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Global stability of an age-structured model of smoking and its treatment

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Smoking is a serious global public health problem. There are serious consequences from smoking and the ability to quit it are closely related to age. Personal determination and education level usually play important roles in quitting smoking. In order to capture such characteristics, we developed a novel age-structured smoking dynamical model. By defining the smoking generation number R_0, the local stability, global stability of the boundary equilibrium and endemic equilibrium are obtained using Lyapunov functions. The uniform persistence and the well-posedness and asymptotic smoothness of the solutions are also studied. Sensitivity analyzes show that the lower the age of onset of smoking and the higher the determination to stop, the greater the likelihood of quitting smoking and numerical studies support the theoretical results.

AQ: pl approve edit of 'There are serious consequences..'

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1. Introduction

Every year a huge number of people die from diseases caused by smoking, such as heart disease, cancers and chronic bronchitis. Smoking has been regarded as a serious global public health problem, which can be spread by social contact [7]. Therefore, in order to prevent the spread of smoking, we first need to explore the transmission mechanisms of smoking. Mathematical modeling is a very useful tool, it cannot only provide a natural description of real problems, but can also reveal the relationships between variables.

Many scholars have tried to use mathematical models to study relationships between smokers. Castillo-Garsow, Sharomi and Gumel *et al.* proposed a series of deterministic giving up smoking dynamical models [4, 19]. They divided the total population into four groups (potential smokers, smokers of temporarily quit, smokers of permanently quit and chain smokers) and studied the local and global stability of smoking-free equilibrium. Zhang *et al.* proposed a stochastic smoking model to study the effects of environmental fluctuations on the dynamics of smoking [12, 29] and showed that the system is ergodic when a noise parameters value was low. Verma *et al.* studied the effects of media campaigns, educational programs and an individual's determination to cease smoking [22, 27] and pointed out that these factors have impacts on quitting smoking. Singh *et al.* constructed a fractional smoking model and the threshold conditions for the existence and uniqueness of its solution were provided [20]. Generally, the diseases caused by smoking became more serious with increasing age of the smokers and the number of smokers depends on the age of smoking initiation [8]. However, these studies have not taken the age of chain smokers into account. Thus, it is of theoretical and practical significance to study age-structured smoking dynamical systems [10, 24].

Age-structured dynamical systems have been widely investigated in epidemics [2, 5], virus dynamics [9, 14, 23], population dynamics [3, 15] and so on. But there are few papers using age-structured models to study the dynamics of smoking. Zeb *et al.* considered the age of potential smokers and formulated an age-structured smoking model [28], in which they studied the properties of the solution and derived conditions for the stability of the smoking free equilibrium. Rahmana *et al.* constructed a novel smoking model concerning the age of chain smokers [18], mainly modeling ages from a light smoking class to s chain smoker class, threshold conditions for the local and global stability of the boundary and endemic equilibria were also studied.

However, in most cases either age was not included in the pivotal threshold condition (i.e. the smoking generation number) or treatments (for example, individual's determination [22, 27]) were excluded [18, 28]. Therefore, we proposed a novel age-structured smoking model concerning the age effect and the effects of individual's determination, mainly focusing on two questions: (1) What is the relationship between age, transmission rate, smoking quitting rate and the smoking generation number? (2) How to evaluate the effectiveness of the important parameters affecting quitting smoking?

The rest of the paper is organized as follows. In Sec. 2, a novel age-structured smoking model is presented. In Sec. 3, useful definitions and lemmas are introduced and the properties of solutions of the proposed system are studied. In Sec. 4, conditions for local and global stability are derived for the smoking free equilibrium and endemic equilibrium. In Sec. 5, numerical simulations are described, followed by discussion and concluding remarks.

2. Mathematical Model

We assume that the total number of population N is constant at all time t and then divide N into four classes: Potential smokers $P(t)$, smokers of who are temporary quitters $Q_t(t)$, smokers who are permanent quitters $Q_p(t)$ and chain smokers $S(t,a)$ at time t with age a, and have the following model:

$$\begin{cases} \dfrac{dP(t)}{dt} = \lambda - \displaystyle\int_0^\infty \beta(a)P(t)S(t,a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da - uP(t), \\ \dfrac{\partial S(t,a)}{\partial t} + \dfrac{\partial S(t,a)}{\partial a} = -\alpha(a)S(t,a) - uS(t,a), \\ \dfrac{dQ_t(t)}{dt} = \displaystyle\int_0^\infty \epsilon_1(1-\epsilon_2)\alpha(a)S(t,a)da - uQ_t(t), \\ \dfrac{dQ_p(t)}{dt} = \displaystyle\int_0^\infty \epsilon_1\epsilon_2\alpha(a)S(t,a)da - uQ_p(t) \end{cases} \tag{2.1}$$

with boundary condition

$$S(t,0) = \int_0^\infty \beta(a)P(t)S(t,a)da, \quad t \geq 0 \tag{2.2}$$

and the initial conditions $P(0) = P_0 > 0, S(0,a) = S_0(a) \geq 0, Q_t(0) = Q_t^0 > 0, Q_p(0) = Q_p^0 > 0$ for $a \geq 0$, while $S_0(a)$ belongs to $L_+^1(0,\infty)$ and satisfies $\int_0^\infty S_0(a)da \leq \infty$ ($L_+^1(0,\infty)$ is defined as the space of all essentially bounded and positive functions that are Lebesgue integrable). λ represents the constant recruitment rate of the population, $\beta(a)$ is the transmission rate at age a. $\alpha(a)$ is the rate that smokers are quitting smoking at age a. ϵ_1 is the measure of determination. So $(1-\epsilon_1)\alpha(a)S(t,a)$ is the fraction of quitters who again become chain smokers because of low determination, while the fraction $\epsilon_1\alpha(a)S(t,a)$ stays in the quitter classes. u is the death rate. ϵ_2 is the efficacy of interventions including education or treatment.

Notice that the first two equations of system (2.1) do not contain the variables Q_t and Q_p. Thereby, to study the dynamics of a giving up smoking model we can ignore the variables Q_t and Q_p and only need to focus on the following subsystem:

$$\begin{cases} \dfrac{dP(t)}{dt} = \lambda - \displaystyle\int_0^\infty \beta(a)P(t)S(t,a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da - uP(t), \\ \dfrac{\partial S(t,a)}{\partial t} + \dfrac{\partial S(t,a)}{\partial a} = -\alpha(a)S(t,a) - uS(t,a) \end{cases} \tag{2.3}$$

with boundary condition

$$S(t,0) = \int_0^\infty \beta(a)P(t)S(t,a)da, \quad t \geq 0 \tag{2.4}$$

and the initial conditions $P(0) = P_0 > 0, S(0,a) = S_0(a) \geq 0$ for $a \geq 0$.

Assumption 1. For the functions $\beta(a)$ and $\alpha(a)$ we assume:

(I) $\beta(a), \alpha(a) \in L^1_+(0,\infty)$ have upper bounds $\hat{\beta}, \hat{\alpha}$, respectively;
(II) $\beta(a), \alpha(a)$ are Lipschitz continuous with Lipschitzians L_β and L_α, respectively;
(III) $\beta(a), \alpha(a) \geq c_0$ for $c_0 \in (0, \bar{c}]$ with $a \geq 0$.

For simplicity, the following notation is very useful in the rest of the paper:

$$\begin{aligned} &K_0(a) = e^{-\int_0^a (u+\alpha(s))ds}, \\ &B(t) = \int_0^\infty \beta(a)S(t,a)da, \quad S(t,0) = P(t)B(t), \\ &K_1 = \int_0^\infty \alpha(a)K_0(a)da, \quad K_2 = \int_0^\infty \beta(a)K_0(a)da. \end{aligned} \tag{2.5}$$

$K_0(a)$ is the probability of a chain smoker remaining smoking at age a, $\beta(a)K_0(a)$ is the product of the age-specific remaining probability of a chain smoker and the transmission rate at which the potential smokers become chain smokers by association with a chain smoker of age a. Thus, K_2 is the total number of new smokers produced by a chain smoker over his or her lifespan.

Integrating the second equation of (2.3) along the characteristic line $t - a =$ const., then

$$S(t,a) = \begin{cases} P(t-a)B(t-a)K_0(a), & 0 \leq a < t, \\ S_0(a-t)\dfrac{K_0(a)}{K_0(a-t)}, & a \geq t \geq 0. \end{cases} \tag{2.6}$$

In order to study the dynamics of system (2.3), we need to define the function space X. Let

$$X = R^+ \times L^1_+(0,\infty),$$

which is endowed with the norm

$$\|(x_1,x_2)\|_X = |x_1| + \int_0^\infty |x_2(a)|da.$$

The initial conditions of system (2.3) in the space X can be denoted by

$$x_0 = (P_0, S(t,\cdot)) \in X. \tag{2.7}$$

For system (2.3), define a continuous semi flow as $\zeta : R^+ \times X \to X$, where

$$\zeta(t,x_0) = \zeta_t(x_0) = (P(t), S(t,\cdot)), \quad t \geq 0 \quad \text{and} \quad x_0 \in X. \tag{2.8}$$

Then we have the following norm for $\zeta_t(x_0)$, i.e.

$$\|\zeta_t(x_0)\|_X = \|(P(t), S(t,\cdot)\| = |P(t)| + \int_0^\infty |S(t,a)|da.$$

3. Main Properties of Solutions for System (2.3)

3.1. *Well-posedness*

By using the methods proposed by Webb [24] and Iannelli [10], it can be shown that model (2.3) exists with the unique and non-negative solution with positive initial conditions. Denote Ω as the state space, i.e.

$$\Omega = \left\{(P(t), S(t,\cdot) \in X \mid P(t) + \int_0^\infty S(t,a)da \le \frac{\lambda}{u}\right\}.$$

We can obtain the following proposition for ζ and Ω.

Proposition 1. *For all $t \ge 0$ and $x_0 \in \Omega$, we obtain $\zeta(t, x_0) \in \Omega$. Moreover, Ω attracts all points in X and ζ is point dissipative.*

Proof. From (2.8) we have

$$\frac{d}{dt}\|\zeta_t(x_0)\|_X = \frac{d}{dt}P(t) + \frac{d}{dt}\int_0^\infty S(t,a)da.$$

It follows from (2.6) and the fact that $K_0(0) = 1$ that we obtain

$$\begin{aligned}\frac{d}{dt}\int_0^\infty S(t,a)da &= \frac{d}{dt}\int_0^t P(t-a)B(t-a)K_0(a)da \\ &\quad + \frac{d}{dt}\int_t^\infty S_0(a-t)\frac{K_0(a)}{K_0(a-t)}da \\ &= P(t)B(t) - \int_0^\infty (u+\alpha(a))S(t,a)da.\end{aligned}$$

Thus,

$$\begin{aligned}&\frac{d}{dt}\left(P(t) + \int_0^\infty S(t,a)da\right) \\ &\quad = \lambda - \int_0^\infty \beta(a)P(t)S(t,a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da - uP(t) \\ &\qquad + P(t)B(t) - \int_0^\infty (u+\alpha(a))S(t,a)da \\ &\quad \le \lambda + \int_0^\infty \alpha(a)S(t,a)da - uP(t) - \int_0^\infty (u+\alpha(a))S(t,a)da \\ &\quad = \lambda - u(P(t) + \int_0^\infty S(t,a)da). \qquad (3.1)\end{aligned}$$

It follows from the variation of the formula that

$$\|\zeta_t(x_0)\| \le \frac{\lambda}{u} - e^{-ut}\left(\frac{\lambda}{u} - \|x_0\|_X\right), \quad t \ge 0,$$

which implies that for any $t \ge 0$ and $x_0 \in \Omega$, we have $\zeta(t, x_0) \in \Omega$, and so the set Ω is positive invariant.

When $t \to \infty$, then

$$\lim_{t\to\infty} \|\zeta_t(x_0)\|_X \le \frac{\lambda}{u}, \quad x_0 \in X,$$

which means that ζ is point dissipative and Ω attracts all points in X. This completes the proof. □

Remark 1. For some constant h that satisfies the condition $h \ge \lambda/u$, it follows from Assumption 1 and Proposition 1 that if for any $x_0 \in X$ and $\|x_0\|_X \le h$ then $P(t)$ and $S(t)$ are bounded above by h and bounded below by zero.

3.2. *Asymptotic smoothness*

We introduce the following two lemmas to show the asymptotic smoothness of the semiflow $\{\zeta(t, \cdot)\}_{t\ge 0}$ [21].

Lemma 1. *The semiflow $\zeta : R^+ \times X \to X$ is asymptotically smooth if there exist the two maps $\zeta_1, \zeta_2 : R^+ \times X \to X$ such that $\zeta(t, x) = \zeta_1(t, x) + \zeta_2(t, x)$, and for any bounded closed set $\mathcal{A} \subset X$ ($\mathcal{A}$ is forward invariant of ζ) the following conditions hold*: (1) $\lim_{t\to+\infty} \mathrm{diam}\zeta_2(t, \mathcal{A}) = 0$; (2) *there is a $t_{\mathcal{A}} \ge 0$ and each $t \ge t_{\mathcal{A}}$ will lead to $\zeta_1(t, \mathcal{A})$ which has compact closure.*

Because X is an infinite dimensional space and $L^1_+(0, +\infty) \subset X$, to guarantee the precompactness we need the following results.

Lemma 2. *Denote $\mathcal{A}_1$ as a bounded subset of $L^1_+(0, +\infty)$. The sufficient and necessary conditions for $\mathcal{A}_1$ having a compact closure are as follows*:

(i) $\sup_{f\in\mathcal{A}_1} \int_0^{+\infty} |f(s)|ds < +\infty$;
(ii) $\sup_{t\to+\infty} \int_t^{+\infty} |f(s)|ds = 0$ *uniformly in* $f \in \mathcal{A}_1$;
(iii) $\sup_{t\to 0^+} \int_0^{+\infty} |f(s + t) - f(s)|ds = 0$ *uniformly in* $f \in \mathcal{A}_1$;
(iv) $\sup_{t\to 0^+} \int_0^{t} |f(s)|ds = 0$ *uniformly in* $f \in \mathcal{A}_1$.

By using the above lemmas, we can show that the semiflow $\zeta(t, x)$ is asymptotically smooth. First of all, we give the definitions of ζ_1 and ζ_2. Let

$$S_1(t, a) = \begin{cases} S(t, a), & 0 \le a < t; \\ 0, & a \ge t \ge 0 \end{cases} \tag{3.2}$$

and

$$S_2(t,a) = \begin{cases} 0, & 0 \le a < t; \\ S(t,a), & a \ge t \ge 0. \end{cases} \tag{3.3}$$

Then ζ_1 *and* ζ_2 *can be defined as* $\zeta_1(t,x_0) = (P(t), S_1(t,\cdot))$ *and* $\zeta_2(t,x_0) = (0, S_2(t,\cdot))$, *respectively. It is clear that the semiflow* $\zeta(t,x_0) = \zeta_1(t,x_0) + \zeta_2(t,x_0)$.

Theorem 1. *The semiflow* ζ *defined by* (2.8) *for system* (2.3) *is asymptotically smooth.*

Proof. It follows from Remark 1 that for each $x_0 \in \mathcal{A}$ (here $\mathcal{A} \subset X$) yields $\|x_0\|_X \le h$. Concerning (2.6) and (3.3),

$$\begin{aligned} \|\zeta_2(t,x_0)\|_X &= \int_t^{+\infty} |S_2(t,a)| da = \int_t^{+\infty} \left| S_0(a-t) \frac{K_0(a)}{K_0(a-t)} \right| da \\ &= \int_t^{+\infty} \left| S_0(\tau) \frac{K_0(t+\tau)}{K_0(\tau)} \right| d\tau = \int_t^{+\infty} \left| S_0(\tau) e^{-\int_\tau^{t+\tau}(u+\alpha(s))ds} \right| d\tau \\ &\le e^{-(u+c_0)t} \int_t^{+\infty} |S_0(\tau)|\, d\tau \le e^{-(u+c_0)t} \|x_0\|_X \le e^{-(u+c_0)t} h. \end{aligned} \tag{3.4}$$

So $\lim_{t\to+\infty} \mathrm{diam}\zeta_2(t,\mathcal{A}) = 0$.

Now, we need to show that $\zeta_1(t,\mathcal{A})$ exists with compact closure for any $t \ge 0$. In the light of Remark 1, $P(t)$ lies in the compact set $[0,h]$ for $t \ge 0$. Moreover, it is necessary to prove that $S_1(t,a)$ remains in a precompact subset of $L_1^+(0,+\infty)$ which is independent of x_0. From (2.6) and (3.2), it is clear that $S_1(t,a)$ is non-negative and

$$S_1(t,a) = \begin{cases} P(t-a)B(t-a)K_0(a), & 0 \le a < t; \\ 0, & a \ge t \ge 0. \end{cases} \tag{3.5}$$

Considering Assumption 1 and Remark 1, we have

$$S_1(t,a) \le he^{-(u+c_0)a} B(t-a) \le he^{-(u+c_0)a} \hat{\beta} \|x_0\| \le h^2 \hat{\beta} e^{-(u+c_0)a}. \tag{3.6}$$

Thus, we conclude that (i), (ii) and (iv) of Lemma 2 hold true. Next, we only need to show that the condition (iii) of Lemma 2 holds. Assume that $\tau \in (0,t)$,

$$\begin{aligned} &\int_0^{+\infty} |S_1(t,a+\tau) - S_1(s,a)| da \\ &\quad = \int_0^{t-\tau} |P(t-a-\tau)B(t-a-\tau)K_0(a+\tau) - P(t-a)B(t-a)K_0(a)| da \\ &\qquad + \int_{t-\tau}^{t} |0 - P(t-a)B(t-a)K_0(a)| da \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t-\tau} |P(t-a-\tau)B(t-a-\tau)||K_0(a+\tau)-K_0(a)|da \\
&\quad + \int_0^{t-\tau} |K_0(a)||P(t-a-\tau)B(t-a-\tau)-P(t-a)B(t-a)|da \\
&\quad + \int_{t-\tau}^{t} |P(t-a)B(t-a)K_0(a)|da \\
&\leq h^2\hat{\beta}\int_0^{t-\tau} |K_0(a+\tau)-K_0(a)|da + h^2\hat{\beta}\tau \\
&\quad + \int_0^{t-\tau} |K_0(a)||P(t-a-\tau)B(t-a-\tau)-P(t-a)B(t-a)|da \\
&\doteq h^2\hat{\beta}\int_0^{t-\tau} |K_0(a+\tau)-K_0(a)|da + h^2\hat{\beta}\tau + \Upsilon.
\end{aligned}
$$

Note that $K_0(a)$ is non-increasing with respect to a and $0 \leq K_0(a) \leq e^{-(u+c_0)a} \leq 1$, so

$$
\begin{aligned}
&\int_0^{t-\tau} |K_0(a+\tau)-K_0(a)|da \\
&\quad = \int_0^{t-\tau} K_0(a)da - \int_0^{t-\tau} K_0(a+\tau)da \\
&\quad = \int_0^{t-\tau} K_0(a)da - \int_\tau^{t} K_0(a)da \\
&\quad = \int_0^{t-\tau} K_0(a)da - \int_\tau^{t-\tau} K_0(a)da - \int_{t-\tau}^{t} K_0(a)da \\
&\quad \leq \int_0^{\tau} K_0(a)da - \int_{t-\tau}^{t} K_0(a)da \leq \tau.
\end{aligned}
$$

In the light of (3.1), we have

$$
\left|\frac{dB(t)}{dt}\right| \leq \hat{\beta}\lambda, \left|\frac{dP(t)}{dt}\right| \leq \lambda + uh + \hat{\beta}h^2 + (1-\epsilon_1)\hat{\alpha}h.
$$

Then

$$
\begin{aligned}
&|P(t-a-\tau)B(t-a-\tau)-P(t-a)B(t-a)| \\
&\quad \leq |P(t-a-\tau)||B(t-a-\tau)-B(t-a)| \\
&\qquad + |B(t-a)||P(t-a-\tau)-P(t-a)| \\
&\quad \leq \left\{h\hat{\beta}\lambda + h\hat{\beta}(\lambda + uh + \hat{\beta}h^2 + (1-\epsilon_1)\hat{\alpha}h)\right\}\tau.
\end{aligned}
$$

Furthermore,

$$\begin{aligned}\Upsilon &\leq \tau\{h\hat{\beta}\lambda + h\hat{\beta}(\lambda + uh + \hat{\beta}h^2 + (1-\epsilon_1)\hat{\alpha}h)\}\int_0^{t-\tau} e^{-(u+c_0)a}da \\ &\leq \frac{\tau}{u+c_0}\{h\hat{\beta}\lambda + h\hat{\beta}(\lambda + uh + \hat{\beta}h^2 + (1-\epsilon_1)\hat{\alpha}h)\}.\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^{+\infty} |S_1(t, a+\tau) - S_1(s, a)|da \\ &\quad \leq \tau\left\{2h^2\hat{\beta} + \frac{1}{u+c_0}\left[h\hat{\beta}\lambda + h\hat{\beta}(\lambda + uh + \hat{\beta}h^2 + (1-\epsilon_1)\hat{\alpha}h)\right]\right\}.\end{aligned}$$

The above inequality converges uniformly to 0 as $\tau \to 0$. It means condition (iii) of Lemma 2 holds, and all conditions of Lemma 2 are satisfied. This indicates that $\zeta_1(t, \mathcal{A})$ has compact closure for any $t_{\mathcal{A}} \geq 0$. Therefore, the semiflow ζ defined by (2.8) for system (2.3) is asymptotically smooth. This completes the proof. □

Concerning Proposition 1 and Theorem 1, the following result holds [6, 16].

Theorem 2. *The semiflow ζ defined by* (2.8) *for system* (2.3) *has a global attractor in X and it attracts any bounded subset of X.*

4. Equilibria and Stability

4.1. *Existence and local stability of the equilibria*

Denote $E^*(P^*, S^*)$ as the equilibria of system (2.3), and E^* satisfies the following equations:

$$\begin{cases}\lambda - \displaystyle\int_0^\infty \beta(a)P^*S^*(a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)S^*(a)da - uP^* = 0, \\ \dfrac{dS^*(a)}{da} + (u + \alpha(a))S^*(a) = 0, \\ S^*(0) = P^*\displaystyle\int_0^\infty \beta(a)S^*(a)da = P^*B^*.\end{cases} \tag{4.1}$$

It is clear that there always exists a smoking free equilibrium $E_0(P_0, 0)$, where $P_0 = \lambda/u$. Solving the second equation of (4.1) yields

$$S^*(a) = S^*(0)e^{-\int_0^a (u+\alpha(s))da} = P^*B^*K_0(a), \tag{4.2}$$

then

$$\beta(a)S^*(a) = P^*B^*\beta(a)K_0(a),$$

i.e.

$$B^* = P^* B^* K_2,$$

where K_2 is defined by (2.5). Thus, $P^* = 1/K_2$. Considering the first equation of (4.1) and making use of (4.2), we have

$$\lambda - P^* B^* + P^*(1-\epsilon_1) B^* K_1 - u P^* = 0, \tag{4.3}$$

where K_1 is defined by (2.5), solving Eq. (4.3) with respect to B^*,

$$B^* = \frac{\lambda - uP^*}{P^* - P^*(1-\epsilon_1)K_1}.$$

Substituting the expressions of P^* and B^* into (4.2) we get

$$S^*(a) = \frac{(\lambda K_2 - u)K_0(a)}{K_2 - (1-\epsilon_1)K_1 K_2}. \tag{4.4}$$

Notice that K_1 is less than 1, so $S^*(a) > 0$ if and only if $\lambda K_2 - u > 0$, or $R_0 > 1$, where

$$R_0 = \frac{\lambda K_2}{u}.$$

System (2.3) exists with a positive endemic equilibrium $E^*(P^*, S^*)$ provided that $R_0 > 1$.

Theorem 3. *System* (2.3) *always exists with a smoking free equilibrium* E_0. *If* $R_0 > 1$, *then there is a positive endemic equilibrium* E^* *for system* (2.3).

Theorem 3 *provided the conditions for the existence of the equilibria, in the following we investigate the local stability of these equilibria.*

Theorem 4. *The equilibrium* E_0 *of system* (2.3) *is locally stable if* $R_0 < 1$.

Proof. To show the local stability of the equilibrium E_0, we need to consider the linearized model of system (2.3) at E_0. To this end, let $y_1(t) = P(t) - P_0$ and $y_2(t,a) = S(t,a)$, then substituting these into system (2.3), we get the corresponding linearized system at E_0:

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -uy_1(t) - P_0 \displaystyle\int_0^\infty \beta(a) y_2(t,a)da + \int_0^\infty (1-\epsilon_1)\alpha(a) y_2(t,a)da, \\ \dfrac{dy_2(t,a)}{dt} + \dfrac{dy_2(t,a)}{da} = -(u+\alpha(a))y_2(t,a), \\ y_2(t,0) = P_0 \displaystyle\int_0^\infty \beta(a) y_2(t,a)da. \end{cases} \tag{4.5}$$

Let $y_1(t) = \tilde{y_1} e^{\theta t}$ and $y_2(t,a) = \tilde{y_2}(a) e^{\theta t}$, from (4.5) we get

$$\begin{cases} \theta \tilde{y_1} = -u\tilde{y_1} - P_0 \displaystyle\int_0^\infty \beta(a)\tilde{y_2}(a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)\tilde{y_2}(a)da, \\ \theta \tilde{y_2}(a) + \dfrac{d\tilde{y_2}(a)}{da} = -(u+\alpha(a))\tilde{y_2}(a), \\ \tilde{y_2}(0) = P_0 \displaystyle\int_0^\infty \beta(a)\tilde{y_2}(a)da. \end{cases} \tag{4.6}$$

Integrating the second equation of (4.6) from 0 to a yields

$$\tilde{y_2}(a) = \tilde{y_2}(0) e^{-\int_0^a (u+\alpha(s)+\theta)ds}. \tag{4.7}$$

Putting (4.7) into the third equation of (4.6),

$$\mathcal{L}(\theta) \doteq \frac{\lambda}{u}\int_0^\infty \beta(a) e^{-\int_0^a (u+\alpha(s)+\theta)ds} da - 1 = 0. \tag{4.8}$$

Obviously,

$$\mathcal{L}(0) = R_0 - 1, \quad \lim_{\theta\to+\infty}\mathcal{L}(\theta) = 0, \quad \lim_{\theta\to-\infty}\mathcal{L}(\theta) = \infty, \quad \frac{d}{d\theta}\mathcal{L}(\theta) < 0.$$

Hence, if $R_0 > 1$, then $\mathcal{L}(\theta)$ has a unique positive real root θ^*, i.e. $R_0 > 1$ implies that E_0 is unstable. If $R_0 < 1$, then $\theta^* < 0$, i.e. E_0 is locally stable. Otherwise, denote $\theta = a_1 + ib_1$ as any complex root of $\mathcal{L}(\theta)$ with real part $a_1 \geq 0$. However,

$$|\mathcal{L}(a_1 + ib_1)| = \frac{\lambda}{u}\int_0^\infty \beta(a) e^{-\int_0^a (u+\alpha(s))ds} da - 1 = R_0 - 1 < 0.$$

It indicates that if $R_0 < 1$, all roots of $\mathcal{L}(\theta)$ exist with negative real parts, then E_0 is locally stable. This completes the proof. $\square$

Theorem 5. *The equilibrium E^* of system* (2.3) *is locally stable if $R_0 > 1$.*

Proof. Let $y_1(t) = P(t) - P^*$ and $y_2(t,a) = S(t,a) - S^*$, then we obtain the linearized system at E_0 of system (2.3):

$$\begin{cases} \dfrac{dy_1(t)}{dt} = -uy_1(t) - P^* \displaystyle\int_0^\infty \beta(a) y_2(t,a)da - \int_0^\infty \beta(a)S^* y_1(t)da \\ \qquad\qquad + \displaystyle\int_0^\infty (1-\epsilon_1)\alpha(a) y_2(t,a)da, \\ \dfrac{dy_2(t,a)}{dt} + \dfrac{dy_2(t,a)}{da} = -(u+\alpha(a))y_2(t,a), \\ y_2(t,0) = P_0 \displaystyle\int_0^\infty \beta(a)y_2(t,a)da + \int_0^\infty \beta(a)S^* y_1(t)da. \end{cases} \tag{4.9}$$

Then taking the exponential solutions $y_1(t) = \tilde{y_1}e^{\theta t}$ and $y_2(t,a) = \tilde{y_2}(a)e^{\theta t}$ into account and putting these into system (4.9)

$$\begin{cases} \theta\tilde{y_1} = -u\tilde{y_1} - \int_0^\infty \beta(a)\tilde{y_1}S^* da - \int_0^\infty \beta(a)P^*\tilde{y_2}(a)da \\ \qquad + \int_0^\infty (1-\epsilon_1)\alpha(a)\tilde{y_2}(a)da, \\ \theta\tilde{y_2}(a) + \dfrac{d\tilde{y_2}(a)}{da} = -(u+\alpha(a))\tilde{y_2}(a), \\ \tilde{y_2}(0) = \int_0^\infty \beta(a)\tilde{y_1}S^* da + \int_0^\infty \beta(a)P^*\tilde{y_2}(a)da. \end{cases} \tag{4.10}$$

Solving the second equation of (4.10) yields

$$\tilde{y_2}(a) = \tilde{y_2}(0)e^{-\int_0^a (u+\alpha(s)+\theta)ds}. \tag{4.11}$$

Combining (4.11) with the third equation of (4.10),

$$\tilde{y_2}(0) = \int_0^\infty \beta(a)\tilde{y_1}S^* da + \int_0^\infty \beta(a)P^*\tilde{y_2}(0)e^{-\int_0^a (u+\alpha(s)+\theta)ds}da. \tag{4.12}$$

From the first equation of (4.10) we have

$$\tilde{y_1} = \frac{-\int_0^\infty \beta(a)P^*\tilde{y_2}(a)da + \int_0^\infty (1-\epsilon_1)\alpha(a)\tilde{y_2}(a)da}{\theta + u + \int_0^\infty \beta(a)\tilde{y_1}S^* da}. \tag{4.13}$$

Substituting (4.13) into (4.12) and after simplification we get

$$\begin{aligned} \mathcal{L}_1(\theta) \doteq\ & \frac{B^* \int_0^\infty (1-\epsilon_1)\alpha(a)e^{-\int_0^a (u+\theta+\alpha(a))ds}da}{\theta + u + B^*} \\ & + \frac{(\theta + u)\int_0^\infty \beta(a)P^* e^{-\int_0^a (u+\theta+\alpha(a))ds}da}{\theta + u + B^*} = 1. \end{aligned}$$

If $R_0 > 1$, then E^* is locally stable. Otherwise, denote $\theta = a_2 + ib_2$ as any complex root of $\mathcal{L}_1(\theta)$ with real part $a_2 \geq 0$. However,

$$\begin{aligned} |\mathcal{L}(a_2 + ib_2)| &\leq \frac{B^*(1-\epsilon_1)\int_0^\infty \alpha(a)K_0(a)da + uP^*\int_0^\infty \beta(a)K_0(a)da}{u + B^*} \\ &= \frac{B^*(1-\epsilon_1)K_1 + uP^*K_2}{u + B^*} \\ &\leq \frac{B^*(1-\epsilon_1) + u}{u + B^*} \leq 1, \end{aligned}$$

which is a contradiction. It implies that if $R_0 > 1$, all roots of $\mathcal{L}_1(\theta) = 1$ exist with negative real parts, then E^* is locally stable. This completes the proof. $\square$

4.2. *Uniform persistence*

This subsection deals with the uniform persistence of system (2.3) when $R_0 > 1$. Define

$$M_0 = \left\{ (P(t), S(t,a))^T \in X \,\middle|\, \int_0^\infty S(t,a)da > 0 \right\},$$

let $\partial M_0 = X \backslash M_0$ and $X = M_0 \cup \partial M_0$.

Proposition 2. *Under the semiflow $\zeta(t,\cdot)$, the sets M_0 and ∂M_0 are both positively invariant.*

Theorem 6. *The equilibrium E_0 of system* (2.3) *is globally asymptotically stable for the semiflow $\{\zeta(t,\cdot)\}_{t\geq 0}$ restricted to ∂M_0.*

Proof. Notice that $P(t) \leq \lambda/u$ as $t \to \infty$. Hence, $S(t,a) \leq \tilde{S}(t,a)$ where $\tilde{S}(t,a)$ satisfies

$$\begin{cases} \dfrac{d\tilde{S}(t,a)}{dt} + \dfrac{d\tilde{S}(t,a)}{da} = -(u + \alpha(a))\tilde{S}(t,a), \\ \tilde{S}(t,0) = \displaystyle\int_0^\infty \beta(a)P(t)\tilde{S}(t,a)da, \quad \tilde{S}(0,a) = S_0(a). \end{cases}$$

It follows from (3.2)–(3.4) and (3.6) that we get $\lim_{t\to+\infty} \tilde{S}(t,a) = 0$, which means $\lim_{t\to+\infty} S(t,a) = 0$. Furthermore, the first equation of (2.3) leads to $\lim_{t\to+\infty} P(t) = P_0$. Therefore, $\lim_{t\to+\infty}(P(t), S(t,a)) = (P_0, 0)$, i.e. the equilibrium E_0 of system (2.3) is globally asymptotically stable for the semiflow $\zeta(t,\cdot)$ restricted to ∂M_0. This completes the proof. $\square$

Theorem 7. *If $R_0 > 1$, then the semiflow $\{\zeta(t,\cdot)\}_{t\geq 0}$ is uniformly persistent with regard to the decomposition $(M_0, \partial M_0)$, and there is a compact subset $\mathfrak{A}_0 \subset M_0$ for $\{\zeta(t,\cdot)\}_{t\geq 0}$ in X.*

Proof. Notice that E_0 is globally asymptotically stable in ∂M_0, let

$$W_s(E_0) = \left\{ x \in X \,\middle|\, \lim_{t\to\infty} \zeta(t,x) = E_0 \right\},$$

then we only need to ensure that $W_s(E_0) \cap M_0 = \emptyset$. Otherwise, there exists a $\tilde{x} \in M_0$ such that $\tilde{x} \in W_s(E_0)$. Thus, there is a list of $\{\tilde{x}_n\} \in M_0$ and it satisfies $\|\zeta(t,\tilde{x}_n) - E_0\|_X < n$ $(t \geq 0)$. Note that $Q_t(t) = 0$ at E_0, let $\zeta(t,\tilde{x}_n) = (P^n(t), S^n(t,\cdot), Q_t^n(t))$. For $t \geq 0$ we have

$$P_0 - \frac{1}{n} < P^n(t) < P_0 + \frac{1}{n}, \quad 0 \leq Q_t^n(t) \leq \frac{1}{n}.$$

Equation (2.6) leads to $S(t,a) \geq P(t-a)B(t-a)K_0(a)$. Concerning these inequalities and the third equation of system (2.1) yields $Q_t^n(t) \geq Q_n(t)$, where

$$\begin{cases} \dfrac{dQ_n(t)}{dt} = \displaystyle\int_0^\infty \epsilon_1(1-\epsilon_2)\alpha(\tau)\left(P_0 - \frac{1}{n}\right)B(t-\tau)K_0(\tau)d\tau - uQ_n(t), \\ Q_n(0) = Q_t^n(0), \end{cases}$$

If $R_0 > 1$, the large $n > 0$ implies that

$$\begin{aligned} &\left(P_0 - \frac{1}{n}\right)\int_0^\infty \epsilon_1(1-\epsilon_2)\alpha(\tau)B(t-\tau)K_0(\tau)d\tau \\ &\quad \geq \left(P_0 - \frac{1}{n}\right)\int_0^\infty uQ_n(t)\alpha(\tau)K_2(\tau)d\tau \geq uQ_n(t). \end{aligned}$$

It follows from [1] that $Q_t^n(t)$ is unbounded, and then $Q_t^n(t)$ is unbounded. It implies that $\zeta(t,\tilde{x}_n)$ is unbounded, which contradicts the boundedness of $Q_t^n(t)$. Therefore, $W_s(E_0) \cap M_0 = \emptyset$ holds, so we conclude that semiflow $\{\zeta(t,\cdot)\}_{t\geq 0}$ is uniformly persistent. Furthermore, from [16] we can find a compact subset $\mathfrak{A}_0 \subset M_0$ for $\{\zeta(t,\cdot)\}_{t\geq 0}$ in X, which is a global attractor. This completes the proof. □

4.3. *Global stability of the equilibria*

This part mainly deals with the global stability of system (2.3), for which we first introduce a very useful function [9].

Proposition 3. *For the Volterra function* $M(x) = x - 1 - \ln x$, *it is clear that* $M(x) \geq 0$ *if* $x > 0$ *and* $M(1) = 0$ *is a global minimum.*

Theorem 8. *If* $R_0 < 1$, *then the equilibrium* E_0 *of system* (2.3) *is globally asymptotically stable.*

Proof. Define a positive function $L_1(a)$ as

$$L_1(a) = \int_a^\infty P_0\beta(\tau)e^{-\int_a^\tau(u+\alpha(s))ds}d\tau,$$

then it is clear that $L_1(a) > 0 (a \geq 0)$ and

$$L_1(0) = P_0\int_0^\infty \beta(\tau)e^{-\int_0^\tau(u+\alpha(s))ds}d\tau = P_0K_2 = R_0.$$

Further, taking the derivative of $L_1(a)$ with respect to a yields

$$\begin{aligned} \frac{dL_1(a)}{da} &= L_1(a)(u+\alpha(a)) - e^{\int_0^a(u+\alpha(s))ds}P_0\beta(a)e^{-\int_0^a(u+\alpha(s))ds} \\ &= L_1(a)(u+\alpha(a)) - P_0\beta(a). \end{aligned} \tag{4.14}$$

Considering any solution $(P(t), S(t,a))$ of system (2.3), we define the Lyapunov function $V(t)$ as follows:

$$V(t) = P_0M\left(\frac{P(t)}{P_0}\right) + \int_0^\infty L_1(a)S(t,a)da \doteq V_1(t) + V_2(t).$$

We calculate the derivative of $V_1(t)$ along with the solutions of system (2.3),

$$\begin{aligned}\frac{dV_1(t)}{dt} &= \left(1-\frac{P_0}{P(t)}\right)\left(\lambda-\int_0^\infty \beta(a)P(t)S(t,a)da\right.\\ &\quad\left.+\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da - uP(t)\right)\\ &= uP_0\left(2-\frac{P_0}{P(t)}-\frac{P(t)}{P_0}\right)-\int_0^\infty \beta(a)P(t)S(t,a)da\\ &\quad+\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da+\int_0^\infty \beta(a)P_0S(t,a)da\\ &\quad-\frac{P_0}{P(t)}\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da. \end{aligned} \tag{4.15}$$

Concerning (4.14) and the derivative of $V_2(t)$ along the solutions of system (2.3) yields

$$\begin{aligned}\frac{dV_2(t)}{dt} &= -\int_0^\infty L_1(a)\frac{\partial}{\partial a}S(t,a)da-\int_0^\infty L_1(a)(u+\alpha(a))S(t,a)da\\ &= -L_1(a)S(t,a)|_{a=0}^{a=\infty}+\int_0^\infty S(t,a)\frac{dL_1(a)}{da}da\\ &\quad-\int_0^\infty L_1(a)(u+\alpha(a))S(t,a)da\\ &= L_1(0)S(t,0)+\int_0^\infty S(t,a)\left(L_1(a)(u+\alpha(a))-P_0\beta(a)\right)da\\ &\quad-\int_0^\infty L_1(a)(u+\alpha(a))S(t,a)da\\ &= L_1(0)S(t,0)-\int_0^\infty S(t,a)P_0\beta(a)da. \end{aligned} \tag{4.16}$$

Hence,

$$\begin{aligned}\frac{dV(t)}{dt} &= uP_0\left(2-\frac{P_0}{P(t)}-\frac{P(t)}{P_0}\right)+L_1(0)S(t,0)-\int_0^\infty \beta(a)P(t)S(t,a)da\\ &\quad+\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da-\frac{P_0}{P(t)}\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da\\ &= uP_0\left(2-\frac{P_0}{P(t)}-\frac{P(t)}{P_0}\right)+(R_0-1)S(t,0)\\ &\quad+\frac{P(t)-P_0}{P(t)}\int_0^\infty (1-\epsilon_1)\alpha(a)S(t,a)da. \end{aligned} \tag{4.17}$$

From Proposition 1, $P(t) \le \lambda/u = P_0$ holds. Thereby, if $R_0 < 1$, then $dV(t)/dt \le 0$. $dV(t)/dt = 0$ holds only for $P(t) = P_0$ and $S(t,a) = 0$, i.e. the equality holds only at the equilibrium E_0. Therefore, $\{E_0\} \in \Omega$ is the largest invariant subset of $\{(P(t), S(t,a)) | dV(t)/dt = 0\}$, it follows from the Lyapunov–LaSalle theorem for semiflows that the equilibrium E_0 is globally asymptotically stable if $R_0 < 1$. This completes the proof. $\square$

Theorem 9. *If $R_0 > 1$ and $\epsilon_1 = 1$, then the equilibrium E^* of system* (2.3) *is globally asymptotically stable.*

Proof. Introduce a function $L_2(a)$ such that $L_2(a) > 0 (a \ge 0)$, where

$$L_2(a) = \int_a^\infty P^* \beta(\tau) e^{-\int_a^\tau (u+\alpha(s))ds} d\tau$$

with

$$L_2(0) = \int_0^\infty P^* \beta(\tau) e^{-\int_0^\tau (u+\alpha(s))ds} d\tau = P^* K_2.$$

The derivative of $L_2(a)$ with respect to a yields

$$\begin{aligned} \frac{dL_2(a)}{da} &= L_2(a)(u+\alpha(a)) - e^{\int_0^a (u+\alpha(s))ds} P^* \beta(a) e^{-\int_0^a (u+\alpha(s))ds} \\ &= L_2(a)(u+\alpha(a)) - P^* \beta(a). \end{aligned} \tag{4.18}$$

Defining the Lyapunov function $U(t)$ as follows:

$$U(t) = P^* M\left(\frac{P(t)}{P^*}\right) + \int_0^\infty L_2(a) S^*(a) M\left(\frac{S(t,a)}{S^*(a)}\right) da \doteq U_1(t) + U_2(t).$$

In the light of system (2.3), the derivative of $U_1(t)$ yields

$$\begin{aligned} \frac{dU_1(t)}{dt} &= \left(1 - \frac{P^*}{P(t)}\right)\left(\lambda - uP(t) - \int_0^\infty \beta(a) P(t) S(t,a) da\right) \\ &= uP^*\left(2 - \frac{P^*}{P(t)} - \frac{P(t)}{P^*}\right) + \int_0^\infty \beta(a) P^* S^*(a) \\ &\quad \times \left(1 - \frac{P(t)S(t,a)}{P^* S^*(a)} - \frac{P^*}{P(t)} + \frac{S(t,a)}{S^*(a)}\right) da. \end{aligned} \tag{4.19}$$

To get the expression of $dU_2(t)/dt$, we first need to calculate the derivative of $M(S(t,a)/S^*(a))$ with respect to a, i.e.

$$\begin{aligned} \frac{\partial}{\partial a} M\left(\frac{S(t,a)}{S^*(a)}\right) &= \frac{\partial}{\partial a}\left(\frac{S(t,a)}{S^*(a)} - 1 - \ln\left(\frac{S(t,a)}{S^*(a)}\right)\right) \\ &= \frac{\partial}{\partial a}\left(\frac{S(t,a)}{S^*(a)}\right) - \frac{S^*(a)}{S(t,a)} \frac{\partial}{\partial a}\left(\frac{S(t,a)}{S^*(a)}\right) \end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{S^*(a)}{S(t,a)}\right)\left(\frac{\frac{\partial S(t,a)}{\partial a}S^*(a) - S(t,a)\frac{\partial S^*(a)}{\partial a}}{(S^*(a))^2}\right)\\
&= \left(1 - \frac{S^*(a)}{S(t,a)}\right)\left(\frac{1}{S^*(a)}\frac{\partial S(t,a)}{\partial a} + \frac{S(t,a)}{S^*(a)}(u+\alpha(a))\right)\\
&= \left(\frac{1}{S^*(a)} - \frac{1}{S(t,a)}\right)\frac{\partial S(t,a)}{\partial t}\\
&= -\frac{\partial}{\partial t}M\left(\frac{S(t,a)}{S^*(a)}\right).
\end{aligned} \tag{4.20}
$$

It follows from (4.20) that

$$
\begin{aligned}
\frac{dU_2(t)}{dt} &= \int_0^\infty L_2(a)S^*(a)\frac{\partial}{\partial t}M\left(\frac{S(t,a)}{S^*(a)}\right)da\\
&= -\int_0^\infty L_2(a)S^*(a)\frac{\partial}{\partial a}M\left(\frac{S(t,a)}{S^*(a)}\right)da\\
&= -L_2(a)S^*(a)M\left(\frac{S(t,a)}{S^*(a)}\right)\Bigg|_{a=0}^{a=\infty}\\
&\quad + \int_0^\infty M\left(\frac{S(t,a)}{S^*(a)}\right)\frac{\partial}{\partial a}(L_2(a)S^*(a))da\\
&= L_2(0)S^*(0)M\left(\frac{S(t,0)}{S^*(0)}\right) - \int_0^\infty P^*S^*(a)\beta(a)M\left(\frac{S(t,a)}{S^*(a)}\right)da\\
&= \int_0^\infty P^*S^*(a)\beta(a)\left(M\left(\frac{P(t)B(t)}{P^*B^*}\right) - M\left(\frac{S(t,a)}{S^*(a)}\right)\right)da.
\end{aligned} \tag{4.21}
$$

From (4.19) and (4.21),

$$
\begin{aligned}
\frac{dU(t)}{dt} &= uP^*\left(2 - \frac{P^*}{P(t)} - \frac{P(t)}{P^*}\right)\\
&\quad + \int_0^\infty \beta(a)P^*S^*(a)\left(M\left(\frac{P(t)B(t)}{P^*B^*}\right) - M\left(\frac{S(t,a)}{S^*(a)}\right)\right)da\\
&\quad + \int_0^\infty \beta(a)P^*S^*(a)\left(1 - \frac{P(t)S(t,a)}{P^*S^*(a)} - \frac{P^*}{P(t)} + \frac{S(t,a)}{S^*(a)}\right)da\\
&= uP^*\left(2 - \frac{P^*}{P(t)} - \frac{P(t)}{P^*}\right) - \int_0^\infty \beta(a)P^*S^*(a)M\left(\frac{P^*}{P(t)}\right)da \le 0.
\end{aligned} \tag{4.22}
$$

Moveover, $dU(t)/dt = 0$ if and only if $P(t) = P^*$ and $S(t,a) = S^*(a)$, i.e. $dU(t)/dt = 0$ holds only at the equilibrium E^*. Therefore, the largest invariant subset of $\{(P(t), S(t,a))|dU(t)/dt = 0\}$ is $\{E^*\} \in \Omega$, according to the Lyapunov–LaSalle theorem for semiflows, the equilibrium E^* is globally asymptotically stable if $R_0 > 1$. This completes the proof. $\square$

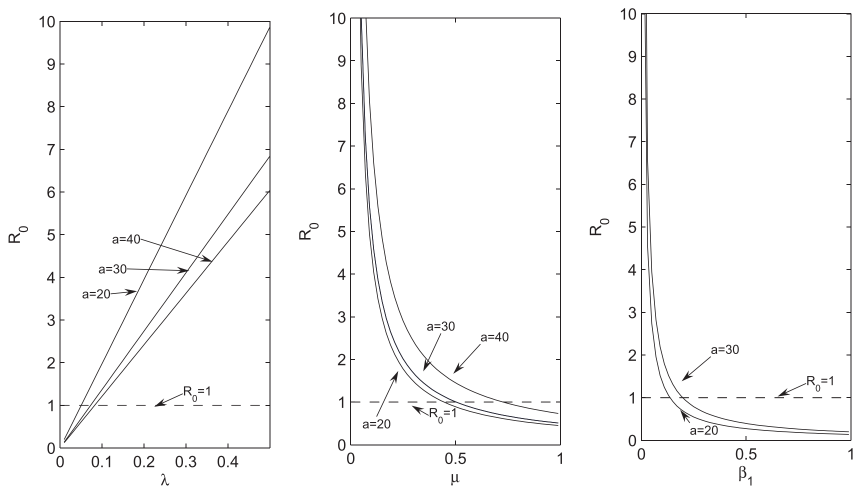
From Theorem 9, the equilibrium E^* of system (2.3) is globally asymptotically stable provided that $R_0 > 1$ and $\epsilon_1 = 1$. However, we cannot determine whether it is also true when $\epsilon_1 \in [0, 1)$. To this end, we will discuss the results when $\epsilon_1 \neq 1$ by means of numerical investigations in the next section.

5. Numerical Investigations and Discussion

In order to carry out numerical analysis to support the theoretical results, we assume that the age-dependent transmission rate $\beta(a)$ and the age-dependent smoking quitting rate $\alpha(a)$ have the following expressions:

$$\beta(a) = \beta_1\left(1 + \sin\frac{(a-10)\pi}{20}\right), \quad \alpha(a) = 0.01\left(1 + \sin\frac{(a-10)\pi}{20}\right).$$

Firstly, it is very important to show how parameter values affect the final states of the chain smokers. Since R_0 is a threshold which determines the stability of the equilibria and contains all important parameters of model (2.3), we carry out sensitivity analysis to address the effects of parameters on the threshold R_0. If we fix all parameter values as shown in Fig. 1, it is found that R_0 is increasing when λ increases, but R_0 decreases once the age a increases. In view of Theorems 4 and 8, this indicates that increasing the lower smoking age a and decreasing the constant



(a) (b) (c)

Fig. 1. Sensitivities of the threshold condition R_0 with respect to key parameters, we set $u = 0.0736$, $\alpha(a) = 0.01(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$ and $\beta(a) = \beta_1(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$. (a) $\beta_1 = 0.1$; (b) $\beta_1 = 0.1$ and $\lambda = 0.5$; (c) $\lambda = 0.1$.

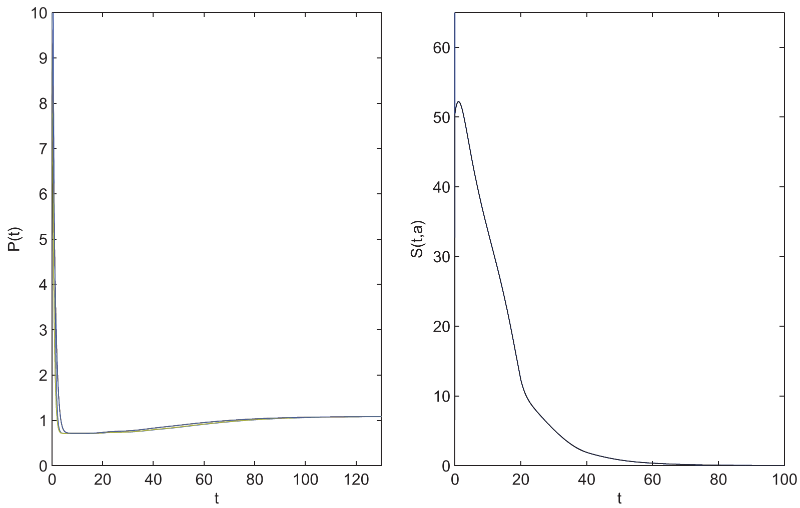
recruitment rate λ will make the smoking free equilibrium E_0 stable (Fig. 1(a)). In Figs. 1(b) and 1(c), it is clear that R_0 is decreasing when u and β_1 increase, R_0 is increasing when u and β_1 increase. Thus, increasing the death rate u and transmission rate β_1 and at the same time decreasing the age a will stabilize E_0. Meanwhile, because

$$S^*(a) = \frac{(\lambda K_2 - u)K_0(a)}{K_2 - (1-\epsilon_1)K_1K_2},$$

so a stronger determination ϵ_1 will decrease the final state of $S^*(a)$, i.e. a larger determination ϵ_1 will lead to a high quitting rate for chain smokers. Thus, feasible ways to give up smoking include: strengthening the determination to quit smoking, decreasing the constant recruitment rate and increasing the age, increasing both the death rate and transmission rate and decreasing the age.

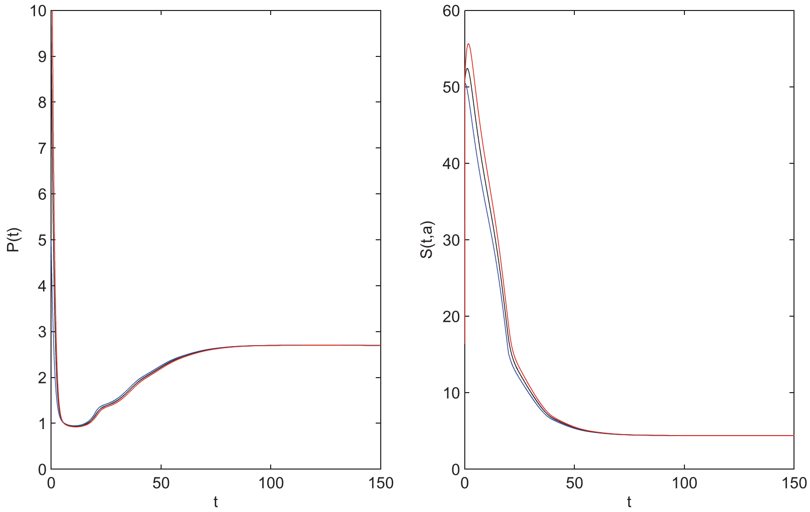
In Fig. 2, with the parameter values fixed as shown in Fig. 1(a) with $a = 20$, it is observed that $R_0 < 1$ holds when $\lambda = 0.05$. In fact, by simple calculation we have $R_0 \approx 0.955 < 1$. It follows from Theorems 4 and 8 that the smoking free equilibrium E_0 is globally asymptotically stable. Because R_0 is a monotone increasing function with respect to λ, when λ is increased, the value of R_0 will increase and be greater than 1. For example, fixing $\lambda = 0.05$ indicates $R_0 \approx 9.865 > 1$, the results of Theorems 5 and 9 imply that the smoking endemic equilibrium E^* is globally asymptotically stable (Fig. 3).

In Theorem 9, we proved that the equilibrium E^* of system (2.3) is globally asymptotically stable when $R_0 > 1$ and $\epsilon_1 = 1$. But we cannot determine whether



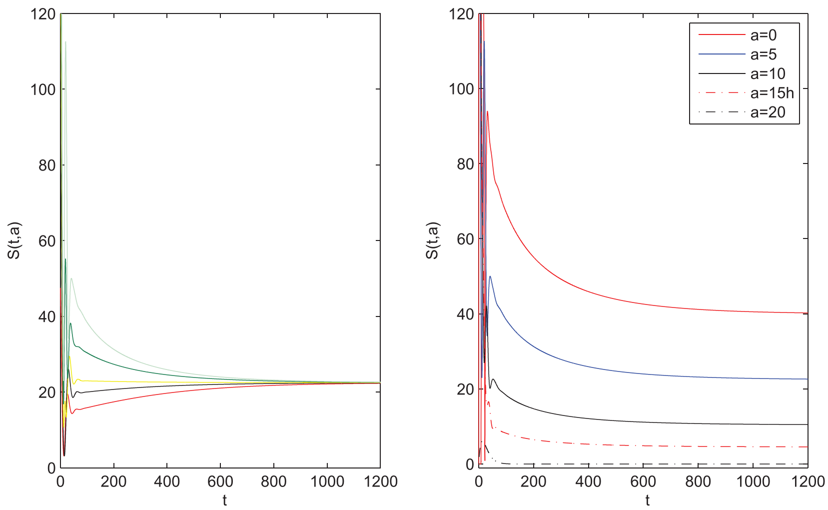
(a) (b)

Fig. 2. Time series of potential smokers $P(t)$ and chain smokers $S(t,a)$ with different initial conditions. The parameters were fixed as: $\lambda = 0.05$, $u = 0.0736$, $a = 20$, $\epsilon_1 = 0.3$, $\alpha(a) = 0.01(1+\sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$ and $\beta(a) = 0.1(1+\sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$.



(a) (b)

Fig. 3. Time series of potential smokers $P(t)$ and chain smokers $S(t,a)$ with different initial conditions. The parameters were fixed as: $\lambda = 0.5$, $u = 0.0736$, $a = 20$, $\epsilon_1 = 0.3$, $\alpha(a) = 0.01(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$ and $\beta(a) = 0.1(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$.



(a) (b)

Fig. 4. Time series of $S(t,a)$ showing the global stability of E^*. (a) for $\epsilon_1 \in [0,1)$ with different initial conditions; (b) at different fixed ages a. The parameters were fixed as: $\lambda = 0.2$, $u = 0.0736$, $a = 20$, $\epsilon_1 = 0.2$, $\alpha(a) = 0.01(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$ and $\beta(a) = 0.1(1 + \sin\frac{(a-10)\pi}{20})(0 \le a \le 20)$.

it is also true when $\epsilon_1 \in [0, 1)$. To this end, fix $\lambda = 0.2$ and $\epsilon_1 = 0.2$ such that $R_0 \approx 3.946 > 1$, it is observed that the solutions of system (2.3) with different initial values tend to the equilibrium E^* when t is large enough (Fig. 4(a)). In this case, we also find that the equilibrium E^* of system (2.3) is also globally asymptotically stable at different smoking ages (Fig. 4(b)), the final states of the chain smokers $S(t, a)$ decreases when the age increases. The main reasons may be that as the smoking age increases, the chain smokers may be affected by many interventions including media reports on the dangers of smoking, education, their determination, expostulations from their families, and so on (as can be seen in Eq. (4.2)).

6. Conclusions

It has been known that smoking has caused a series of public health problems [7], and many scholars have tried to construct different types of mathematical models to explore the internal transmission mechanisms of smoking [4, 8, 18, 20, 22, 29]. In this study, we proposed a more generalized age-structured smoking dynamical model with interventions to evaluate the effectiveness of the important parameters on giving up smoking.

We first studied the main properties of the solutions including well-posedness and asymptotic smoothness, by defining the semiflow of system (2.3) and showing that it is globally attractive. Then we derived the explicit expression of the smoking generation number R_0 which determines the global stability of the boundary equilibrium E_0 and the endemic equilibrium E^*. If $R_0 < 1$, then the smoking free equilibrium E_0 is globally asymptotically stable, if $R_0 > 1$, then the endemic equilibrium E^* is globally asymptotically stable. Biologically, numerical simulation not only verified the theoretical results but also suggested feasible ways to give up smoking such as strengthening the determination to quit smoking, decreasing the constant recruitment rate and increasing the age, increasing both the death rate and transmission rate and decreasing the age. On the one hand, we discussed the relationship among age, transmission rate, smoking quitting rate and smoking generation number. On the other hand, the effectiveness of the key parameters for quitting smoking was evaluated. Therefore, we have solved two problems raised in the introduction.

Compared to the previous studies [18, 28], highlights of this paper included (1) consideration of a more generalized age-structured smoking model with treatments; (2) an age parameter was included in the threshold condition R_0, which indicated that the age effect has a substantial effect on the stability of the system; (3) discussion of the effects of the treatment parameters and biological significance.

There are still many problems worthy of further study. For example, it is believed that there is a relationship between media reports and smoking cessation [22], but how to consider the role of media reports in the proposed model is challenging. Media reports may raise individuals' awareness of quitting smoking [13], and closely related to the final states of the chain smokers [11, 17]. Another very challenging question is whether we could investigate media impact by employing a piecewise smooth function to model the individuals' awareness depending on the number of chain smokers [25, 26].

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References

[1] C. J. Browne and S. S. Pilyugin, Global analysis of age-structured within-host virus model, *Discrete Contin. Dyn. Syst. Ser. B* **18** (2013) 1999–2017.
[2] B. Cao and H. F. Huo and H. Xiang, Global stability of an age-structure epidemic model with imperfect vaccination and relapse, *Phys. A* **486** (2017) 638–655.
[3] J. M. Cushing, Existence and stability of equilibria in age-structured population dynamics, *J. Math. Biol.* **20** (1984) 259–276.
[4] G. C. Castillo, S. G. Jordan and H. A. Rodriguez, Mathematical models for dynamics of tobacco use, recovery and relapse, Technical Report Series BU-1505-M, Cornell University (2000).
[5] Y. X. Dang, Z. P. Qiu and X. Z. Li, Global dynamics of a vector-host epidemic model with age of infection, *Math. Biosci. Eng.* **14** (2017) 1159–1186.
[6] E. M. C. D'Agata, P. Magal and S. Ruan, Asymptotic behavior in nosocomial epidemic models with antibiotic resistance, *Differ. Integral Equ.* **19** (2006) 573–600.
[7] O. K. Ham, Stages and processes of smoking cessation among adolescents, *West J. Nurs. Res.* **29** (2007) 301–315.
[8] G. Howe, C. Westhoff, M. Vessey and D. Yeates, Effects of age, cigarette smoking, and other factors on fertility: Findings in a large prospective study, *Br. Med. J.* **290** (1985) 1697–1700.
[9] G. Huang, X. Liu and Y. Takeuchi, Lyapunov functions and global stability for age-structured HIV infection model, *SIAM J. Appl. Math.* **72** (2012) 25–38.
[10] M. Iannelli, *Mathematical Theory of Age-Structured Population Dynamics* (Giadini Editorie Stampatori, Pisa, 1994).
[11] I. Z. Kiss, J. Cassell, M. Recker and P. L. Simon, The impact of information transmission on epidemic outbreaks, *Math. Biosci.* **255** (2010) 1–10.
[12] A. Lahrouz, L. Omari, D. Kiouach and A. Belmaati, Deterministic and stochastic stability of a mathematical model of smoking, *Stat. Probab. Lett.* **81** (2011) 1276–1284.
[13] H. Liu and W. Tan, The effect of anti-smoking media campaign on smoking behavior: The California experience, *Ann. Econ. Finance* **10** (2009) 29–47.
[14] Z. J. Liu, J. Chen, J. H. Pang, P. Bi and S. G. Ruan, Modeling and analysis of a nonlinear age-structured model for tumor cell populations with quiescence, *J. Nonlinear Sci.* **28** (2018) 1763–1791.

[15] P. Magal, Compact attractors for time-periodic age-structured population models, *Electron. J. Differ. Equ.* **65** (2001) 229–262.
[16] P. Magal and X. Q. Zhao, Global attractors and steady states for uniformly persistent dynamical systems, *SIAM J. Appl. Math.* **37** (2005) 251–275.
[17] A. K. Misra, A. Sharma and J. B. Shukla, Modeling and analysis of effects of awareness programs by media on the spread of infectious diseases, *Math. Comput. Model.* **53** (2011) 1221–1228.
[18] G. Rahmana, R. P. Agarwal, L. L. Liu and A. Khana, Threshold dynamics and optimal control of an age-structured giving up smoking model, *Nonlinear Anal. Real World Appl.* **43** (2018) 96–120.
[19] O. Sharomi and A. B. Gumel, Curtailing smoking dynamics: A mathematical modeling approach, *Appl. Math. Comput.* **195** (2008) 475–499.
[20] J. Singh, D. Kumar, M. A. Qurash and D. Baleanu, A new fractional model for giving upsmoking dynamics, *Adv. Differ. Equ.* **2017** (2017) 88.
[21] H. L. Smith and H. R. Thieme, *Dynamical Systems and Population Persistence* (American Mathematical Society, Providence, RI, 2011).
[22] V. Verma and M. Agarwal, Global dynamics of a mathematical model on smoking with media campaigns, *Res. Desk* **4** (2015) 500–512.
[23] J. L. Wang, R. Zhang and T. Kuniya, Mathematical analysis for an age-structured HIV infection model with saturation infection rate, *Electron. J. Differ. Equ.* **33** (2015) 1–19.
[24] G. Webb, *Theory of Nonlinear Age-Dependent Population Dynamics* (Marcel Dekker, New York, 1985).
[25] Y. N. Xiao, S. Y. Tang and J. H. Wu, Media impact switching surface during an infectious disease outbreak, *Sci. Rep.* **5** (2015) 7838.
[26] Y. N. Xiao, T. T. Zhao and S. Y. Tang, Dynamics of an infectious disease with media psychology induced non-smooth incidence, *Math. Biosci. Eng.* **10** (2013) 445–461.
[27] A. Yadav, P. K. Srivastava and A. Kumar, Mathematical model for smoking: Effect of determination and education, *Int. J. Biomath.* **8** (2015) 1550001.
[28] A. Zeb, S. Hussain, O. J. Algahtani and G. Zaman, Global aspects of age-structured cigarette smoking model, *Mediterr. J. Math.* **15** (2018) 72.
[29] X. K. Zhang, Z. Z. Zhang, J. Y. Tong and M. Dong, Ergodicity of stochastic smoking model and parameter estimation, *Adv. Differ. Equ.* **2016** (2016) 274.
[30] H. Yang, Y. S. Tan and J. Yang, Extinction and persistence of a tumor-immune model with white noise and pulsed comprehensive therapy, *Math. Comput. Simul.* **182** (2021) 456–470.
[31] H. Yang and Y. S. Tan, Dynamic behavior of prostate cancer cells under antitumor immunity and pulse vaccination in a random environment, *Nonlinear Dyn.* (in press).
[32] Y. S. Tan, F. M. Tao and L. S. Chen, Dynamics of a nonautonomous system with impulsive output, *Int. J. Biomath.* **1** (2008) 02.
[33] Y. S. Tan, J. H. Liang and S. Y Tang, The dynamical behavior of non-smooth system with impulsive control strategies, *Int. J. Biomath.* **5** (2012) 1260018.
[34] H. Inaba, Mathematical analysis of an age-structured SIR epidemic model with vertical transmission, *Discrete. Contin. Dyn. Syst. Ser. B* **6** (2012) 69–96.
[35] Z. H. Liu and N. W. Li, Stability and bifurcation in a predator-prey model with age structure and delays, *J. Nonlinear Sci.* **25** (2015) 937–957.

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