A_{\alpha,\beta}^{(j)}$ by $e_{i,j}$ for all $1 \leq i \leq 6$.

As usual we denote by t_i the canonical image of T_i in $A_{\alpha,\beta}^{(2)}$ for each $1 \leq i \leq 6$. For each $3 \leq j \leq 6$, define $S_j := \left\{\lambda t_j^{i_j} t_{j+1}^{i_{j+1}} \dots t_6^{i_6} \mid i_j, \dots, i_6 \in \mathbb{N} \text{ and } \lambda \in \mathbb{K}^*\right\}$. One can observe that S_j is a multiplicative system of non-zero divisors (or regular elements) of $A_{\alpha,\beta}^{(j)}$. Furthermore; $t_j, \dots, t_6$ are all normal elements of $A_{\alpha,\beta}^{(j)}$ and so S_j is an Ore set in $A_{\alpha,\beta}^{(j)}$. One can localize $A_{\alpha,\beta}^{(j)}$ at S_j as follows:

$$\mathcal{R}_j := A_{\alpha,\beta}^{(j)} S_j^{-1}.$$

Let $3 \leq j \leq 6$, and set $\Sigma_j := \{t_j^k \mid k \in \mathbb{N}\}$. By [4, Lemme 5.3.2], Σ_j is an Ore set in both $A_{\alpha,\beta}^{(j)}$ and $A_{\alpha,\beta}^{(j+1)}$, and

$$A_{\alpha,\beta}^{(j)}\Sigma_j^{-1} = A_{\alpha,\beta}^{(j+1)}\Sigma_j^{-1}.$$

As a consequence, similar to (8), we have that

$$\mathcal{R}_j = \mathcal{R}_{j+1}\Sigma_j^{-1}, \tag{15}$$

for all $2 \leq j \leq 6$. By convention, $\mathcal{R}_7 := A_{\alpha,\beta}$. We also construct the following tower of algebras in a manner similar to (9):

$$\mathcal{R}_7 = A_{\alpha,\beta} \subset \mathcal{R}_6 = \mathcal{R}_7\Sigma_6^{-1} \subset \mathcal{R}_5 = \mathcal{R}_6\Sigma_5^{-1} \subset \mathcal{R}_4 = \mathcal{R}_5\Sigma_4^{-1} \subset \mathcal{R}_3. \tag{16}$$

Note that $\mathcal{R}_3 = A_{\alpha,\beta}^{(3)}S_3^{-1} = \mathcal{R}_4\Sigma_3^{-1}$ is the quantum torus $\mathscr{A}_{\alpha,\beta} = \mathbb{K}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$ studied in Section 4.1.

Our strategy to compute the derivations of $\mathcal{R}_7$ is to extend these derivations to derivations of the quantum torus $\mathcal{R}_3$. Then we can use the description of the derivations of a quantum torus obtained by Osborn and Passman in [19]. Once this is done, we will have a "nice" description but involving elements of $\mathcal{R}_3$ and we will then use the fact that these derivations fix (globally) all $\mathcal{R}_i$ to obtain a description only involving elements of $\mathcal{R}_7$. This is a step by step process requiring knowing linear bases for $\mathcal{R}_i$. We find such bases in the next subsection.

Before doing so, we note from [4, Lemme 5.3.2] that the DDA theory predicts the following relations between the elements $e_{i,j}$:

$$\begin{aligned}
e_{1,6} &= e_1 + re_5e_6^{-1}\\
e_{2,6} &= e_2 + te_3e_5e_6^{-1} + ue_4e_6^{-1} + ne_5^3e_6^{-2}\\
e_{3,6} &= e_3 + se_5^2e_6^{-1}\\
e_{4,6} &= e_4 + be_5^3e_6^{-1}\\
e_{1,5} &= e_{1,6} + he_{3,6}e_{5,6}^{-1} + ge_{4,6}e_{5,6}^{-2}\\
e_{2,5} &= e_{2,6} + fe_{3,6}^2e_{5,6}^{-1} + pe_{3,6}e_{4,6}e_{5,6}^{-2} + ee_{4,6}^2e_{5,6}^{-3}\\
e_{3,5} &= e_{3,6} + ae_{4,6}e_{5,6}^{-1}\\
e_{1,4} &= e_{1,5} + se_{3,5}^2e_{4,5}^{-1}\\
e_{2,4} &= e_{2,5} + be_{3,5}^3e_{4,5}^{-1}\\
e_{1,3} &= e_{1,4} + ae_{2,4}e_{3,4}^{-1}\\
t_1 &:= e_{1,2} = e_{1,3}\\
t_2 &:= e_{2,2} = e_{2,3} = e_{2,4}\\
t_3 &:= e_{3,2} = e_{3,3} = e_{3,4} = e_{3,5}\\
t_4 &:= e_{4,2} = e_{4,3} = e_{4,4} = e_{4,5} = e_{4,6}\\
t_5 &:= e_{5,2} = e_{5,3} = e_{5,4} = e_{5,5} = e_{5,6} = e_5
\end{aligned}$$

$$t_6 := e_{6,2} = e_{6,3} = e_{6,4} = e_{6,5} = e_{6,6} = e_6,$$

where, as usual, the necessary parameters can be found in Appendix A.2.

We also note that we have complete control over the centres of the algebras $\mathcal{R}_i$.

Lemma 5.1. *Let $Z(\mathcal{R}_i)$ denote the centre of $\mathcal{R}_i$, then $Z(\mathcal{R}_i) = \mathbb{K}$ for each $3 \leq i \leq 7$.*

Proof. One can easily verify that $Z(\mathcal{R}_3) = \mathbb{K}$. Note that $\mathcal{R}_7 = A_{\alpha,\beta}$. Since $\mathcal{R}_i$ is a localization of $\mathcal{R}_{i+1}$ (see (15)), we have that $\mathbb{K} \subseteq Z(\mathcal{R}_7) \subseteq Z(\mathcal{R}_6) \subseteq Z(\mathcal{R}_5) \subseteq Z(\mathcal{R}_4) \subseteq Z(\mathcal{R}_3) = \mathbb{K}$. Therefore, $Z(\mathcal{R}_7) = Z(\mathcal{R}_6) = Z(\mathcal{R}_5) = Z(\mathcal{R}_4) = Z(\mathcal{R}_3) = \mathbb{K}$. $\square$

5.2. Linear bases for $\mathcal{R}_3$, $\mathcal{R}_4$ and $\mathcal{R}_5$

Let $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$. We aim to find a basis of $\mathcal{R}_j$ for each $j = 3, 4, 5$. Since $\mathcal{R}_3 = \mathscr{A}_{\alpha,\beta}$, the set $\{t_3^i t_4^j t_5^k t_6^l \mid i, j, k, l \in \mathbb{Z}\}$ is a $\mathbb{K}$-basis of R_3.

For simplicity, we set

$$\begin{aligned} f_1 &:= e_{1,4} & F_1 &:= E_{1,4} \\ z_1 &:= e_{1,5} & Z_1 &:= E_{1,5} \\ z_2 &:= e_{2,5} & Z_2 &:= E_{2,5}. \end{aligned}$$

Basis for $\mathcal{R}_4$. Observe that

$$A_{\alpha,\beta}^{(4)} = \frac{A^{(4)}}{\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle},$$

where $\Omega_1 = F_1 T_3 T_5 + a T_2 T_5$ and $\Omega_2 = T_2 T_4 T_6$ in $A^{(4)}$. Recall from Section 4.2 that finding a basis for the algebra A_β served as a good ground for finding a basis for $A_{\alpha,\beta}$. In a similar manner, to find a basis for $\mathcal{R}_4$, we will first and foremost find a basis for the algebra

$$A_\beta^{(4)} S_4^{-1} = \frac{A^{(4)} S_4^{-1}}{\langle \Omega_2 - \beta \rangle} = \frac{A^{(4)} S_4^{-1}}{\langle T_2 T_4 T_6 - \beta \rangle},$$

where $\beta \in \mathbb{K}^*$. We will denote the canonical images of $E_{i,4}$ (resp. T_i) in $A_\beta^{(4)}$ by $\widehat{e_{i,4}}$ (resp. $\widehat{t_i}$) for all $1 \leq i \leq 6$. Observe that $\widehat{t_2} = \beta \widehat{t_6}^{-1} \widehat{t_4}^{-1}$ in $A_\beta^{(4)} S_4^{-1}$. Note that when $\beta = 0$, then one can easily deduce that $A_\beta^{(4)} S_4^{-1} = A^{(4)} S_4^{-1} / \langle T_2 \rangle$, hence, $\widehat{t_2} = 0$.

Proposition 5.2. *The set $\mathfrak{S}_4 = \left\{ \widehat{f_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} \mid (i_1, i_3, i_4, i_5, i_6) \in \mathbb{N}^2 \times \mathbb{Z}^3 \right\}$ is a $\mathbb{K}$-basis of $A_\beta^{(4)} S_4^{-1}$, where $\beta \in \mathbb{K}$.*

Proof. We begin by showing that $\mathfrak{S}_4$ is a spanning set for $A_\beta^{(4)}S_4^{-1}$. It is sufficient to do this by showing that $\widehat{f_1}^{k_1}\widehat{t_2}^{k_2}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}$ can be written as a finite linear combination of the elements of $\mathfrak{S}_4$ for all $(k_1,\dots,k_6)\in\mathbb{N}^3\times\mathbb{Z}^3$. This can easily be done through an induction on k_2 using the fact that $\widehat{t_2}=\beta\widehat{t_6}^{-1}\widehat{t_4}^{-1}$ (note that, if $\beta=0$, then $\widehat{t_2}=0$).

We now prove that $\mathfrak{S}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}=0.$$

This implies that

$$\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}=(\Omega_2-\beta)\nu,$$

for some $\nu\in A^{(4)}S_4^{-1}$. Write $\nu=\sum_{\underline{j}\in J} b_{\underline{j}}F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}$, where $\underline{j}=(i_1,i_2,i_3,i_4,i_5,i_6)$ $\in J\subset\mathbb{N}^3\times\mathbb{Z}^3$ and $b_{\underline{j}}$ is a family of scalars. Given that $\Omega_2=T_2T_4T_6$, it follows from the above equality that

$$\begin{aligned}\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}&=\sum_{\underline{j}\in J} q^\bullet b_{\underline{j}}F_1^{i_1}T_2^{i_2+1}T_3^{i_3}T_4^{i_4+1}T_5^{i_5}T_6^{i_6+1}\\&\quad-\sum_{\underline{j}\in J}\beta b_{\underline{j}}F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}.\end{aligned}$$

We denote by $<_2$ the total order on $\mathbb{Z}^6$ defined by $(i_1,i_2,i_3,i_4,i_5,i_6)<_2(w_1,w_2,w_3,w_4,w_5,w_6)$ if $[w_2>i_2]$ or $[w_2=i_2,\ w_1>i_1]$ or $[w_2=i_2,\ w_1=i_1,\ w_3>i_3]$ or ... or $[w_l=i_l,\ w_6\geq t_6,\ l=2,1,3,4,5]$.

Suppose that there exists $(i_1,\dots,i_6)\in J$ such that $b_{(i_1,\dots,i_6)}\neq 0$. Let $(w_1,\dots,w_6)\in J$ be the greatest element of J with respect to $<_2$ such that $b_{(w_1,\dots,w_6)}\neq 0$. Note that $\left(F_1^{i_1}T_2^{i_2}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}\right)_{(i_1,\dots,i_6)\in J}$ is a basis of $A^{(4)}S_4^{-1}$. Identifying the coefficients of $F_1^{w_1}T_2^{w_2+1}T_3^{w_3}T_4^{w_4+1}T_5^{w_5}T_6^{w_6+1}$, we have that $b_{(w_1,\dots,w_6)}=0$. This is a contradiction to our assumption, hence $b_{(i_1,\dots,i_6)}=0$ for all $(i_1,\dots,i_6)\in J$. This implies that

$$\sum_{\underline{i}\in I} a_{\underline{i}}F_1^{i_1}T_3^{i_3}T_4^{i_4}T_5^{i_5}T_6^{i_6}=0.$$

Consequently, $a_{\underline{i}}=0$ for all $\underline{i}\in I$. Therefore, $\mathfrak{S}_4$ is a linearly independent set. □

In $\mathcal{R}_4=A_{\alpha,\beta}^{(4)}S_4^{-1}$, we have the following two relations: $f_1t_3t_5+at_2t_5=\alpha$ and $t_2t_4t_6=\beta$. This implies that $f_1t_3=\alpha t_5^{-1}-at_2$ and $t_2=\beta t_6^{-1}t_4^{-1}$. Putting these two relations together, we have that

$$f_1t_3=\alpha t_5^{-1}-a\beta t_6^{-1}t_4^{-1}.\tag{17}$$

Note that we will usually identify $\mathcal{R}_4$ with $A_\beta^{(4)} S_4^{-1}/\langle \widehat{\Omega}_1 - \alpha\rangle$.

Proposition 5.3. *The set* $\mathcal{B}_4 = \left\{ f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6}, t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid i_1, i_3 \in \mathbb{N} \text{ and } i_4, i_5, i_6 \in \mathbb{Z} \right\}$ *is a* $\mathbb{K}$*-basis of* $\mathcal{R}_4$*.*

Proof. Since $\left(\widehat{f_1}^{k_1} \widehat{t_3}^{k_3} \widehat{t_4}^{k_4} \widehat{t_5}^{k_5} \widehat{t_6}^{k_6}\right)_{(k_1,k_3,\dots,k_6)\in \mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $A_\beta^{(4)} S_4^{-1}$ (Proposition 5.2), the set $\left(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}\right)_{(k_1,k_3,\dots,k_6)\in \mathbb{N}^2\times\mathbb{Z}^3}$ spans $\mathcal{R}_4$. We show that $\mathcal{B}_4$ is a spanning set of $\mathcal{R}_4$ by showing that $f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}$ can be written as a finite linear combination of the elements of $\mathcal{B}_4$ for all $(k_1, k_3, \dots, k_6) \in \mathbb{N}^2 \times \mathbb{Z}^3$. By Proposition 5.2, it is sufficient to do this by induction on k_1. The result is clear when $k_1 = 0$. Assume that the statement is true for $k_1 \geq 0$. That is,

$$f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = \sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6},$$

where $\underline{i} = (i_1, i_4, i_5, i_6) \in I_1 \subset \mathbb{N} \times \mathbb{Z}^3$ and $\underline{j} = (i_3, i_4, i_5, i_6) \in I_2 \subset \mathbb{N} \times \mathbb{Z}^3$. Note that $a_{\underline{i}}$ and $b_{\underline{j}}$ are all scalars. It follows that

$$f_1^{k_1+1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6} = f_1 \left(f_1^{k_1} t_3^{k_3} t_4^{k_4} t_5^{k_5} t_6^{k_6}\right) = \sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6}.$$

Clearly, the monomial $f_1^{i_1+1} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \operatorname{Span}(\mathcal{B}_4)$. We have to also show that $f_1 t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \in \operatorname{Span}(\mathcal{B}_4)$ for all $i_3 \in \mathbb{N}$ and $i_4, i_5, i_6 \in \mathbb{Z}$. This can easily be achieved by an induction on i_3, and the use of the relation $f_1 t_3 = \alpha t_5^{-1} - a\beta t_6^{-1} t_4^{-1}$. Therefore, by the principle of mathematical induction, $\mathcal{B}_4$ is a spanning set of $\mathcal{R}_4$ over $\mathbb{K}$.

We prove that $\mathcal{B}_4$ is a linearly independent set. Suppose that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0.$$

It follows that there exists $\nu \in A_\beta^{(4)} S_4^{-1}$ such that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}} \widehat{f_1}^{i_1} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6} = \left(\widehat{\Omega}_1 - \alpha\right)\nu.$$

Write $\nu = \sum_{\underline{l}\in J} c_{\underline{l}} \widehat{f_1}^{i_1} \widehat{t_3}^{i_3} \widehat{t_4}^{i_4} \widehat{t_5}^{i_5} \widehat{t_6}^{i_6}$, where $\underline{l} = (i_1, i_3, i_4, i_5, i_6) \in J \subset \mathbb{N}^2 \times \mathbb{Z}^3$ and $c_{\underline{l}} \in \mathbb{K}$. Note that $\widehat{t_2} = \beta \widehat{t_6}^{-1} \widehat{t_4}^{-1}$. We have that $\widehat{\Omega}_1 = \widehat{f_1}\widehat{t_3}\widehat{t_5} + a\widehat{t_2}\widehat{t_5} = \widehat{f_1}\widehat{t_3}\widehat{t_5} + a\beta \widehat{t_6}^{-1} \widehat{t_4}^{-1} \widehat{t_5}$.

Therefore,

$$\begin{aligned}\sum_{\underline{i}\in I_1} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} =& \sum_{\underline{l}\in J} q^{\bullet}c_{\underline{l}}\widehat{f_1}^{i_1+1}\widehat{t_3}^{i_3+1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6}\\ &+\sum_{\underline{l}\in J} q^{\bullet}\beta a c_{\underline{l}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4-1}\widehat{t_5}^{i_5+1}\widehat{t_6}^{i_6-1}\\ &-\sum_{\underline{l}\in J}\alpha c_{\underline{l}}\widehat{f_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6}.\end{aligned}$$

Suppose that there exists $(i_1, i_3, i_4, i_5, i_6) \in J$ such that $c_{(i_1,i_3,i_4,i_5,i_6)} \neq 0$.
Let $(w_1, w_3, w_4, w_5, w_6) \in J$ be the greatest element (in the lexicographic order on $\mathbb{N}^2 \times \mathbb{Z}^3$) of J such that $c_{(w_1,w_3,w_4,w_5,w_6)} \neq 0$. Since $\left(\widehat{f_1}^{k_1}\widehat{t_3}^{k_3}\widehat{t_4}^{k_4}\widehat{t_5}^{k_5}\widehat{t_6}^{k_6}\right)_{(k_1,k_3,\dots,k_6)\in\mathbb{N}^2\times\mathbb{Z}^3}$ is a basis of $A^{(4)}S_4^{-1}$, it follows that the coefficients of $\widehat{f_1}^{w_1+1}\widehat{t_3}^{w_3+1}\widehat{t_4}^{w_4}\widehat{t_5}^{w_5+1}\widehat{t_6}^{w_6}$ in the above equality can be identified as: $q^{\bullet}c_{(w_1,w_3,w_4,w_5,w_6)} = 0$. Hence, $c_{(w_1,w_3,w_4,w_5,w_6)} = 0$, a contradiction! Therefore, $c_{(i_1,i_3,i_4,i_5,i_6)} = 0$ for all $(i_1, i_3, i_4, i_5, i_6) \in J$. This further implies that

$$\sum_{\underline{i}\in I_1} a_{\underline{i}}\widehat{f_1}^{i_1}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} + \sum_{\underline{j}\in I_2} b_{\underline{j}}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} = 0.$$

It follows from the previous proposition that $a_{\underline{i}}$ and $b_{\underline{j}}$ are all zero. In conclusion, $\mathcal{B}_4$ is a linearly independent set. $\square$

Basis for $\mathcal{R}_5$. We will identify $\mathcal{R}_5$ with $A_\alpha^{(5)}S_5^{-1}/\langle\widehat{\Omega}_2 - \beta\rangle$, where $A_\alpha^{(5)}S_5^{-1} = \dfrac{A^{(5)}S_5^{-1}}{\langle\Omega_1 - \alpha\rangle}$. Note that the canonical images of $E_{i,j}$ (resp. T_i) in $A_\alpha^{(5)}$ will be denoted by $\widehat{e_{i,j}}$ (resp. $\widehat{t_i}$). We now find a basis for $A_\alpha^{(5)}S_5^{-1}$. Recall that $\Omega_1 = Z_1T_3T_5 + aZ_2T_5$ and $\Omega_2 = Z_2T_4T_6 + bT_3^3T_6$ in $A^{(5)}$ (remember that $Z_1 := E_{1,5}$ and $Z_2 := E_{2,5}$). Since $z_2t_4t_6 + bt_3^3t_6 = \beta$ and $\widehat{z_1}\widehat{t_3}\widehat{t_5} + a\widehat{z_2}\widehat{t_5} = \alpha$ in $\mathcal{R}_5$ and $A_\alpha^{(5)}S_5^{-1}$ respectively, we have the relation $\widehat{z_2} = \frac{1}{a}\left(\alpha\widehat{t_5}^{-1} - \widehat{z_1}\widehat{t_3}\right)$ in $A_\alpha^{(5)}S_5^{-1}$ and, in $\mathcal{R}_5$, we have the following two relations:

$$z_2 = \frac{1}{a}\left(\alpha t_5^{-1} - z_1t_3\right), \tag{18}$$

$$t_3^3 = \frac{1}{b}\left(\beta t_6^{-1} - z_2t_4\right) = \frac{\beta}{b}t_6^{-1} - \frac{q^3\alpha}{ab}t_4t_5^{-1} + \frac{1}{ab}z_1t_3t_4. \tag{19}$$

Proposition 5.4. *The set $\mathfrak{S}_5 = \left\{\widehat{z_1}^{i_1}\widehat{t_3}^{i_3}\widehat{t_4}^{i_4}\widehat{t_5}^{i_5}\widehat{t_6}^{i_6} \mid (i_1, i_3, \dots, i_6) \in \mathbb{N}^3 \times \mathbb{Z}^2\right\}$ is a $\mathbb{K}$-basis of $A_\alpha^{(5)}S_5^{-1}$, where $\alpha \in \mathbb{K}$.*

Proof. The proof is similar to that of Proposition 5.2, and so it is left to the reader. Details can be found in [18, Sec. 4.1]. □

Proposition 5.5. *The set* $\mathcal{B}_5 = \left\{ z_1^{i_1} t_3^{\xi} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid (\xi, i_1, i_4, i_5, i_6) \in \{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2 \right\}$ *is a* $\mathbb{K}$*-basis of* $\mathcal{R}_5$*.*

Proof. The proof is similar to that of Proposition 5.3, and so it is left to the reader. Details can be found in [18, Sec. 4.1]. □

We note for future reference the following immediate corollary.

Corollary 5.6. *Let* I *be a finite subset of* $\{0,1,2\} \times \mathbb{N} \times \mathbb{Z}^3$ *and* $(a_{(\xi,\underline{i})})_{\underline{i}\in I}$ *be a family of scalars. If*

$$\sum_{(\xi,\underline{i})\in I} a_{(\xi.\underline{i})} z_1^{i_1} t_3^{\xi} t_4^{i_4} t_5^{i_5} t_6^{i_6} = 0,$$

then $a_{(\xi,\underline{i})} = 0$ *for all* $(\xi, \underline{i}) \in I$*.*

Remark 5.7. We were not successful in finding a basis for $\mathcal{R}_6$. However, this has no effect on our main results in this section. Since $\mathcal{R}_7 = A_{\alpha,\beta}$, we already have a basis for $\mathcal{R}_7$ (Proposition 4.3).

5.3. Derivations of $A_{\alpha,\beta}$

We are now going to study the derivations of $A_{\alpha,\beta}$. We will only treat the case when both α and β are non-zero, and mention results when either α or β is zero without details.

Throughout this subsection, we assume that α and β are non-zero. Let $\mathrm{Der}(A_{\alpha,\beta})$ denote the set of $\mathbb{K}$-derivations of $A_{\alpha,\beta}$ and $D \in \mathrm{Der}(A_{\alpha,\beta})$. Via localization, D extends uniquely to a derivation of each of the series of algebras in (16). Therefore, D extends to a derivation of the quantum torus $\mathcal{R}_3 = \mathbb{K}_{q^N}[t_3^{\pm 1}, t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$. It follows from [19, Corollary 2.3] that D can uniquely be written as:

$$D = \mathrm{ad}_x + \delta,$$

where $x \in \mathcal{R}_3$, and δ is a scalar derivation of $\mathcal{R}_3$ defined as $\delta(t_i) = \lambda_i t_i$ for each $i = 3,4,5,6$. Note that $\lambda_i \in Z(\mathcal{R}_3) = \mathbb{K}$. Also, ad_x is an inner derivation of $\mathcal{R}_3$ defined as $\mathrm{ad}_x(u) = xu - ux$ for all $u \in \mathcal{R}_3$.

We aim to describe D as a derivation of $A_{\alpha,\beta} = \mathcal{R}_7$. We do this in several steps.

Before starting the process we note the following relations that will be used in this section. They all follow from [4, Lemme 5.3.2].

Remark 5.8. Recall the notations:

$$\begin{aligned} f_1 &:= e_{1,4} & F_1 &:= E_{1,4} \\ z_1 &:= e_{1,5} & Z_1 &:= E_{1,5} \\ z_2 &:= e_{2,5} & Z_2 &:= E_{2,5}. \end{aligned}$$

Then

$$\begin{aligned} f_1 &= t_1 - at_2t_3^{-1} & e_{3,6} &= t_3 - at_4t_5^{-1} \\ z_1 &= f_1 - st_3^2t_4^{-1} & e_1 &= e_{1,6} - rt_5t_6^{-1} \\ z_2 &= t_2 - bt_3^3t_4^{-1} & e_3 &= e_{3,6} - st_5^2t_6^{-1} \\ e_{1,6} &= z_1 - he_{3,6}t_5^{-1} - gt_4t_5^{-2} & e_4 &= t_4 - bt_5^3t_6^{-1}. \end{aligned}$$

We first describe D as a derivation of $\mathcal{R}_4$.

Lemma 5.9.

1. $x \in \mathcal{R}_4$.
2. $\lambda_5 = \lambda_4 + \lambda_6$, $\delta(f_1) = -(\lambda_3 + \lambda_5)f_1$ *and* $\delta(t_2) = -\lambda_5 t_2$.
3. *Set* $\lambda_1 := -(\lambda_3 + \lambda_5)$ *and* $\lambda_2 := -\lambda_5$. *Then,* $D(e_{\kappa,4}) = ad_x(e_{\kappa,4}) + \lambda_\kappa e_{\kappa,4}$ *for all* $\kappa \in \{1, \dots, 6\}$.

Proof. 1. Set $\mathcal{Q}_q := \mathbb{K}_{q^{N'}}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$, where N' is the skew-symmetric sub-matrix of N (see Section 4.1) obtained by deleting the first row and first column of N. Observe that $\mathcal{Q}_q$ is a subalgebra of both $\mathcal{R}_3$ and $\mathcal{R}_4$ with central element

$$z := t_4t_5^{-1}t_6.$$

Furthermore, since $\mathcal{R}_3$ is a quantum torus, we can present it as a free left $\mathcal{Q}_q$-module with basis $(t_3^s)_{s\in\mathbb{Z}}$. With this presentation, $x \in \mathcal{R}_3$ can be written as

$$x = \sum_{s\in\mathbb{Z}} y_s t_3^s,$$

where $y_s \in \mathcal{Q}_q$. Set

$$x_+ := \sum_{s\geq 0} y_s t_3^s \quad \text{and} \quad x_- := \sum_{s<0} y_s t_3^s.$$

Clearly, $x = x_+ + x_-$. Obviously, $x_+ \in \mathcal{R}_4$, hence we aim to also show that x_- belongs to $\mathcal{R}_4$ by following a pattern similar to [11, Proposition 7.1.2]. As D is a derivation of $\mathcal{R}_4$, we have that $D(z^j) \in \mathcal{R}_4$ for all $j \in \mathbb{N}_{\geq 1}$. Now $D(z^j) = \text{ad}_x(z^j) + \delta(z^j) =$

$\mathrm{ad}_{x_+}(z^j) + \mathrm{ad}_{x_-}(z^j) + \delta(z^j)$. Observe that $\mathrm{ad}_{x_+}(z^j) \in \mathcal{R}_4$; since $x_+, z^j \in \mathcal{R}_4$. Also, $\delta(z) = \delta(t_4 t_5^{-1} t_6) = (\lambda_4 - \lambda_5 + \lambda_6) t_4 t_5^{-1} t_6 = (\lambda_4 - \lambda_5 + \lambda_6) z$, where $\lambda_4, \lambda_5, \lambda_6 \in \mathbb{K}$. It follows that $\delta(z^j) = j(\lambda_4 - \lambda_5 + \lambda_6) z^j \in \mathcal{R}_4$. We can therefore conclude that each $\mathrm{ad}_{x_-}(z^j)$ belongs to $\mathcal{R}_4$ since $D(z^j), \mathrm{ad}_{x_+}(z^j), \delta(z^j) \in \mathcal{R}_4$. We have:

$$\mathrm{ad}_{x_-}(z^j) = x_- z^j - z^j x_- = \sum_{s=-1}^{-n} y_s t_3^s z^j - \sum_{s=-1}^{-n} y_s z^j t_3^s.$$

One can verify that $z t_3 = q^{-2} t_3 z$. Therefore,

$$\mathrm{ad}_{x_-}(z^j) = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s z^j, \text{ hence, } \mathrm{ad}_{x_-}(z^j) z^{-j} = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s.$$

Set $\nu_j := \mathrm{ad}_{x_-}(z^j) z^{-j} \in \mathcal{R}_4$. It follows that

$$\nu_j = \sum_{s=-1}^{-n} (1 - q^{-2js}) y_s t_3^s,$$

for each $j \in \{1, \dots, n\}$. One can therefore write the above equality as a matrix equation as follows:

$$\begin{bmatrix} (1-q^2) & (1-q^4) & (1-q^6) & \cdots & (1-q^{2n}) \\ (1-q^4) & (1-q^8) & (1-q^{12}) & \cdots & (1-q^{4n}) \\ (1-q^6) & (1-q^{12}) & (1-q^{18}) & \cdots & (1-q^{6n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-q^{2n}) & (1-q^{4n}) & (1-q^{6n}) & \cdots & (1-q^{2n^2}) \end{bmatrix} \begin{bmatrix} y_{-1} t_3^{-1} \\ y_{-2} t_3^{-2} \\ y_{-3} t_3^{-3} \\ \vdots \\ y_{-n} t_3^{-n} \end{bmatrix} = \begin{bmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \vdots \\ \nu_n \end{bmatrix}.$$

We already know that each ν_j belongs to $\mathcal{R}_4$. We want to show that $y_s t_3^s$ also belongs to $\mathcal{R}_4$ for each $s \in \{-1, \dots, -n\}$. To establish this, it is sufficient to show that the coefficient matrix of the above matrix equation is invertible. Let U represent this matrix. Thus,

$$U = \begin{bmatrix} (1-q^2) & (1-q^4) & (1-q^6) & \cdots & (1-q^{2n}) \\ (1-q^4) & (1-q^8) & (1-q^{12}) & \cdots & (1-q^{4n}) \\ (1-q^6) & (1-q^{12}) & (1-q^{18}) & \cdots & (1-q^{6n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (1-q^{2n}) & (1-q^{4n}) & (1-q^{6n}) & \cdots & (1-q^{2n^2}) \end{bmatrix}.$$

Apply row operations: $-r_{n-1} + r_n \to r_n, \dots, -r_2 + r_3 \to r_3, -r_1 + r_2 \to r_2$ to U to obtain:

$$U' = \begin{bmatrix} l_1 & l_2 & l_3 & \cdots & l_n \\ q^2 l_1 & q^4 l_2 & q^6 l_3 & \cdots & q^{2n} l_n \\ q^4 l_1 & q^8 l_2 & q^{12} l_3 & \cdots & q^{4n} l_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q^{2(n-1)} l_1 & q^{4(n-1)} l_2 & q^{6(n-1)} l_3 & \cdots & q^{2n(n-1)} l_n \end{bmatrix},$$

where $l_i := 1 - q^{2i}$; $i \in \{1, 2, \dots, n\}$. Clearly, U' is similar to a Vandermonde matrix (since the terms in each column form a geometric sequence) which is well known to be invertible when all parameters are pairwise distinct (this is the case here as q is not a root of unity). This further implies that U is invertible. So each $y_s t_3^s$ is a linear combination of the $\nu_j \in \mathcal{R}_4$. As a result, $y_s t_3^s \in \mathcal{R}_4$ for all $s \in \{-1, \dots, -n\}$. Consequently, $x_- = \sum_{s=-1}^{-n} y_s t_3^s \in \mathcal{R}_4$ as desired.

2. Recall that $\delta(t_\kappa) = \lambda_\kappa t_\kappa$ for all $\kappa \in \{3,4,5,6\}$ and $\lambda_\kappa \in \mathbb{K}$. From Remark 5.8, we have that $f_1 = t_1 - a t_2 t_3^{-1}$. Recall from Section 4.1 that $t_1 = \alpha t_5^{-1} t_3^{-1}$ and $t_2 = \beta t_6^{-1} t_4^{-1}$ in $\mathcal{R}_3 = \mathscr{A}_{\alpha,\beta}$. As a result, $f_1 = \alpha t_5^{-1} t_3^{-1} - a\beta t_6^{-1} t_4^{-1} t_3^{-1}$. Hence,

$$\delta(f_1) = -(\lambda_5 + \lambda_3)\alpha t_5^{-1} t_3^{-1} + (\lambda_6 + \lambda_4 + \lambda_3) a\beta t_6^{-1} t_4^{-1} t_3^{-1}. \tag{20}$$

From Proposition 5.3, the set $\mathcal{B}_4 = \left\{ f_1^{i_1} t_4^{i_4} t_5^{i_5} t_6^{i_6}, t_3^{i_3} t_4^{i_4} t_5^{i_5} t_6^{i_6} \mid i_1, i_3 \in \mathbb{N} \text{ and } i_4, i_5, i_6 \in \mathbb{Z} \right\}$ is a $\mathbb{K}$-basis of $\mathcal{R}_4$. Since t_4, t_5 and t_6 q-commute with f_1 and t_3, one can also write $\delta(f_1) \in \mathcal{R}_4$ in terms of $\mathcal{B}_4$ as follows:

$$\delta(f_1) = \sum_{r>0} a_r f_1^r + \sum_{s \geq 0} b_s t_3^s, \tag{21}$$

where a_r and b_s belong to $\mathcal{Q}_q = \mathbb{K}_{q^{N'}}[t_4^{\pm 1}, t_5^{\pm 1}, t_6^{\pm 1}]$.

$$\begin{aligned} f_1^r &= (\alpha t_5^{-1} t_3^{-1} - a\beta t_6^{-1} t_4^{-1} t_3^{-1})^r = \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} (\alpha t_5^{-1} t_3^{-1})^i (-a\beta t_6^{-1} t_4^{-1} t_3^{-1})^{r-i} \\ &= \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} \alpha^i (-a\beta)^{r-i} q^{\frac{1}{2}i(i-1) + \frac{3}{2}(r-i)(r-i-1) + 3i(i-r)} t_5^{-i} (t_6^{-1} t_4^{-1})^{r-i} t_3^{-r} \\ &= c_r t_3^{-r}, \end{aligned} \tag{22}$$

where

$$c_r = \sum_{i=0}^{r} \binom{r}{i}_{q^\bullet} q^{\frac{1}{2}i(i-1) + \frac{3}{2}(r-i)(r-i-1) + 3i(i-r)} \alpha^i (-a\beta)^{r-i} t_5^{-i} (t_6^{-1} t_4^{-1})^{r-i} \in \mathcal{Q}_q \setminus \{0\}. \tag{23}$$

Substitute (22) into (21) to obtain;

$$\delta(f_1) = \sum_{r>0} a_r c_r t_3^{-r} + \sum_{s \geq 0} b_s t_3^s. \tag{24}$$

One can rewrite (20) as

$$\delta(f_1) = dt_3^{-1}, \tag{25}$$

where $d = -(\lambda_5 + \lambda_3)\alpha t_5^{-1} + (\lambda_6 + \lambda_4 + \lambda_3)a\beta t_6^{-1}t_4^{-1} \in \mathcal{Q}_q$. Comparing (24) to (25) shows that $b_s = 0$ for all $s \geq 0$, and $a_r c_r = 0$ for all $r \neq 1$. Therefore $\delta(f_1) = a_1 c_1 t_3^{-1}$. Moreover, from (23), $c_1 = -a\beta t_6^{-1}t_4^{-1} + \alpha t_5^{-1}$. Hence,

$$\delta(f_1) = a_1 c_1 t_3^{-1} = a_1(-a\beta t_6^{-1}t_4^{-1} + \alpha t_5^{-1})t_3^{-1} = a_1\alpha t_5^{-1}t_3^{-1} - a_1 a\beta t_6^{-1}t_4^{-1}t_3^{-1}. \tag{26}$$

Comparing (26) to (20) reveals that $a_1 = -(\lambda_5 + \lambda_3) = -(\lambda_6 + \lambda_4 + \lambda_3)$. Consequently, $\lambda_5 = \lambda_6 + \lambda_4$. Hence, $\delta(f_1) = -(\lambda_5+\lambda_3)\alpha t_5^{-1}t_3^{-1} + (\lambda_5+\lambda_3)a\beta t_6^{-1}t_4^{-1}t_3^{-1} = -(\lambda_5+\lambda_3)f_1$. Finally, since $t_2 = \beta t_6^{-1}t_4^{-1}$ in $\mathcal{R}_4$, it follows that $\delta(t_2) = -(\lambda_6 + \lambda_4)\beta t_6^{-1}t_4^{-1} = -(\lambda_6 + \lambda_4)t_2 = -\lambda_5 t_2$.

3. This easily follows from parts 1 and 2. $\square$

We proceed to describe D as a derivation of $\mathcal{R}_5$.

Lemma 5.10.

1. *$x \in \mathcal{R}_5$.*
2. *$\lambda_4 = 3\lambda_3 + \lambda_5$, $\lambda_6 = -3\lambda_3$, $\delta(z_1) = -(\lambda_3 + \lambda_5)z_1$ and $\delta(z_2) = -\lambda_5 z_2$.*
3. *Set $\lambda_1 := -(\lambda_3 + \lambda_5)$ and $\lambda_2 := -\lambda_5$. Then, $D(e_{\kappa,5}) = ad_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa \in \{1, \dots, 6\}$.*

Proof. In this proof, we denote $\underline{v} := (i, j, k, l) \in \mathbb{N} \times \mathbb{Z}^3$.

1. We already know that $x \in \mathcal{R}_4 = \mathcal{R}_5[t_4^{-1}]$. Given the basis $\mathcal{B}_5$ of $\mathcal{R}_5$ (Proposition 5.5), x can be written as $x = \sum_{(\xi,\underline{v})\in I} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l$, where I is a finite subset of $\{0, 1, 2\} \times \mathbb{N} \times \mathbb{Z}^3$ and the $a_{(\xi,\underline{v})}$ are scalars. Write $x = x_- + x_+$, where

$$x_+ = \sum_{\substack{(\xi,\underline{v})\in I \\ j\geq 0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l \quad \text{and} \quad x_- = \sum_{\substack{(\xi,\underline{v})\in I \\ j<0}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Suppose that $x_- \neq 0$. Then, there exists a minimum $j_0 < 0$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$ for some $(\xi, i, j_0, k, l) \in I$ and $a_{(\xi,i,j,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$ with $j < j_0$. Given this assumption, write

$$x_- = \sum_{\substack{(\xi,\underline{v})\in I \\ j_0\leq j\leq -1}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

Now, $D(t_6) = \mathrm{ad}_{x_+}(t_6) + \mathrm{ad}_{x_-}(t_6) + \delta(t_6) \in \mathcal{R}_5$. This implies that $\mathrm{ad}_{x_-}(t_6) \in \mathcal{R}_5$, since $\mathrm{ad}_{x_+}(t_6)+\delta(t_6) = \mathrm{ad}_{x_+}(t_6)+\lambda_6 t_6 \in \mathcal{R}_5$. We aim to show that $x_- = 0$. Since t_6 is normal in $\mathcal{R}_5$, one can easily verify that

$$\mathrm{ad}_{x_-}(t_6) = \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} \left(q^{3(i-j-k)} - 1\right) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1}.$$

Set $\underline{w} := (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$. One can equally write $\mathrm{ad}_{x_-}(t_6) \in \mathcal{R}_5$ in terms of the basis $\mathcal{B}_5$ of $\mathcal{R}_5$ (Proposition 5.5) as:

$$\mathrm{ad}_{x_-}(t_6) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l,$$

where J is a finite subset of $\{0,1,2\} \times \mathbb{N}^2 \times \mathbb{Z}^2$ and $b_{(\xi,\underline{w})}$ are all scalars. It follows that

$$\sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} \left(q^{3(i-j-k)} - 1\right) a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^{l+1} = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

As $\mathcal{B}_5$ is a basis for $\mathcal{R}_5$, we deduce from Corollary 5.6 that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Now, at $j = j_0$, denote $\underline{v} = (i,j,k,l)$ by $\underline{v}_0 := (i,j_0,k,l)$. Since $\underline{v}_0 \in \mathbb{N} \times \mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2 \times \mathbb{Z}^2$ (with $j \ge 0$), it follows from the above equality that, at $\underline{v}_0$, we must have

$$\left(q^{3(i-j_0-k)} - 1\right) a_{(\xi,\underline{v}_0)} = 0.$$

From our initial assumption, the coefficients $a_{(\xi,\underline{v}_0)}$ are all not zero, therefore $q^{3(i-j_0-k)} - 1 = 0$. This implies that

$$k = i - j_0, \tag{27}$$

for some $(\xi,\underline{v}_0) \in I$.

In a similar manner, $D(t_3) = \mathrm{ad}_{x_+}(t_3) + \mathrm{ad}_{x_-}(t_3) + \delta(t_3) \in \mathcal{R}_5$. This implies that $\mathrm{ad}_{x_-}(t_3) \in \mathcal{R}_5$, since $\mathrm{ad}_{x_+}(t_3) + \delta(t_3) = \mathrm{ad}_{x_+}(t_3) + \lambda_3 t_3 \in \mathcal{R}_5$. We have that

$$\mathrm{ad}_{x_-}(t_3) = \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} a_{(\xi,\underline{v})} z_1^i t_3^\xi t_4^j t_5^k t_6^l t_3 - \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} a_{(\xi,\underline{v})} t_3 z_1^i t_3^\xi t_4^j t_5^k t_6^l.$$

One can deduce from Lemma A.1(3a) that

$$t_3 z_1^i = q^{-i} z_1^i t_3 + d_2[i] z_1^{i-1} z_2,$$

where $d_2[i] = q^{1-i} d_2[1] \left(\frac{1-q^{-2i}}{1-q^{-2}}\right)$, $d_2[1] = -(q+q^{-1}+q^{-3})$ and $d_2[0] = 0$. Therefore, the above expression for $\mathrm{ad}_{x_-}(t_3)$ can be expressed as:

$$\begin{aligned}\mathrm{ad}_{x_-}(t_3) = &\sum_{\substack{(0,\underline{v})\in I\\ j_0\le j\le -1}} f[i,j,k] a_{(0,\underline{v})} z_1^i t_3 t_4^j t_5^k t_6^l + \sum_{\substack{(1,\underline{v})\in I\\ j_0\le j\le -1}} f[i,j,k] a_{(1,\underline{v})} z_1^i t_3^2 t_4^j t_5^k t_6^l\\ &+ \sum_{\substack{(2,\underline{v})\in I\\ j_0\le j\le -1}} f[i,j,k] a_{(2,\underline{v})} z_1^i t_3^3 t_4^j t_5^k t_6^l - \sum_{\substack{(\xi,\underline{v})\in I\\ j_0\le j\le -1}} a_{(\xi,\underline{v})} d_2[i] z_1^{i-1} z_2 t_3^\xi t_4^j t_5^k t_6^l,\end{aligned}$$

where $f[i,j,k] := q^{-(k+3j)} - q^{-i}$. Recall from (18) and (19) that

$$z_2 = \frac{1}{a}\left(\alpha t_5^{-1} - z_1 t_3\right) \quad \text{and} \quad t_3^3 = \frac{\beta}{b} t_6^{-1} - \frac{q^3\alpha}{ab} t_4 t_5^{-1} + \frac{1}{ab} z_1 t_3 t_4,$$

where a and b are non-zero scalars (Appendix A.2). Using these two expressions, one can write $\mathrm{ad}_{x_-}(t_3)$ in terms of the basis of $\mathcal{R}_5$ as:

$$\begin{aligned}\mathrm{ad}_{x_-}(t_3) = \mathcal{K} &+ \sum_{(0,\underline{v}_0)\in I} g[i,j_0,k] a_{(0,\underline{v}_0)} z_1^i t_3 t_4^{j_0} t_5^k t_6^l + \sum_{(1,\underline{v}_0)\in I} g[i,j_0,k] a_{(1,\underline{v}_0)} z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l\\ &+ \sum_{(2,\underline{v}_0)\in I} \frac{q^\bullet\beta}{b} a_{(2,\underline{v}_0)} g[i,j_0,k] z_1^i t_4^{j_0} t_5^k t_6^{l-1} - \sum_{(\xi,\underline{v}_0)\in I} \frac{q^\bullet\alpha}{a} d_2[i] a_{(\xi,\underline{v}_0)} z_1^{i-1} t_3^\xi t_4^{j_0} t_5^{k-1} t_6^l\\ &= \sum 1/b\left(q^\bullet\beta g[i,j_0,k] a_{(2,i,j_0,k,l+1)} + (q^\bullet\alpha b d_2[i+1]/a) a_{(0,i+1,j_0,k+1,l)}\right) z_1^i t_4^{j_0} t_5^k t_6^l\\ &+ \sum \left(g[i,j_0,k] a_{(0,i,j_0,k,l)} + (q^\bullet\alpha d_2[i+1]/a) a_{(1,i+1,j_0,k+1,l)}\right) z_1^i t_3 t_4^{j_0} t_5^k t_6^l\\ &+ \sum \left(g[i,j_0,k] a_{(1,i,j_0,k,l)} + (q^\bullet\alpha d_2[i+1]/a) a_{(2,i+1,j_0,k+1,l)}\right) z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K},\end{aligned} \tag{28}$$

where $g[i,j_0,k] := q^{-(k+3j_0)} - q^{-i} + d_2[i]/a$ and

$$\mathcal{K} \in \mathrm{Span}\left(\mathcal{B}_5 \setminus \{z_1^i t_3^\xi t_4^{j_0} t_5^k t_6^l \mid (\xi,i,j_0,k,l) \in \{0,1,2\}\times\mathbb{N}\times\mathbb{Z}^3\}\right).$$

One can also write $\mathrm{ad}_{x_-}(t_3) \in \mathcal{R}_5$ in terms of the basis $\mathcal{B}_5$ of $\mathcal{R}_5$ (Proposition 5.5) as:

$$\mathrm{ad}_{x_-}(t_3) = \sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l, \tag{29}$$

where J is a finite subset of $\{0,1,2\}\times\mathbb{N}^2\times\mathbb{Z}^2$, and $b_{(\xi,\underline{w})} \in \mathbb{K}$. Recall: $\underline{w} = (i,j,k,l) \in \mathbb{N}^2\times\mathbb{Z}^2$. Now, (28) and (29) imply that

$$\sum_{(\xi,\underline{w})\in J} b_{(\xi,\underline{w})} z_1^i t_3^\xi t_4^j t_5^k t_6^l =$$
$$\sum 1/b \left(q^\bullet \beta g[i,j_0,k] a_{(2,i,j_0,k,l+1)} + (q^\bullet \alpha b d_2[i+1]/a) a_{(0,i+1,j_0,k+1,l)}\right) z_1^i t_4^{j_0} t_5^k t_6^l$$
$$+ \sum \left(g[i,j_0,k] a_{(0,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(1,i+1,j_0,k+1,l)}\right) z_1^i t_3 t_4^{j_0} t_5^k t_6^l$$
$$+ \sum \left(g[i,j_0,k] a_{(1,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(2,i+1,j_0,k+1,l)}\right) z_1^i t_3^2 t_4^{j_0} t_5^k t_6^l + \mathcal{K}.$$

We have already established that $\left(z_1^i t_3^\xi t_4^j t_5^k t_6^l\right)_{(i\in\mathbb{N};j,k,l\in\mathbb{Z};\xi\in\{0,1,2\})}$ is a basis for $\mathcal{R}_5[t_4^{-1}]$. Given that $\underline{v}_0 = (i,j_0,k,l) \in \mathbb{N}\times\mathbb{Z}^3$ (with $j_0 < 0$) and $\underline{w} = (i,j,k,l) \in \mathbb{N}^2\times\mathbb{Z}^2$ (with $j \geq 0$), it follows that

$$q^\bullet \beta g[i,j_0,k] a_{(2,i,j_0,k,l+1)} + (q^\bullet \alpha b d_2[i+1]/a) a_{(0,i+1,j_0,k+1,l)} = 0; \tag{30}$$
$$g[i,j_0,k] a_{(0,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(1,i+1,j_0,k+1,l)} = 0; \tag{31}$$
$$g[i,j_0,k] a_{(1,i,j_0,k,l)} + (q^\bullet \alpha d_2[i+1]/a) a_{(2,i+1,j_0,k+1,l)} = 0. \tag{32}$$

Suppose that there exists $(\xi,i,j_0,k,l) \in I$ such that $g[i,j_0,k] = 0$. Then,

$$g[i,j_0,k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = 0.$$

Note that $d_2[i] = d_2[1]q^{1-i}\left(\dfrac{1-q^{-2i}}{1-q^{-2}}\right)$, where $d_2[1] = -(q+q^{-1}+q^{-3})$ and $d_2[0] = 0$. Again, recall from Appendix A.2 that $a = (q^2+1+q^{-2})/(q^{-2}-1) = \dfrac{qd_2[1]}{1-q^{-2}}$. Given these expressions for $d_2[i]$ and a, we have that

$$g[i,j_0,k] = q^{-(k+3j_0)} - q^{-i} + d_2[i]/a = q^{-3j_0-k} - q^{-3i} = 0.$$

Since q is not a root of unity, we get

$$k = 3(i - j_0). \tag{33}$$

Comparing (33) to (27) shows that $i - j_0 = 0$ which implies that $i = j_0 < 0$, a contradiction (note that $i \geq 0$). Therefore, $g[i,j_0,k] \neq 0$ for all $(\xi,i,j,k,l) \in I$.

Now, observe that if there exists $\xi \in \{0,1,2\}$ such that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(i,j_0,k,l) \in \mathbb{N}\times\mathbb{Z}^3$, then one can easily deduce from equations (30), (31) and (32) that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi,i,j_0,k,l) \in I$. This will contradict our initial assumption. Therefore, for each $\xi \in \{0,1,2\}$, there exists some $(i,j_0,k,l) \in \mathbb{N}\times\mathbb{Z}^3$ such that $a_{(\xi,i,j_0,k,l)} \neq 0$. Without loss of generality, let (u,j_0,v,w) be the greatest element in the lexicographic order on $\mathbb{N}\times\mathbb{Z}^3$ such that $a_{(0,u,j_0,v,w)} \neq 0$ and $a_{(0,i,j_0,k,l)} = 0$ for all $i > u$.

From (31), at $(i,j_0,k,l) = (u,j_0,v,w)$, we have:

$$g[u,j_0,v] a_{(0,u,j_0,v,w)} + (q^\bullet \alpha d_2[u+1]/a) a_{(1,u+1,j_0,v+1,w)} = 0.$$

From (32), at $(i, j_0, k, l) = (u+1, j_0, v+1, w)$, we have:

$$g[u+1, j_0, v+1]a_{(1,u+1,j_0,v+1,w)} + (q^{\bullet}\alpha d_2[u+2]/a)a_{(2,u+2,j_0,v+2,w)} = 0.$$

Finally, from (30), at $(i, j_0, k, l) = (u+2, j_0, v+2, w-1)$, we have:

$$q^{\bullet}\beta g[u+2, j_0, v+2]a_{(2,u+2,j_0,v+2,w)} + (q^{\bullet}\alpha b d_2[u+3]/a)a_{(0,u+3,j_0,v+3,w-1)} = 0.$$

Note that $a, b, \alpha, \beta, q^{\bullet} \neq 0$; $g[i, j_0, k] \neq 0$ for all $(\xi, i, j_0, k, l) \in I$; and $d_2[i] \neq 0$ for $i > 0$. Since $u + 3 > u$, it follows from the above list of equations (starting from the last one) that

$$\begin{aligned} a_{(0,u+3,j_0,v+3,w-1)} = 0 \Rightarrow a_{(2,u+2,j_0,v+2,w)} = 0 \Rightarrow a_{(1,u+1,j_0,v+1,w)} = 0 \\ \Rightarrow a_{(0,u,j_0,v,w)} = 0, \end{aligned}$$

a contradiction! Hence, $a_{(0,i,j_0,k,l)} = 0$ for all $(i, j_0, k, l) \in \mathbb{N} \times \mathbb{Z}^3$. From (30), (31) and (32), one can easily conclude that $a_{(\xi,i,j_0,k,l)} = 0$ for all $(\xi, i, j_0, k, l) \in I$. This contradicts our initial assumption, hence $x_- = 0$. Consequently, $x = x_+ \in \mathcal{R}_5$ as desired.

2. From Remark 5.8, we have $z_2 = t_2 - bt_3^3t_4^{-1}$. Since $\delta(t_\kappa) = \lambda_\kappa t_\kappa,\ \kappa \in \{2, \dots, 6\}$, with $\lambda_2 := -\lambda_5$ (see Lemma 5.9), it follows that

$$\delta(z_2) = -\lambda_5 t_2 - b(3\lambda_3 - \lambda_4)t_3^3t_4^{-1} = -\lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1}.$$

Furthermore,

$$D(z_2) = \mathrm{ad}_x(z_2) + \delta(z_2) = \mathrm{ad}_x(z_2) - \lambda_5 z_2 - b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in \mathcal{R}_5.$$

Hence $b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3t_4^{-1} \in \mathcal{R}_5$, since $\mathrm{ad}_x(z_2) - \lambda_5 z_2 \in \mathcal{R}_5$. This implies that $b(3\lambda_3 - \lambda_4 + \lambda_5)t_3^3 \in \mathcal{R}_5 t_4$ (note that from Appendix A.2, $b \neq 0$). Set $w := 3\lambda_3 - \lambda_4 + \lambda_5$. Suppose that $w \neq 0$. From (19), we have:

$$t_3^3 = \frac{\beta}{b}t_6^{-1} - \frac{q^3\alpha}{ab}t_4t_5^{-1} + \frac{1}{ab}z_1t_3t_4.$$

It follows that

$$wbt_3^3 = w\beta t_6^{-1} - \frac{q^3 w\alpha}{a}t_4t_5^{-1} + \frac{w}{a}z_1t_3t_4 \in \mathcal{R}_5t_4.$$

Since $t_3^3,\ t_4t_5^{-1}$ and $z_1t_3t_4$ are all elements of $\mathcal{R}_5t_4$, it follows that $t_6^{-1} \in \mathcal{R}_5t_4$. Hence, $1 \in \mathcal{R}_5t_4t_6$. Using the basis $\mathcal{B}_5$ of $\mathcal{R}_5$ (Proposition 5.5), this leads to a contradiction. Therefore, $w = 0$. That is, $3\lambda_3 - \lambda_4 + \lambda_5 = 0$, and so $\lambda_4 = 3\lambda_3 + \lambda_5$. This further implies that $\delta(z_2) = -\lambda_5 z_2$ as desired.

Again, from Lemma 5.9, we have that $\delta(f_1) = -(\lambda_3 + \lambda_5)f_1$. Recall from Remark 5.8 that $z_1 = f_1 - st_3^2t_4^{-1}$. It follows that

$$\begin{aligned}\delta(z_1) =& -(\lambda_3+\lambda_5)f_1 - s(2\lambda_3-\lambda_4)t_3^2t_4^{-1} = -(\lambda_3+\lambda_5)z_1 - s(3\lambda_3-\lambda_4+\lambda_5)t_3^2t_4^{-1}\\ =& -(\lambda_3+\lambda_5)z_1 - s(3\lambda_3-(3\lambda_3+\lambda_5)+\lambda_5)t_3^2t_4^{-1} = -(\lambda_3+\lambda_5)z_1.\end{aligned}$$

Finally, we know that $\delta(t_6)=\lambda_6 t_6$. This implies that $\delta(t_6^{-1}) = -\lambda_6 t_6^{-1}$. From (19), we have that

$$t_3^3 = \frac{\beta}{b}t_6^{-1} - \frac{q^3\alpha}{ab}t_4t_5^{-1} + \frac{1}{ab}z_1t_3t_4,$$

where a and b are non-zero scalars (Appendix A.2). This implies that

$$t_6^{-1} = \frac{b}{\beta}t_3^3 + \frac{q^3\alpha}{a\beta}t_4t_5^{-1} - \frac{1}{a\beta}z_1t_3t_4.$$

Given that $\delta(z_1) = -(\lambda_3+\lambda_5)z_1,\ \delta(t_3)=\lambda_3t_3,\ \delta(t_4) = (3\lambda_3+\lambda_5)t_4$ and $\delta(t_5)=\lambda_5t_5$, applying δ to the above relation gives

$$-\lambda_6t_6^{-1} = 3\lambda_3\left(\frac{b}{\beta}t_3^3 + \frac{q^3\alpha}{a\beta}t_4t_5^{-1} - \frac{1}{a\beta}z_1t_3t_4\right).$$

It follows that $\lambda_6 = -3\lambda_3$ as desired.

3. Set $\lambda_1 := -(\lambda_3+\lambda_5)$ and $\lambda_2 := -\lambda_5$. Remember: $z_1 = e_{1,5},\ z_2 = e_{2,5}$ and $t_i = e_{i,5}$ $(3\le i\le 6)$. It follows from points (1) and (2) that $D(e_{\kappa,5}) = \mathrm{ad}_x(e_{\kappa,5}) + \delta(e_{\kappa,5}) = \mathrm{ad}_x(e_{\kappa,5}) + \lambda_\kappa e_{\kappa,5}$ for all $\kappa\in\{1,\dots,6\}$. In conclusion, $D = \mathrm{ad}_x+\delta$ with $x\in\mathcal{R}_5$. □

We are now ready to describe D as a derivation of $A_{\alpha,\beta}$.

Lemma 5.11.

1. *$x\in A_{\alpha,\beta}$.*
2. *$\delta(e_\kappa)=0$ for all $\kappa\in\{1,\dots,6\}$.*
3. *$D = ad_x$.*

Proof. The strategy of the proof is similar to that of Lemma 5.10, hence we omit it here and refer the interested reader to check it out in [18, Section 4.2]. □

Using similar techniques, one can describe the derivations of $A_{\alpha,0}$ and $A_{0,\beta}$. Details can be found in [18, Section 4.2]. There are fundamental differences in these two cases. Indeed, there exist in both cases derivations which are not inner. More precisely, one can check that the linear map θ on $A_{\alpha,0}$ defined by

$$\theta(e_1)=-e_1,\quad \theta(e_2)=-e_2,\quad \theta(e_3)=0,\quad \theta(e_4)=e_4,\quad \theta(e_5)=e_5,\quad \theta(e_6)=2e_6$$

and extended to $A_{\alpha,0}$ using the Leibniz rule is a $\mathbb{K}$-derivation of $A_{\alpha,0}$.

Similarly, the linear map $\tilde\theta$ on $A_{0,\beta}$ defined by

$$\tilde\theta(e_1)=-2e_1,\quad \tilde\theta(e_2)=-3e_2,\quad \tilde\theta(e_3)=-e_3,\quad \tilde\theta(e_4)=0,\quad \tilde\theta(e_5)=e_5,\quad \tilde\theta(e_6)=3e_6$$

and extended to $A_{0,\beta}$ using the Leibniz rule is a $\mathbb{K}$-derivation of $A_{0,\beta}$.

We summarize our main results in the theorem below.

Theorem 5.12. *Given $A_{\alpha,\beta} = U_q^+(G_2)/\langle \Omega_1 - \alpha, \Omega_2 - \beta \rangle$, with $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0,0)\}$, we have the following results:*

1. *if $\alpha, \beta \neq 0$; then every derivation D of $A_{\alpha,\beta}$ can uniquely be written as $D = ad_x$, where $x \in A_{\alpha,\beta}$.*
2. *if $\alpha \neq 0$ and $\beta = 0$, then every derivation D of $A_{\alpha,0}$ can uniquely be written as $D = ad_x + \lambda\theta$, where $\lambda \in \mathbb{K}$ and $x \in A_{\alpha,0}$.*
3. *if $\alpha = 0$ and $\beta \neq 0$, then every derivation D of $A_{0,\beta}$ can uniquely be written as $D = ad_x + \lambda\tilde{\theta}$, where $\lambda \in \mathbb{K}$ and $x \in A_{0,\beta}$.*
4. *$HH^1(A_{\alpha,0}) = \mathbb{K}[\theta]$ and $HH^1(A_{0,\beta}) = \mathbb{K}[\tilde{\theta}]$, where $[\theta]$ and $[\tilde{\theta}]$ respectively denote the classes of θ and $\tilde{\theta}$ modulo the space of inner derivations.*
5. *if $\alpha, \beta \neq 0$; then $HH^1(A_{\alpha,\beta}) = \{[0]\}$, where $[0]$ denotes the class of 0 modulo the space of inner derivations.*

The above theorem shows that $A_{\alpha,\beta}$ when both α and β are nonzero shares a number of properties with the second Weyl algebra over $\mathbb{K}$: it is a simple noetherian domain with GKdim 4, units are reduced to scalars, and all derivations are inner.

It would be interesting to compute the automorphism group of these algebras and verify if all endomorphisms are automorphisms, i.e. an analogue of the celebrated Dixmier Conjecture [6].

In general, the present work and [13] suggest that the primitive quotients of $U_q^+(\mathfrak{g})$ by primitive ideals from the 0-stratum provide algebras that could (should?) be regarded (and studied) as quantum analogues of Weyl algebras.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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Appendix A. Relations of $U_q^+(G_2)$, and definition of parameters used

A.1. Some selected general relations of $U_q^+(G_2)$

Lemma A.1. *For any $n \in \mathbb{Z}_{\geq 0}$, we have that:*

1(a) $E_jE_i^n = q^{-3n}E_i^nE_j$ *(b)* $E_j^nE_i = q^{-3n}E_iE_j^n$ *for all* $1 \leq i, j \leq 6$, *with* $j - i = 1$.

2(a) $E_6E_4^n = q^{-3n}E_4^nE_6 + d_1[n]E_4^{n-1}E_5^3$ *(b)* $E_6^nE_4 = q^{-3n}E_4E_6^n + d_1[n]E_5^3E_6^{n-1}$
(c) $E_4E_2^n = q^{-3n}E_2^nE_4 + d_1[n]E_2^{n-1}E_3^3$ *(d)* $E_4^nE_2 = q^{-3n}E_2E_4^n + d_1[n]E_3^3E_4^{n-1}$,
where $d_1[n] = q^{3(1-n)}d_1[1]\left(\dfrac{1-q^{-6n}}{1-q^{-6}}\right)$; $d_1[1] = -\dfrac{q^4 - 2q^2 + 1}{q^4 + q^2 + 1}$ *and* $d_1[0] := 0$.

3(a) $E_3E_1^n = q^{-n}E_1^nE_3 + d_2[n]E_1^{n-1}E_2$ *(b)* $E_3^nE_1 = q^{-n}E_1E_3^n + d_2[n]E_2E_3^{n-1}$
(c) $E_5E_3^n = q^{-n}E_3^nE_5 + d_2[n]E_3^{n-1}E_4$ *(d)* $E_5^nE_3 = q^{-n}E_3E_5^n + d_2[n]E_4E_5^{n-1}$,
where $d_2[n] = q^{1-n}d_2[1]\left(\dfrac{1-q^{-2n}}{1-q^{-2}}\right)$; $d_2[1] = -(q + q^{-1} + q^{-3})$ *and* $d_2[0] := 0$.

4(a) $E_6^nE_3 = E_3E_6^n + d_3[n]E_5^2E_6^{n-1}$ *(b)* $E_5E_2^n = E_2^nE_5 + d_3[n]E_2^{n-1}E_3^2$,
where $d_3[n] = d_3[1]\left(\dfrac{1-q^{-6n}}{1-q^{-6}}\right)$; $d_3[1] = 1 - q^2$ *and* $d_3[0] := 0$.

Proof. This is an easy proof by induction, left to the reader. □

A.2. Definition of parameters used throughout

In this subsection, we define some parameters/scalars used in this article. Any other scalars not defined here must be defined in/before the context in which it is found.

$$a = \frac{q^2 + 1 + q^{-2}}{q^{-2} - 1} \qquad b = \frac{-q^7 + 2q^5 - q^3}{(q^4 + q^2 + 1)(1 - q^{-6})} \qquad d' = \frac{q^{12}}{q^6 - 1}$$

$$c' = -\frac{q^9}{q^4 + q^2 + 1}.$$

$$g = \frac{q + q^{-1} + q^{-3}}{(1 - q^{-2})^2} \qquad f = \frac{1 - q^2}{1 - q^{-2}} \qquad b' = \frac{q^{13} - q^{11}}{(q^4 + q^2 + 1)^2}$$

$$h = \frac{q + q^{-1}}{q^{-2} - 1} \qquad s = \frac{1 - q^2}{1 - q^{-6}} \qquad a' = af + hq = \frac{q^6}{q^2 - 1}$$

$$t = \frac{q^{-1} - q}{1 - q^{-6}} \qquad u = \frac{q + q^{-1} - q^{-3}}{1 - q^{-6}} \qquad n = \frac{q^{12}}{(q^4 + q^2 + 1)^3}$$

$$p = \frac{q^4 + q^2 + 1}{q^2 - 1} \qquad r = \frac{-1}{1 - q^{-6}} \qquad e = \frac{-(q^7 + q^5 + q^3)}{q^4 - 2q^2 + 1}$$

References

[1] J. Bell, K. Casteels, S. Launois, Primitive ideals in quantum Schubert cells: dimension of the strata, Forum Math. 26 (3) (2014) 703–721.

[2] K.A. Brown, K.R. Goodearl, Lectures on Algebraic Quantum Groups, Advanced Courses in Mathematics CRM Barcelona, Birkhäuser, Basel, 2002.

[3] P. Caldero, Sur le centre de $U_q(\mathfrak{n}^+)$, Beitr. Algebra Geom. 35 (1) (1994) 13–24.

[4] G. Cauchon, Effacement des dérivations et spectres premiers des algèbres quantiques, J. Algebra 260 (2003) 476–518.

[5] W.A. De Graaf, Computing with quantized enveloping algebras: PBW-type bases, highest-weight modules and R-matrices, J. Symb. Comput. 32 (2001) 475–490.

[6] J. Dixmier, Sur les algèbres de Weyl, Bull. Soc. Math. Fr. 96 (1968) 209–242.

[7] J. Dixmier, Algèbres enveloppantes, Les Grands Classiques Gauthiers-Villars, Éditions Jacques Gabay, Paris, 1996.

[8] K.R. Goodearl, S. Launois, T. Lenagan, Tauvel's height formula for quantum nilpotent algebras, Commun. Algebra 47 (2019) 4194–4209.

[9] K.R. Goodearl, E.S. Letzter, Prime and primitive spectra of multiparameter quantum affine spaces, in: Trends in Ring Theory, Miskolc, 1996, vol. 22, 1998, pp. 39–58.

[10] K.R. Goodearl, M. Yakimov, From quantum Ore extensions to quantum tori via noncommutative UFDs, Adv. Math. 300 (2016) 672–716.

[11] A. Kitchin, On the quantum analogues of the Weyl algebras, PhD Thesis, University of Kent, 2016.

[12] G.R. Krause, T.H. Lenagan, Growth of Algebras and Gelfand-Kirillov Dimension, Graduate Studies in Mathematics, vol. 22, American Mathematical Society, 2000.

[13] S. Launois, Primitive ideals and automorphism group of $U_q^+(B_2)$, J. Algebra Appl. 6 (2007) 21–47.

[14] S. Launois, T.H. Lenagan, L. Rigal, Quantum unique factorisation domains, J. Lond. Math. Soc. (2) 74 (2006) 321–340.

[15] S. Launois, S.A. Lopes, Automorphisms and derivations of $U_q(sl_4^+)$, J. Pure Appl. Algebra 211 (2007) 249–264.

[16] S.A. Lopes, Primitive ideals of $U_q(\mathfrak{sl}_n^+)$, Commun. Algebra 34 (2006) 4523–4550.

[17] A. Mériaux, Cauchon diagrams for quantized enveloping algebras, J. Algebra 323 (2010) 1060–1097.

[18] I. Oppong, A Quantum Deformation of the Second Weyl Algebra: Its derivations and Poisson Derivations, PhD Thesis, University of Kent, 2021, Available at https://kar.kent.ac.uk/92766/.

[19] J.M. Osborn, D. Passman, Derivations of skew polynomial rings, J. Algebra 176 (1995) 417–448.

[20] C.M. Ringel, PBW-bases of quantum groups, J. Reine Angew. Math. 470 (1996) 51–88.

[21] M. Yakimov, Invariant prime ideals in quantizations of nilpotent lie algebras, Lond. Math. Soc. 101 (2010) 454–476.

[22] M. Yakimov, Strata of prime ideals of De Concini-Kac-Procesi algebras and Poisson geometry, in: New Trends in Noncommutative Algebra, in: Contemp. Math., vol. 562, Amer. Math. Soc., Providence, RI, 2012, pp. 265–278.