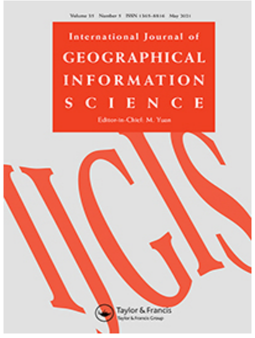

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Qualitative-geometric 'surrounds' relations between disjoint regions

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ABSTRACT

This paper explores a class of qualitative-geometric relations between disjoint regions, embedded in the surface of a sphere or in the plane. These relations combine topological information about the configuration of the regions themselves as well as of the geometric Voronoi regions surrounding them. The method uses maptrees to construct a rigorous and exhaustive framework for *n*-ary relations between ensembles of regions. The resulting uniquely defined relations are akin to the verbal spatial relation 'surrounds,' in one common interpretation. The paper provides an exhaustive formal classification of such extended spatial relations between up to three regions. A further exhaustive computational exploration of up to seven regions in the sphere also provides an algorithm to enumerate all possible configurations. The results demonstrate how the approach can be used to provide fine-grained and salient descriptions of qualitative-geometric relations for complex scenes involving ensembles of multiple regions.

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Qualitative spatial reasoning; topological relations; region ensembles; surrounds; Voronoi regions

1. Introduction

This paper describes the systematic exploration of a class of qualitative-geometric relations between disjoint regions. We classify relations between regions embedded in the surface of a sphere and in the Euclidean plane by representing not only the topological relations between the regions themselves, but also topological relations between the Voronoi regions surrounding them. Combining topology and geometry in this way allows us to capture more finely grained spatial relations, akin to those often described informally using verbs such as 'surrounds' (Jackendoff 1976, Rosenfeld and Klette 1985, Vandeloise 1991, Cohn *et al*. 1997, Both *et al*. 2018).

Our approach has been initiated in earlier publications (cf. Worboys 2012, Both *et al*. 2018). However, the central contribution of the current paper is to provide the tools to rigorously describe and systematically classify all possible qualitative-geometric region (so-called 'surrounds') relations. The paper demonstrates this approach exhaustively up to the ternary case, with further exploration of such relations in the sphere up to the septenary case (i.e. involving seven regions).

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Figure 1. Some islands in the Vasörarna (Finnish Valassaaret) archipelago in the Gulf of Bothnia, part of Korsholm municipality of Finland. (Data source: National Land Survey of Finland under a Creative Commons 4.0 license.).

Figure 1 provides a motivating example of where purely topological relations need to be supplemented with geometric knowledge. The islands in the illustration form part of the Vasörarna (Finnish Valassaaret) archipelago, composed of more than 150 islands out of almost 200,000 islands in Finland. From a purely topological viewpoint, all we can say about the relations between these islands is that they are pairwise disjoint and disjoint from the mainland.

However, we might wish to be able to reason about the various more subtle ways the islands interact with each other, the mainland, and other objects, where some islands may be considered to surround others. For example, do the more northern islands of Äbbskärsgrynnorna, Oxygrynnan, Voråboarnas hällan, Båtslaggrynnan, and Trekantbȯden together surround Båtslaget? Is Båtslagbådan also part of the surrounding islands of Båtslaget. Or is it itself, like Båtslaget, surrounded by the others? And how do all these islands relate to the other nearby islands and the wider Gulf of Bothnia, in which the entire archipelago is located?

Having said that, this paper is careful *not* to attempt to resolve the precise semantics of terms like 'surrounds' in natural language. Nor does it aim to supplant or replace existing work in the area of spatial reasoning that already defines related qualitative spatial relations akin to 'surrounds' (for example, based on the convex hull). Instead, our paper provides the first systematic treatment of n-ary qualitative-geometric relations – akin to 'surrounds' – where questions such as those above might be unambiguously computed and resolved. Our relations may involve any number of participating (simple) regions (both surrounding and surrounded) and may involve arbitrary levels of nesting of relations (e.g. islands surrounded by other islands themselves in bays and in gulfs and so on).

Section 2 reviews the related work on relations that similarly go further than the purely topological relations as well as research on higher-order (beyond binary) spatial relations. Section 3 summarizes the two formal structures that underpin our work – the combinatorial map and the maptree – before introducing and defining the extension of the maptree developed in this work: the region maptree (R-maptree). Sections 4 and 5 apply our method to an exhaustive classification of qualitative-geometric relations between two and three regions, respectively. Section 6 extends the analysis to provide a computational technique for the automated exploration of relations between any number of regions. The Appendix A includes the results of applying our method to an exhaustive classification of relations between up to seven regions. The final section discusses our results and contribution, arguing for the potential significance and importance of further exploration of our construction, such as in defining conceptual neighborhoods based on our relations.

2. Background

The major focus of early research advances in qualitative spatial reasoning was an analysis of binary topological region relations, most notably the region connection calculus (RCC) (Randell *et al*. 1992, Cui *et al*. 1993), the 4-intersection model (4IM) and 9-intersection model (9IM) (Egenhofer and Franzosa 1991, Egenhofer and Herring 1992), and the calculus-based method (CBM) of Clementini *et al*. (1993), (1995).

These widely cited foundations gave rise to a rich body of related research on region relations. The so-called 'egg-yolk' representation, for example, combined a certain region core with an associated uncertain penumbra (Cohn and Gotts 1996, Roy and Stell 2001, Cohn 2017). Similarly, other researchers have explored related models of regions with indeterminate or broad boundaries, including extending CBM and 4/9IM (Clementini and Di Felice 1996, 1997) amongst other approaches (e.g. Schneider 1996).

The region relations explored in this paper similarly have their roots in this established tradition of binary qualitative region relations. However, the relations described in this paper differ from these influential models in two important ways. First, our focus concerns combined topological and geometric region relations. Combining geometry and topology enables us to discern apart finer-grained spatial distinctions than is possible with topology alone. Hence, although all regions covered by our representation are all topologically disjoint, they exhibit a rich variety of distinct configurations. Second, unlike the binary topological relations between pairs of regions, our region relations are *n*-ary, potentially involving relations between ensembles of multiple regions in unison.

2.1. Qualitative geometric 'surrounds' relations

There exists no single agreed definition of 'surrounds' in natural language. In some accounts, 'surrounds' is closest in meaning to 'around,' i.e. forming a barrier that completely encloses an object (Jackendoff 1976, Talmy 1983, see Figure 2(a)). In other accounts, a looser definition is discussed where an object may be only 'partially included' in a surrounding region (Vandeloise 1991, Correa-Beningfield *et al*. 2008).

The partial rather complete enclosure is a common feature of computational definitions of 'surrounds.' In artificial intelligence, knowledge representation, and qualitative spatial reasoning most forms of the complete enclosure are more commonly classified as 'contains.' One exception is Liu *et al*. (2008) who argue that 'contains' involves the intersection between region interiors, with 'surrounds' reserved for a completely enclosed region but with non-intersecting interiors with the enclosing region.

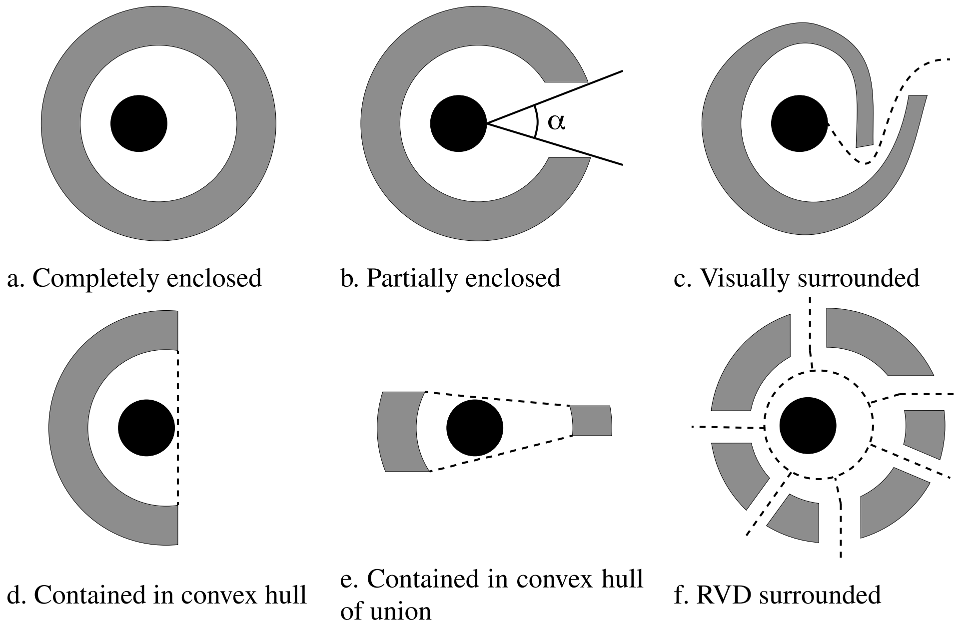


Figure 2. Six different versions of 'surrounds' in the literature, including a. complete enclosure (e.g., Jackendoff 1976); b. partial inclusion (e.g., Vandeloise 1991) or partial enclosure (e.g., Krishnapuram *et al*. 1993); c. visually surrounded (e.g., Rosenfeld and Klette 1985); d. contained in convex hull of singleton region (e.g., Cohn *et al*. 1997); e. contained in convex hull of union of regions (e.g., Hahmann and Brodaric 2013), degenerate case more akin to 'between'; RVD-surrounded used in this research, following Both *et al*. (2018).

Rosenfeld and Klette (1985) discuss the degree of 'surroundedness' in two senses: firstly, as the size of the angle subtended by the field of view from points on the surrounded object through the gap in the surrounding object (see also Krishnapuram *et al*. 1993, Miyajima and Ralescu 1994, Figure 2(b)); secondly, in terms of the sinuosity of the path from the surrounded region needed to 'escape' the surrounding region (Figure 2(b), termed here 'visually surrounded,' but see Rosenfeld and Klette 1985).

In qualitative spatial representations, several authors have used the convex hull of a region as the basis of the definition of a 'surrounds' relation (Cohn *et al*. 1997, Donnelly 2005, Schultz *et al*. 2006, Bittner *et al*. 2008, Bennett *et al*. 2013). The idea is that one region 'surrounds' a second if the second is a subset of the convex hull of the first (see Figure 2(d)).

This same idea – of a surrounded region being contained within the convex hull of a surrounding region – has also been applied to multiple regions. In this case, a region is be surrounded if it is contained within the convex hull of the union of a set of surrounding regions Hahmann and Brodaric (2013). However, as noted in Both *et al*. (2018), the extension is not a natural one, and leads to counter-intuitive cases that might more normally be termed 'between' rather than 'surrounds' (Figure 2(e)). Indeed, the ternary spatial relation 'between' (i.e., *A* is between *B* and *C*) has itself been the subject of study. The approach taken in Bloch *et al*. (2006) acknowledges the diversity of meanings of the term, by exploring the properties of 10 different yet related definitions of 'between.'

One further example in the literature of a 'surrounds' relation that allows for a set of surrounding regions is the *surroundsAttach* relation of Dube and Egenhofer (2014). Although this approach sits closest in spirit to our qualitative-geometric relation, *surroundsAttach* is designed for the quite different case where regions are pairwise disjoint and at the same time fill the space.

With the exception of the purely topological complete enclosure and the *surroundsAttach* relation of Dube and Egenhofer (2014), geometry plays a central role in the definition of all the relations discussed above. While derived from distinct geometric bases, the definitions are geometrically related, and so some instances may satisfy multiple criteria. For example, the partially enclosed figure (Figure 2(b)) is also contained in the convex hull, as is the visually surrounded figure (Figure 2(c)). In general, for relations between just two regions, any object visually surrounded or partially enclosed with angle $\alpha < 180^\circ$ must also be contained in the convex hull.

Instead of the geometry of the field of view, sinuous paths, or the convex hull, the qualitative-geometric relation explored in this paper relies on the *region Voronoi diagram* (RVD) associated with our spatial 'generator' regions (see Figure 2(f)). As for the familiar point Voronoi diagram, the RVD partitions an embedding space into so-called *Voronoi regions*, where the Voronoi region of each generator region is the set of all points in the space nearer to that generator region than to any other (Geiger 1993, Okabe *et al*. 1994, 2000). We note that in the case of just two regions (binary relations), the RVD-based and convex hull-based configurations are equivalent (i.e., in the case of two regions, if one is contained within the convex hull of the other, then that region is also RVD-surrounded, termed 'engulfed' in Both *et al*. 2018).

Beyond observing that there is no agreement on a single definition of 'surrounds' – neither in natural language nor in artificial intelligence, knowledge representation, or spatial computing – our aim in this paper is not to promulgate any one definition over another. We do argue that the qualitative-geometric relation explored in this paper does capture an important aspect of 'surroundedness,' akin to 'surrounds' in the sense described in Figures 1 and 2(f). However, we contend that none of the spatial relations discussed above has necessarily greater (or lesser) claim to capture the human concept and usage of 'surrounds.'

Instead, the contribution of this paper lies in the rigorous definition of a qualitative-geometric relation between regions – akin to the RVD-based 'surrounds' in Figure 2(f) – that uniquely defines the relations between arbitrary sets of regions, including multiple 'surrounding' and 'surrounded' regions, and arbitrary levels of nesting of relations, discussed further below.

2.2. Higher-order spatial relations

The majority of previous research on spatial relations, including the majority of research discussed above, has been focused on binary relations. Not all calculi are based upon binary relations. In the single-cross calculus, Freska (1992) proposed a ternary relation between points to represent the qualitative orientation of a point with respect to a given line segment. Freksa (1992); Zimmermann and Freksa (1996); Moratz *et al*. (2002) extended those relations in the double-cross calculus, further refining the discernible relations with a second line segment. These and other formulations extend topological representation to qualitative orientation. An excellent survey of qualitative direction and orientation calculi is found in Freksa *et al*. (2018). In a similar vein, Schlieder (1995) introduced a ternary logic, where each triple of points maps to one of three values, depending whether the points are in clockwise or anticlockwise orientation, or are collinear.

Bloch *et al*. (2006) and, more recently, Majic *et al*. (2020) explore ternary 'betweenness' relations. In Clementini and Billen (2006), an expressive framework for ternary projective relations involving three regions in the plane is defined and explored, based on tangents to the two reference regions. The approach leads to 34 jointly exhaustive, pairwise disjoint relations between three regions, several of which accord to subtly different interpretations of 'between.' Work in Clementini (2008) extends the approach to ternary region relations in the sphere. In the current paper, we also address region relations in the sphere, but as as a steppingstone to relations in the Euclidean plane.

Importantly, ternary – and in general *n*-ary – spatial relations can not necessarily be decomposed into binary relations. This point is made forcefully by Lewis *et al*. (2013); Lewis (2019) in their investigations into *ensembles* of regions, as illustrated in Figure 3. In the figure, the binary spatial regions between any pair of regions in the ensemble on the left are unchanged in the ensemble on the right. In particular, the highlighted orange region is pairwise disjoint to each of the blue regions in both cases. However, the two configurations are evidently topologically distinct, with the orange region being 'completely enclosed' (in the sense used in Figure 2) inside the union of blue regions only in the ensemble on the left.

Nesting of regions also becomes salient for *n*-ary surrounds relations. For example, a region may be surrounded by a one or more regions that is itself surrounded by other regions, and so on. The qualitative spatial *n*-ary relations developed and explored in this paper provide a rigorous and uniquely defined answer to questions, such as those proposed in connection with Figure 1 and such as: Which islands (collectively) surround Båtslagbådan? Which islands are surrounded by (collections of islands that include) Båtslagbådan? and Which islands(s) surround the islands that surround Båtslagbådan? The convex hull notion of 'surrounds' alone cannot provide unique answers to such questions about ensembles of regions, because using the convex hull of the union of a set of regions itself requires knowledge of which regions are in or out of the set. Hence, to provide

Figure 3. Two topologically distinct region ensembles with identical pairwise binary spatial region relations (Lewis 2019).

uniquely defined answers to questions such as these, the RVD-based notion of surrounds is a natural choice for further exploration.

3. Preliminaries

This research investigates spatial relations between a finite number of pairwise-disjoint regions, each homeomorphic to a closed disk. We consider the first regions embedded in the surface of a sphere. We also consider punctures to the sphere which will then result in embeddings in the Euclidean plane. Topologically, the sphere is bounded and closed, resulting in a simpler construction for our relations. However, relations in the Euclidean plane are conceptually more intuitive and clearly important in a mapping context.

The formal framework of this research is based on two principal structures: the combinatorial map and the maptree.

3.1. Combinatorial maps

Combinatorial maps, developed by Edmonds (1960) and Tutte (1973), provide formal representations of *2-cell connected graph embeddings*. The combinatorial map is the basis of the *maptree* (introduced in the following section). In turn, the maptree is the basis of the formal structure developed in this paper: the *R-maptree* (Section 3.3).

More specifically, we are concerned with a specific case of a graph: *directed multi-graphs*, also called *mutidigraphs* or *quivers*. A directed multigraph $G = (N, E, s, t)$ is a set of nodes N together with a set of E of edges and two functions, $s : E \rightarrow N$ and $t : E \rightarrow N$, which map each edge to its starting and terminal nodes, respectively. Unlike the more familiar directed graphs, a directed multigraph may have multiple distinct edges between the same pair of nodes.

An *embedding* of a graph in a surface is a mapping of the graph into a surface such that: (a) graph nodes map to points on the surface and (b) graph edges map to nonintersecting arcs in the surface, with no arc overlapping allowed. If the embedding has the property that each face of the embedded graph is topologically equivalent to a disk, it is called a *two-cell embedding*. In this paper, we are concerned with two-cell embeddings into the surface of a sphere or the *projective plane* (a plane augmented with points at infinity where parallel lines intersect) and their corresponding embeddings in the Euclidean plane. In the case that the embedding is in the Euclidean plane, then one of the faces must be external and unbounded, and so strictly speaking not a two-cell embedding.

Finally, a graph is *connected* if any pair of nodes may be linked by a path of edges. In the case of combinatorial maps we are interested in *weakly connected* graphs: where the underlying undirected graph – with directed edges replaced with undirected edges – is connected.

Figure 4 illustrates a combinatorial map with four nodes and seven edges, labeled *a*–*g*.

3.2. Maptrees

The tool used to provide unique symbolic representations of such embeddings is the *maptree*. Developed Worboys (2012, 2013), the maptree combines the combinatorial map with the adjacency tree (Buneman 1970, Stell and Worboys 2011), resulting in construction

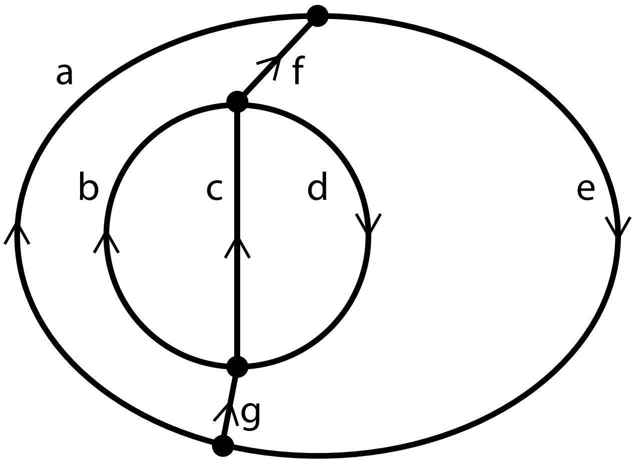
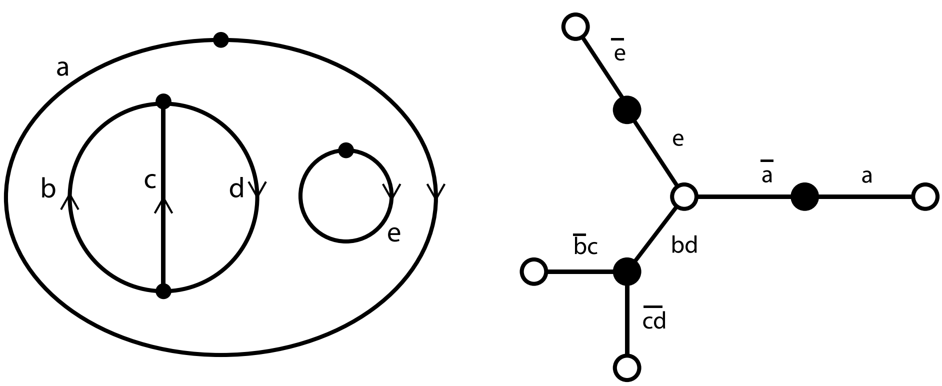


Figure 4. A combinatorial map: a two-cell (weakly) connected graph-embedding in the projective plane.

that can also represent unconnected graphs. Unconnected graphs are required to deal with spatial relations between disjoint regions. With each disjoint spatial region represented by a combinatorial map, the maptree provides a unified graph-based structure that combines these embeddings.

Figure 5 illustrates the maptree construction. The graph embedding in Figure 5(a) has three connected components. Each connected component of the graph embedding is represented by a combinatorial map. The maptree in Figure 5(b) unifies these, with each combinatorial map represented in the maptree by a black node and incident edges; and a white node corresponding to a face of that embedding.

Each edge in the maptree is labeled by a cycle of symbols, where each cycle represents a traversal of arcs around the corresponding face in the combinatorial map. Each traversal is arranged so that by convention its corresponding face is kept on its left. The overline notation $\overline{\bullet}$ indicates that the arc labeled $\bullet$ is traversed in the opposite direction to its



a. b.

Figure 5. A graph-embedding (a) and its maptree (b).

orientation. So in Figure 5, for example, the black node of the maptree with incident edges labeled with cycles *bd*, $\overline{b}c$, and $\overline{cd}$ in Figure 5(b) represents the boundary between the three faces of the component with edges b, c and d in Figure 5(a).

Finally, white nodes are identified when they represent the same face of different components. While all faces in each individual combinatorial map will be topologically equivalent to a disk, faces in the maptree formed by bridging two or more combinatorial maps will not. For example, the central white node of the maptree in Figure 5(b) is a face of all three connected components, with bounding arcs represented by cycles *e*, *bd* and $\overline{a}$ in Figure 5(a). As a consequence, this resulting face is not topologically equivalent to a disk, and maptrees are in general not two-cell embeddings.

When the graph embedding is into a sphere, then the associated maptree provides a unique representation (up to homeomorphism of the sphere) of that graph embedding, discussed in more detail in Worboys (2012). Representing planar, rather than spherical, embeddings simply require additional information about which is the unbounded external face of the embedding. The corresponding node in the maptree then becomes the root node of a rooted maptree. So, for example, embedding the configuration in Figure 5 (a) in the Euclidean plane as shown requires only that the external face be identified by 'planting' the corresponding node (the white node connected to edge labeled *a*) as the root of the rooted maptree. As we shall see later, it is also possible to embed maptrees into the plane by 'planting' an edge or node of the embedded graph (effectively 'puncturing' the sphere at a point), with the maptree transformed in a similar manner.

3.3. Voronoi regions and R-maptrees

As stated above, our goal is to classify spatial relations between regions by representing not only the topological relations between primary 'generator' regions themselves but also topological relations between the derived Voronoi regions surrounding generator regions. Combining geometric and topological relations allows us to capture more finely grained spatial relations than purely topological, as discussed in Section 1. Further, as argued in Section 2 and illustrated in Figure 2, the RVD-based notion of 'surrounds' is a natural choice for ensembles of multiple regions.

We need to make a modification to the maptree so that it can represent pairwise-disjoint, disk-like regions and take into account their Voronoi regions. To this end, the *region maptree* (R-maptree, for short) is a maptree extended by distinguishing two types of white node, the I-node and VI-node, and two types of black node, the B-node and the VB-node. Each I-node and B-node represents the interior and boundary of one of the disk-like 'generator' regions. Each VI-node represents the interior of the corresponding generator region's Voronoi region. Each VB-node represents a single connected component formed by the shared boundaries to two or more Voronoi regions.

Figure 6 shows the graphical notation for the four node types. The four node types are subject to a collection of constraints, namely:

C1 Each I-node is of degree 1, being adjacent to exactly one B-node.

C2 Each B-node is of degree 2, being adjacent to exactly one I-node and exactly one VI-node.

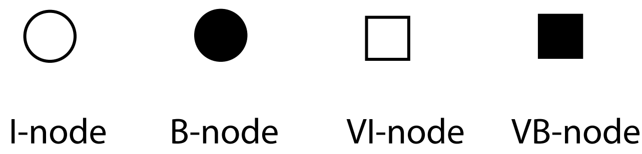


Figure 6. Node types, with I-nodes representing the interior of a generator region; B-nodes representing the boundary of a generator region; VI-nodes representing the interior of a Voronoi region; and VB-nodes representing the boundary of a Voronoi region.

C3 Each VI-node must be adjacent to exactly one B-node and at least one VB-node.

C4 Each VB-node has a degree of at least 2 and is adjacent only to VI-nodes.

Constraint C1 follows from the fact that generator regions are always disk-like and pairwise disjoint (even if the Voronoi regions themselves may be unbounded in the case embedding in the Euclidean plane). For constraint C2, each generator region boundary must separate the generator region interior it encloses from the unique Voronoi region interior surrounding it. Constraint C3 follows because each Voronoi region must by definition contain the boundary of one and only one generator region. Constraint C4 arises because Voronoi regions on each side of a Voronoi boundary must be distinct (Voronoi regions cannot have cuts) and must be boundaries only of Voronoi regions. (From now on, for the sake of conciseness we usually omit the adjective 'generator' from 'generator region.' Unless explicitly Voronoi regions, regions are assumed generator regions.)

From these constraints, we can deduce the following:

C5 The numbers of I-nodes, B-nodes, and VI-nodes are equal.

C6 Each white node is adjacent only to black nodes, and each black node is adjacent only to white nodes.

It turns out that the constraints, along with structural properties of the maptree enable us to determine all possible region/Voronoi configurations, up to topological invariance. (However, it is not yet clear whether every V-maptree results in some region/Voronoi configuration.) We next prove a theorem that allows us to more easily generate V-maptrees.

Theorem 3.1. *Each region will form a leaf of a component of the V-maptree as shown in Figure 7.*

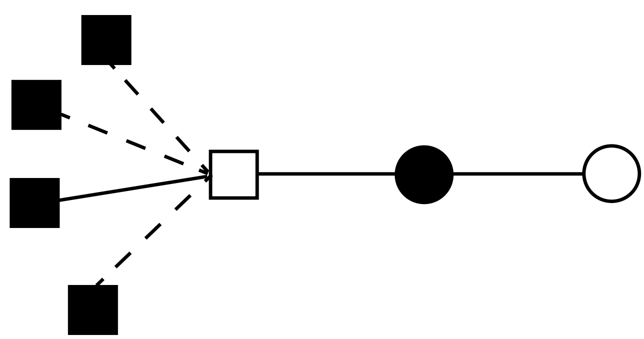


Figure 7. The configuration of nodes, including a V-I-VI node 'block' of the maptree.

Proof. From the right of Figure 7, the I-node is a leaf, by constraint C1, and by constraints C4 and C6 must be adjacent to a single B-node. Continuing to move left, this B-node, by constraint C2, is also adjacent to a single VI-node. This VI-node, by constraint C3, must be adjacent to at least one VB-node, and maybe more (shown with dashed-line connecting edges).

It follows that every I-node is adjacent to a unique B-node, which is in turn adjacent to a unique VI-node. It is convenient to refer to those units as 'blocks.'

Definition 1. *Any connected I-B-VI node component in the maptree, as in Figure 7, is called a block. The valency of a block is the number of VB-nodes adjacent to that block's VI-node.*

3.4. Motivating example revisited

Although our primary focus in this paper is on the representation and analysis of qualitative-geometric relations using the maptree, before moving forward we return briefly to our running motivating example of 'surrounds' relations between islands in the Gulf of Bothnia. Figure 8 shows an example configuration of 11 islands in the vicinity of the island Stora Bergöran in the Korsholm municipality of Finland. The map on the left-hand side of Figure 8 has been augmented with Voronoi boundaries W_1 and W_2 and islands ('generator' regions) R_1–R_{11}. Labels for generator regions boundaries B_1–B_{11} and Voronoi region interiors V_1–V_{11} have been omitted from the map on the left-hand side for reasons of clarity, but are included in the partial maptree on the right-hand side for generator regions R_1–R_6.

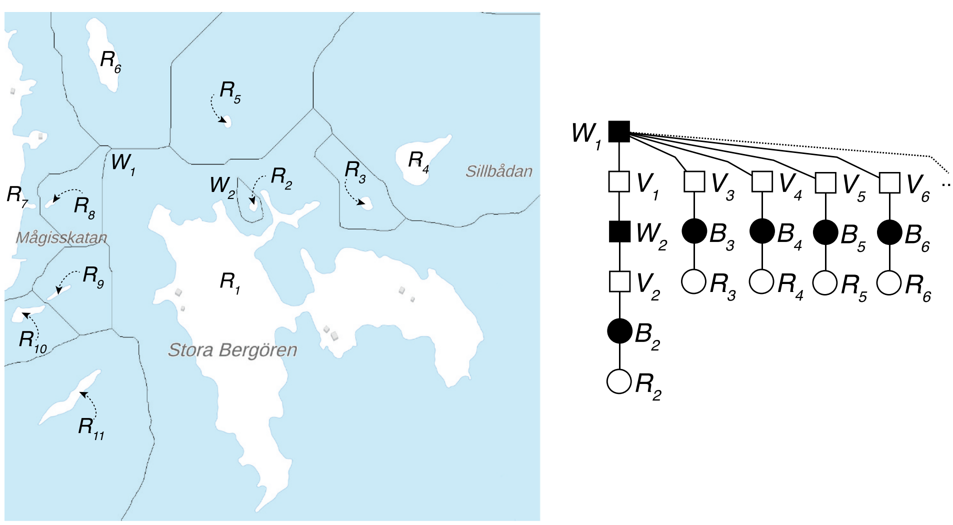


Figure 8. An example configuration of 11 islands in the vicinity of Stora Bergöran in the Korsholm municipality of Finland (left-hand side), with corresponding maptree representation resulting from generator regions R_1–R_{11} (right-hand side, only R_1–R_6 shown). (Data source: National Land Survey of Finland under a Creative Commons 4.0 license.).

The figure shows how some surrounds-like relations can be inferred directly from the maptree even without edge labels. For example, the surrounding of region R_2 by R_1 (i.e., contained in the convex hull, termed *engulfing* in Both *et al*. (2018)) can be inferred from the structure of the maptree, with Voronoi boundary W_2 a child of W_1. However, ensemble surrounds relations, such as regions R_1 and R_4 together surrounding regions R_3, requires inspection of further information about the edge labels of the maptree. The analysis in the following sections steps into more detail about the possible distinct configurations of maptrees in the case of two regions, three regions, and beyond, before finally returning to our motivating example in the discussion and conclusions.

4. Two-region analysis

We begin by providing an exhaustive analysis of two-region configurations. We shall see that on the surface of a sphere, there is only one possible configuration. In the planar case, there are two possibilities, corresponding to: (a) a singleton surrounding region (sometimes termed 'engulfing,' Both *et al*. 2018) and (b) the separation between the two.

4.1. Two regions on the surface of a sphere

The only possibility here, is two blocks, each of valency 1, adjacent to the same VB-node. The resulting R-maptree is shown in Figure 9, with I-nodes labeled R (region interior); B-nodes labeled B (region boundary); VI-nodes labeled V (Voronoi region interior); and VB-nodes labeled W (Voronoi region boundary). Note that the edges are not labeled in Figure 9. This is because each of the black nodes has degree two, and therefore bounded by a single edge. Thus, no ambiguity arises by omitting the labels.

4.2. Two regions on the plane

In theory, the surface of the sphere can be punctured at a point in any of the regions or boundaries represented by the maptree shown in Figure 9. However, since we constrain our regions to be bounded, we can only puncture the sphere at a point in a Voronoi region or on a Voronoi boundary. Hence, punctures may only occur at corresponding VI- or VB-nodes. The VI- or VB-node chosen for the puncture will then form the root of a rooted maptree embedded in the plane. In cases where the puncture occurs in a Voronoi region, that Voronoi region will be then unbounded in the Euclidean plane. Where the puncture occurs at a Voronoi boundary, at least two unbounded Voronoi regions will result (and possibly more than two as we step up to examine relations between more than two regions). In cases where the embedding is into the projective plane, the exterior face will remain bounded and topologically equivalent to a disk.

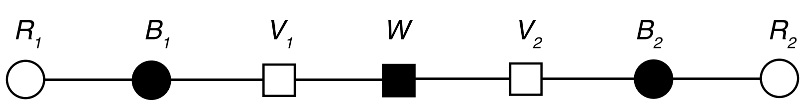


Figure 9. The configuration of nodes for two regions on the surface of a sphere.

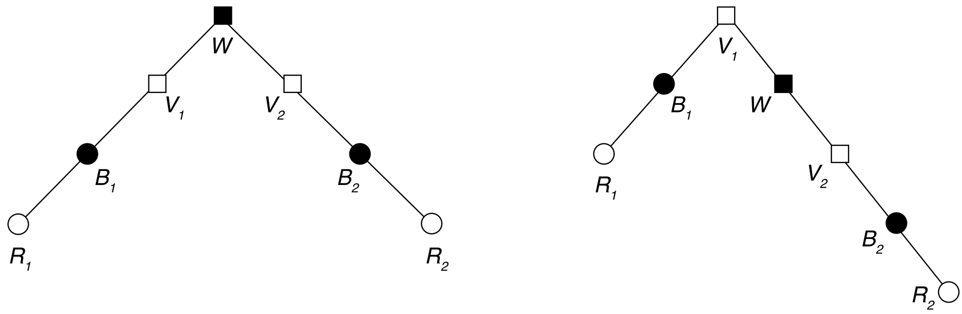


Figure 10. The configuration of nodes for two regions embedded in the Euclidean plane.

As a result, in the two-region case in Figure 9, only nodes labeled W, V_1, and V_2 are candidates for root nodes of the maptree. The W and V_1 cases are shown in Figure 10. The V_2 case is identical to the V_1 case, by symmetry.

Figure 11 shows the two embeddings represented by the planar maptrees in Figure 10. In the right-hand case, we might describe the region R_1 'engulfing' region R_2 (i.e. the case where a singleton region is 'surrounding'), while this is not the case with the regions in the left-hand case. We may also note that from a purely topological viewpoint, the relations between the two regions are identical, both being that of disjointness.

5. Three-region analysis

We next provide an exhaustive analysis of three-region configurations, beginning with the analysis of embeddings on the surface of a sphere.

5.1. Three regions on the surface of a sphere

There must be exactly three blocks in this case. There are two cases: three blocks of valency 1; and one block of valency 2 with two blocks of valency 1. In the case that all blocks are of valency 1, then the tree must be connected by a single VB-node (Figure 12, Case B). In the second case, one brick has valency 2 and connects to two other bricks, each

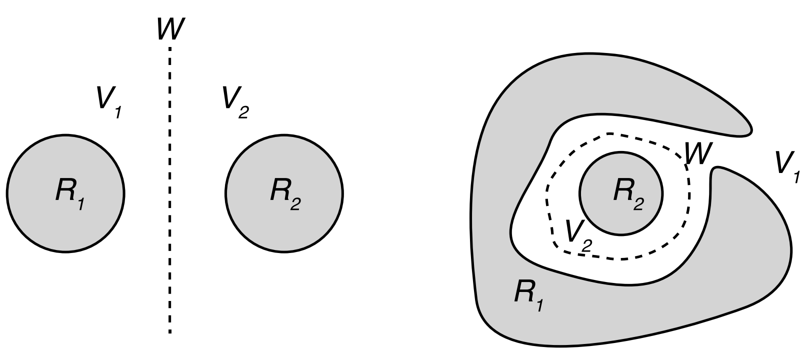


Figure 11. The two embeddings of two regions on the surface of the Euclidean plane (cf. Figure 10).

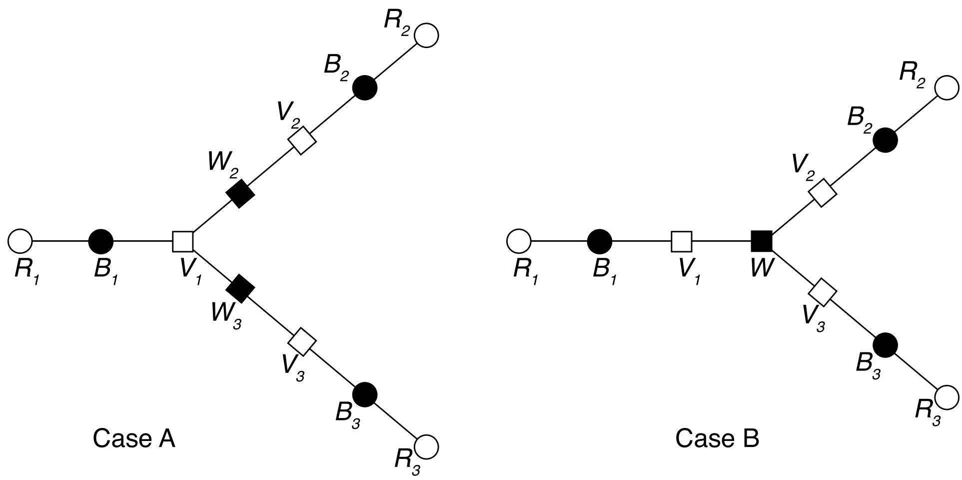


Figure 12. The R-maptrees for cases A and B when three regions are embedded on the surface of a sphere.

of valency 1, by identifying VI-nodes (Figure 12 Case A). These are the only possibilities. However in Case B, because the black VB-node has a degree greater than 2, we do have to consider the labeling of edges around it.

Let us consider the labeling around the VB node W in Case B. We assume that the arc represented by W contains no extraneous vertices: that is all vertices are of degree at least 3. We can make the following observations regarding the structure around W.

(1) Clearly, there are exactly three nodes around W.

(2) Each of the three regions partially bounded by the arc represented by W is a Voronoi region. Hence, there can be no edge of the boundary with the same region on both sides.

(3) The arc represented by W cannot comprise more than two edges. If it had three edges, as we can see from Figure 13, there would have to be more than three regions. If it comprised more than three edges, then the two neighboring regions could not be disk-like. Therefore, each region is bound by exactly two edges of the boundary W, and so the total number of semi-edges is at most six.

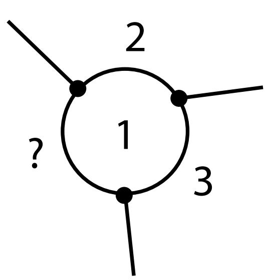


Figure 13. If a boundary of a region comprises three edges, then there cannot be three regions in total.

The above constraints lead to just two (up to relabeling) minimal configurations, namely Case B1, where W bounds: ab, $\bar{a}$, and $\bar{b}$; and Case B2, where W bounds ab, $c\bar{a}$, $\overline{bc}$). Therefore, Cases A, B1, and B2 are the only possible configurations leading to embeddings on the sphere. The three R-maptrees are shown in Figure 14.

5.2. Three regions on the Euclidean plane

Figure 15 shows the three configurations of Voronoi boundaries (indicated with hatched lies) corresponding to the three Cases A, B1, B2 derived above. As in the two-region case, the surface of the sphere can be punctured only at a point in a Voronoi region or on a Voronoi boundary. Figure 15 further shows for each case the topologically distinct (up to symmetry) locations in which the sphere can be punctured. We consider each case separately.

Case A

For Case A in Figure 14, as Figure 15 shows, there are three topologically distinct locations in which the embedding sphere may be punctured. These are labeled α, β, and γ and give rise to nodes V_1, W_2, V_2 being elevated to roots of the resulting planar maptrees. The configurations with W_2 and W_3 are identical, by symmetry, as are the configurations V_2 and V_3.

These maptrees are shown in Figure 16. Their corresponding embeddings are shown in Figure 17. We can see that Case A.α is the surrounding of two separated regions by

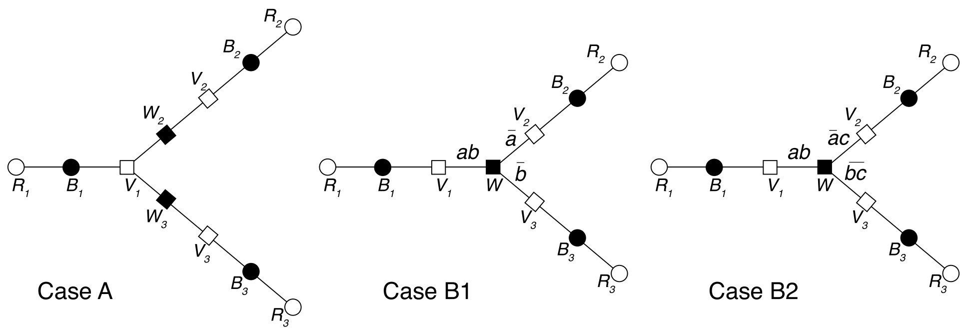


Figure 14. The three cases for three regions on the surface of a sphere.

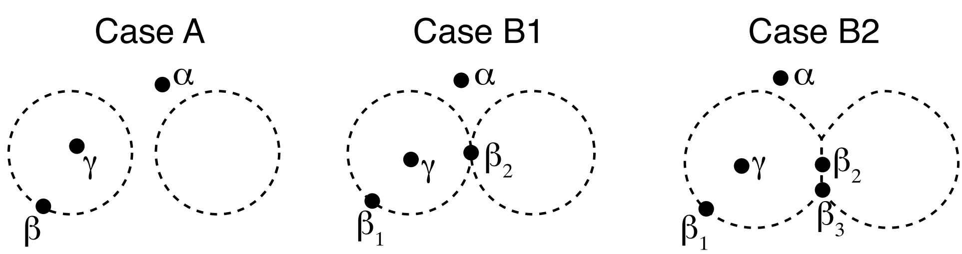


Figure 15. The topologically distinct (up to symmetry) places of puncture of the sphere at a Voronoi region or Voronoi boundary. Topologically distinct Voronoi boundary punctures are identified with the letter β. Topologically distinct Voronoi region punctures are identified with the letters α and γ (with the exception of Case B2, where punctures at α and γ are shown for completeness, but result in topologically equivalent planar configurations).

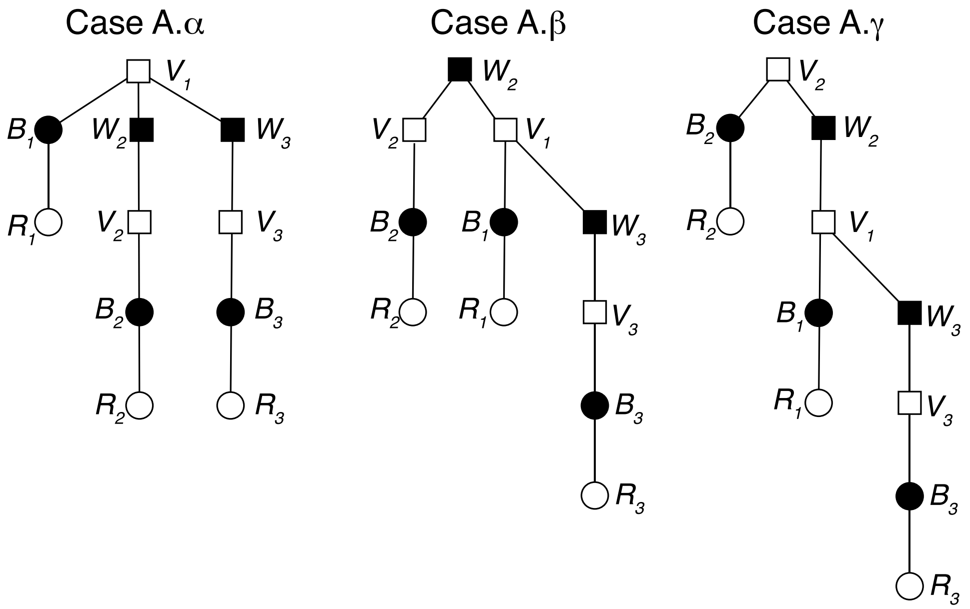


Figure 16. The three Case A planar maptrees for three regions on the surface of the Euclidean plane.

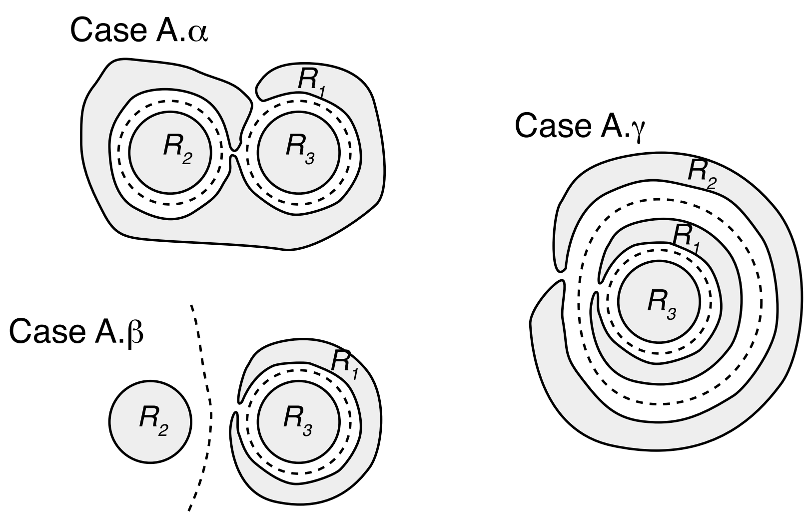


Figure 17. The three Case A planar configurations for three regions on the surface of the Euclidean plane.

a single region (i.e. 'engulfing'); Case A.β is one region separated by two other regions, one of which surrounds the other; and Case A.γ is the nested surrounding of one region by another and both by a third.

Case B

Cases B1 and B2 provide the limiting cases of Case A. In Case B1, the two Voronoi regions meet at a point; in Case B2, the two Voronoi regions meet in a line.

Considering the first Case B1, Figure 15 shows there are four topologically distinct locations in which the embedding sphere may be punctured. These are labeled α, β_1, β_2, and γ, with notation corresponding to that for Case A. Cases β_1, β_2 result from puncturing the sphere at topologically distinct points on W and are subsumed in the single Case B.β. These maptrees are shown in Figure 18. All corresponding embeddings are shown in Figure 19. We can see that case B1.α is a limiting case of the surrounding of two separated regions by a single region; Case B1.β_1 is a limiting case of one region separated by two other regions, one of which engulfs the other; Case B1.β_2 is the case where the Voronoi

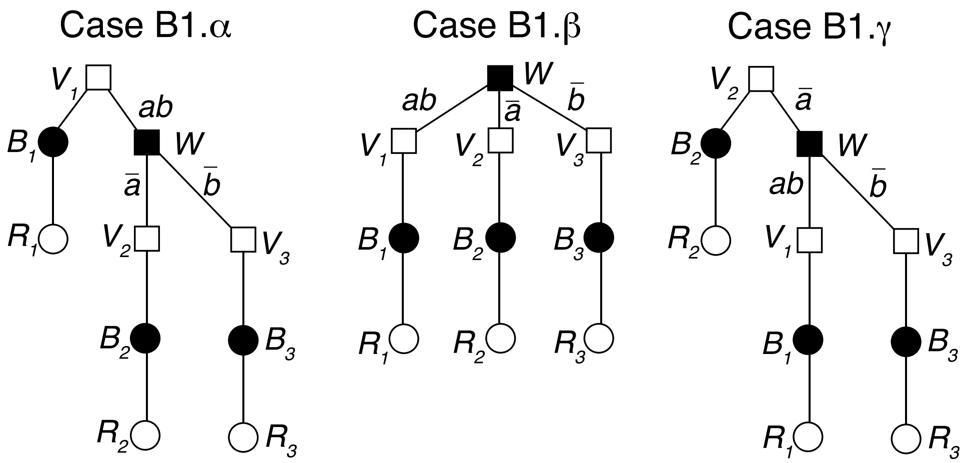


Figure 18. The three Case B1 planar maptrees for three regions on the surface of the Euclidean plane.

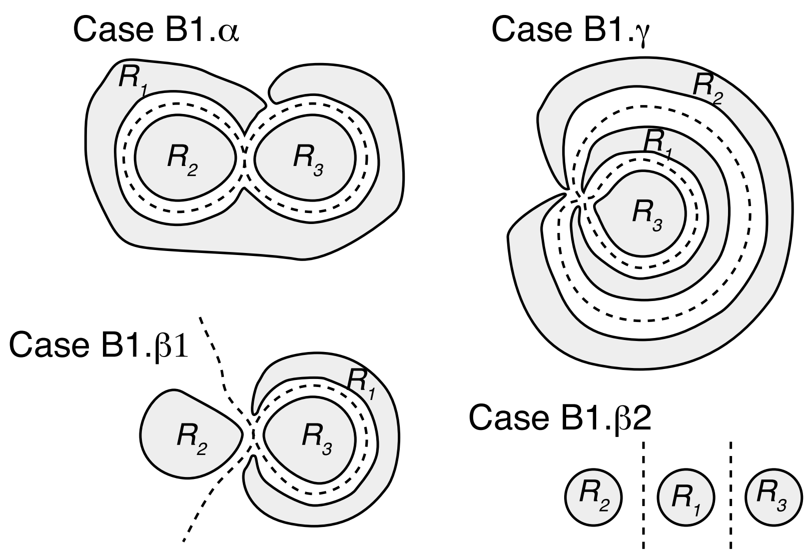


Figure 19. The four Case B1 planar configurations for three regions on the surface of the Euclidean plane.

boundary W splits into two discrete lines with the result shown; and Case B1.γ is the limiting case of the nested surrounding of one region by another and both by a third.

For Case B2, Figure 15 shows that there appear to be five topologically distinct locations in which the embedding sphere may be punctured. These are labeled α, β_1, β_2, β_3, and γ, with notation corresponding to that for Case A. However, punctures α and γ result in equivalent planar configurations, so we may consider only α, β_1, β_2, β_3. Cases β_1, β_2, β_3 result from puncturing the sphere at topologically distinct points on W and are subsumed in the single Case B.β. These maptrees are shown in Figure 20. All corresponding embeddings are shown in Figure 21. We can see that Case B2.α is a limiting case of the surrounding of two separated regions by a single region; Case B2.β_1 is a limiting case of one region separated by two other regions, one of which surrounds the other; Case B2.β_2 is the case where two regions in combination surround a third; and Case B2.β_3 is the case where three regions are located pairwise symmetrically opposed.

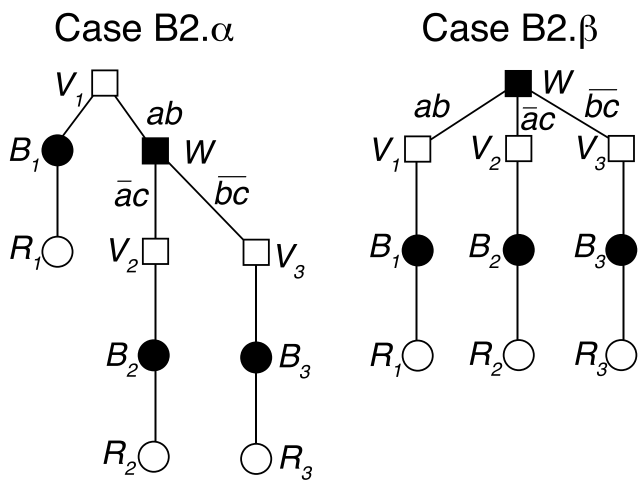


Figure 20. The three Case B2 planar maptrees for three regions on the surface of the Euclidean plane.

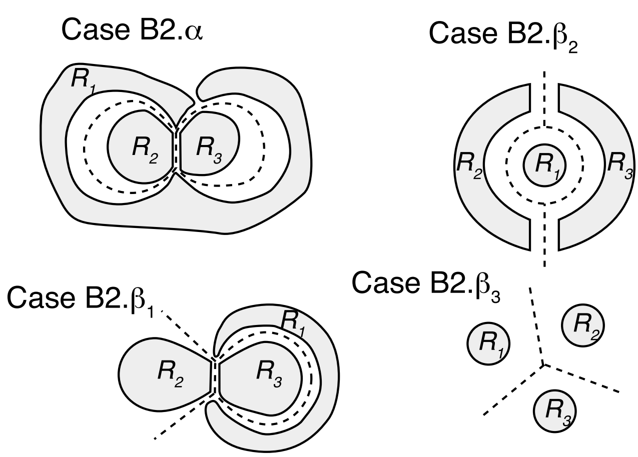


Figure 21. The three Case B2 planar configurations for three regions on the surface of the Euclidean plane.

6. Beyond three regions

The discussion thus far provides a first glimpse of the richness of finer granularity topological distinctions afforded by the R-maptree, even with just three regions. As the number of interacting regions increases, so too does the number of qualitative-geometric distinctions that are possible to discern with the R-maptree.

This section takes the next step into exploring that diversity computationally. By representing the essential information of the R-maptree using a portion of its adjacency matrix, we present an algorithm for enumerating all R-maptrees between n Voronoi regions, examining in more detail those cases up to $n = 7$.

6.1. R-maptree adjacency submatrix (RMAS)

Given n regions in the sphere, there must exist at least 1 and possibly up to $n - 1$ disjoint Voronoi boundaries that separate them. Hence, for any configuration of n regions, we must consider the R-maptrees with *order* (number of nodes) i where $i \in [n + 1, 2n - 1]$.

In all cases, the R-maptree can be represented as a binary adjacency matrix of the form shown in Figure 22. The central insight behind our algorithm for enumerating all possibilities is that all the information contained in the R-maptree is encoded by the top right-hand subsection of the full adjacency matrix, termed here the *R-maptree adjacency submatrix* (RMAS).

6.1.1. RMAS canonical form

The labels of nodes in an R-maptree are arbitrary. Hence the labels of rows and columns of an RMAS are also arbitrary, and can reordered without changing the underlying structure of the R-maptree the RMAS represents.

In order to more easily discuss and compare RMAS, we adopt a 'greedy' canonical form for RMAS where rows and columns are ordered such that:

(1) rows are firstly in descending order of number of 1s across the row;

(2) rows are secondarily in descending order of the magnitude of the binary number formed by that row; and

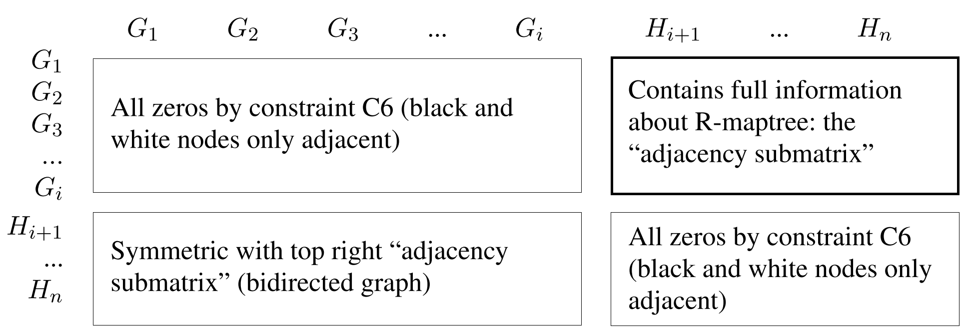


Figure 22. R-maptree adjacency submatrix (RMAS) construction (top right), based on adjacency matrix of I-B-VI blocks $G_1..G_i$ and VB nodes $H_{i+1}..H_n$.

(3) columns are ordered such that they yield the largest possible binary numbers in the highest rows.

Figure 23 provides an illustration of an R-maptree and three equivalent adjacency submatrices, one in canonical form. To simplify Figure 23(a), and all R-maptree figures below, a diagonal-striped box node is used to represent a B-I-VI block.

6.2. R-maptree adjacency submatrix constraints

Representing the R-maptree as an adjacency submatrix enables the structural constraints of R-maptrees to be re-expressed in a format more amenable to computation. More specifically, not all $i \times n - i$ binary matrices encode valid R-maptrees. The following four conditions must be met in order for an $i \times n - i$ binary adjacency matrix to be a valid encoding of an R-maptree:

$\mathcal{C}2$ The number of 1s in each row of the binary matrix is strictly greater than 0.

$\mathcal{C}3$ The number of 1s in each column of the binary matrix is strictly greater than 1.

$\mathcal{C}1$ The number of all 1s in the binary matrix is equal to $o - 1$, where o is the order (number of nodes) in the corresponding R-maptree.

$\mathcal{C}4$ There exists a path between any pair of 1s in the binary matrix. A path through the tree may be found by following sequences of cells, with each adjacent cell in the sequence containing a 1 in the same row or column of the binary matrix.

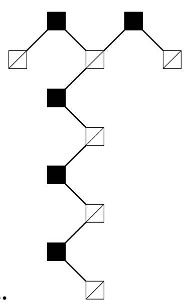
With reference to the R-maptree constraints C1–4 in Section 3.3:

• constraints C1 and C2 concern I- and B-nodes and are unaffected by the reformulation of VB and I-B-VI blocks as an adjacency submatrix;

• constraint C3 is captured by constraint $\mathcal{C}1$ above;

• constraint C4 maps to constraint $\mathcal{C}2$ above; and

• constraints $\mathcal{C}3$ and $\mathcal{C}4$ above effectively encode that the adjacency submatrix describes a tree, with size (number of edges) equal to order o minus 1 and a (by implication unique) path between every pair of edges.



a.

$$\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

b.

$$\begin{vmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

c.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix}$$

d.

Figure 23. Three equivalent RMAS describing the same (unlabeled) R-maptree. Subfigure 'a' shows an R-maptree for six regions separated by five Voronoi boundaries. The diagram introduces the graphical notation of using unfilled diagonally striped boxes to represent B-I-VI blocks. This notation is used in all subsequent diagrams for conciseness. Subfigure 'b' shows the RMAS for the R-maptree in Subfigure 'a' in canonical form. Subfigure 'c' shows an equivalent RMAS for the R-maptree in Subfigure 'a,' not in canonical form and formed by swapping columns 3 and 6 in the RMAS in Subfigure 'b.' Subfigure 'd' shows a further equivalent RMAS, not in canonical form, and formed by further swapping columns 1 and 2 and rows 1 and 5 from the RMAS in Subfigure 'c.'

Hence, these constraints are sufficient to ensure the adjacency submatrix satisfying those constraints describes a valid R-maptree.

Theorem 6.1. *All four constraints $\mathcal{C}1$–4 are necessary to ensure the adjacency submatrix describes a valid R-maptree.*

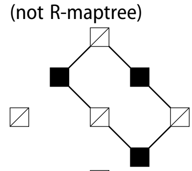
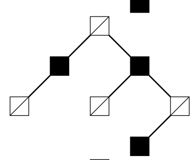
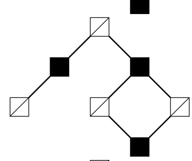
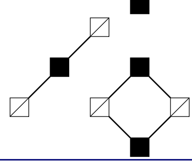
Proof. Proof by contradiction. Table 1 provides four adjacency submatrices, each of which satisfy three of the constraints $\mathcal{C}1$–4 but not the fourth. Each adjacency submatrix violates a different constraint, and leads to a counterexample graph that is not a valid R-maptree, illustrated in Table 1.

6.3. Algorithm

Given the constraints and structures defined above, it is straightforward to write a brute-force algorithm to exhaustively generate all possible R-maptrees for n Voronoi regions (Algorithm 1). The algorithm iterates through all the $n \times 1$, $n \times 2$, … $n \times n - 1$ binary matrices (Algorithm 1 lines 2–3) storing any binary matrices that satisfy the constraints in the previous section (Algorithm 1 lines 4–5).

Algorithm 1: Brute-force enumeration of R-maptrees for n Voronoi regions
Input: Number n of Voronoi trees
1 Initialize set $S \leftarrow \emptyset$;
2 **for** $i \in [1, n-1]$ **do**

Table 1. Proof by contradiction of Theorem 6.1.

	Constraint				Adjacency		Graph
Case	$\mathcal{C}1$	$\mathcal{C}2$	$\mathcal{C}3$	$\mathcal{C}4$	submatrix	Comment	(not R-maptree)
1	$\times$				$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix}$	Σ row 4 < 1	
2		$\times$					
3			$\times$		$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}$	Σ of 1s > 6	
4				$\times$	$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}$	No path to cells in column 3	

3 **foreach** *binary matrix b with n rows and i columns* **do**
4 **if** *b satisfies constraints* $\mathcal{C}1$–4 **then**
5 $S \leftarrow S \cup \{b\}$
6 **end**
7 **end**
8 **end**

Unfortunately, there are in general 2^{ni} permutations of an $n \times i$ binary matrix. So, whilst simple, this brute force algorithm is not computationally tractable, $\approx O(n!)$. The search space can be reduced by excluding any binary matrices that cannot possibly satisfy constraints $\mathcal{C}1$–4, such as binary matrices with empty rows or columns and most with sums other than $n + i - 1$. For example, a standard result in combinatorics gives the number of $n \times i$ binary matrices with exactly k 1s and no empty rows or columns as:

$$\sum_{u=0}^{i}\sum_{v=0}^{n}(-1)^{u+v}\binom{m}{u}\binom{n}{v}\binom{(m-u)(n-v)}{k}$$

While such exclusions significantly reduce the search space (for example, reducing the number of permutations of a 6×5 binary matrix from 1 073 741 824 to 7 926 790 by considering only those matrices with 10 1s and no empty rows or columns), this leaves the underlying computational complexity unchanged, $O(n!)$.

Heuristics to reduce the search space further are likely possible, but left as a focus for future research. Instead, the open-source code accompanying this paper implements a brute force search with some pruning of the search space, as described above. Using standard desktop computers, the approach is tractable for up to $i = 7$ (i.e. all possible topological configurations in the sphere of up to seven regions and their corresponding Voronoi boundaries).

6.4. Results

The Appendix A contains the 111 canonical adjacency submatrices for every possible R-maptree for between 2 to 7 regions in the sphere: 1 $\times$ 2-region R-maptree, 2 $\times$ 3-region R-maptrees, 4 $\times$ 4-region R-maptrees, and 9, 23, and 72 $\times$ 5-, 6-, and 7-region R-maptrees, respectively). Figure 24 details example configurations from all nine possible five-region R-maptrees in the sphere.

7. Discussion and conclusions

This paper has demonstrated that the geometrical information provided by a Voronoi partition of a surface can considerably enrich a purely topological representation. Indeed, in the cases discussed above, apart from the number of regions, all would reduce to the single topological case of pairwise disjoint. We have thus found a convenient and qualitative half-way house between the paucity of information that a purely topological framework provides and the uncountably infinite cases if we were to assume full Euclidean geometry.

Throughout, we have been careful to emphasize that our RVD-based qualitative-geometric relations corresponds to one of a number of possible interpretations of the ill-

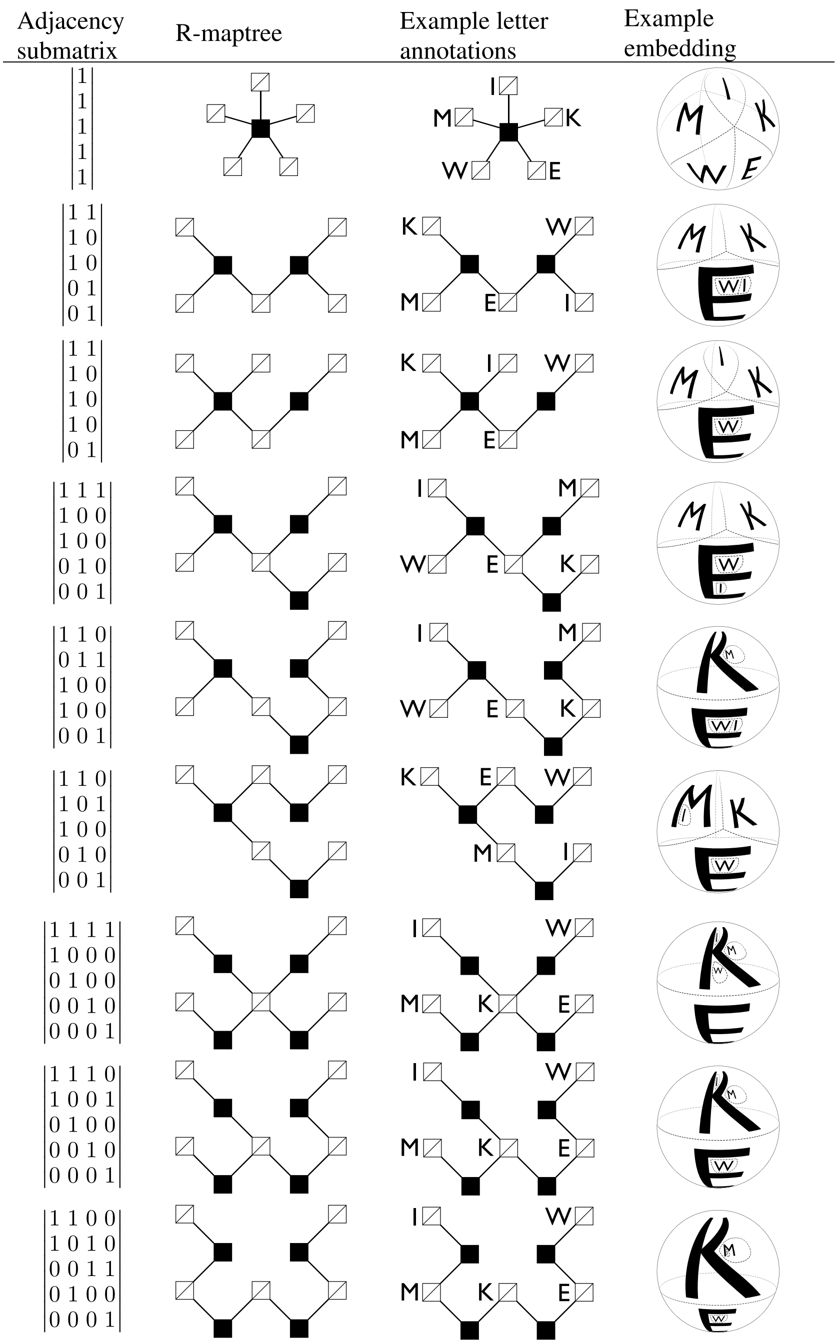


Figure 24. All 9 possible adjacency submatrices and R-maptrees corresponding to five regions on the sphere, with example letter-region R-maptree annotations, using the five letters 'M-I-K-E-W' as regions, and example embeddings in the sphere (dashed lines show approximate Voronoi boundaries).

defined linguistic notion of 'surrounds' (see Figure 2). Nevertheless, returning to the motivational example in Figure 1, our qualitative-geometric can be used to provide a uniquely answer to the questions posed in the introduction, as illustrated in Figure 25. According to the definition of ensemble 'surrounds' proposed in Both *et al.* (2018) (intuitively, a region is surrounded by the set of generator regions with adjacent Voronoi regions), Båtslaget is surrounded by the combination of Äbbskärsgrynnorna, Oxygrynnan, Voråboarnas hällan, Båtslaggrynnan, Trekantbȯden, *and* Båtslagbådan. Båtslagbådan on the other hand is surrounded by Båtslaget, Äbbskärsgrynnorna, and Trekantbȯden in combination. And Båtslagbådan and Båtslaget are also together surrounded by the combination of Äbbskärsgrynnorna, Oxygrynnan, Voråboarnas hällan, Båtslaggrynnan, and Trekantbȯden.

However, building on the firm formal foundations set out in this paper, our targets for future work can be much more ambitious than simply rigorous and uniquely defined descriptions of configurations of ensembles as in Figure 25. The R-maptree provides a basis for future work into the similarity between scenes; changes and events amongst moving regions; and conceptual neighborhoods in sets of *n*-ary qualitative-geometric relations.

As a final glimpse into this exciting future, Figure 26 illustrates just a few of the changes to a simplified R-maptree that accompany the voyage of a sailing boat, Hayen, in the Gulf of Bothnia (Figure 26(a)), into a bay in the island group of Replot (Figure 26(b)), before mooring in a cove in the island of Stora Bergöran Figure 26(c). In purely topological terms nothing has changed in any of the scenes: the islands and the boat all remain pairwise disjoint throughout. However, even for the simplified R-maptree in Figure 26, changes occur in sequence (i.e. underpinned by an implicit conceptual neighborhood, yet to be explored) and correspond to salient events, such as 'entering' the bay formed by the island group of Replot, itself

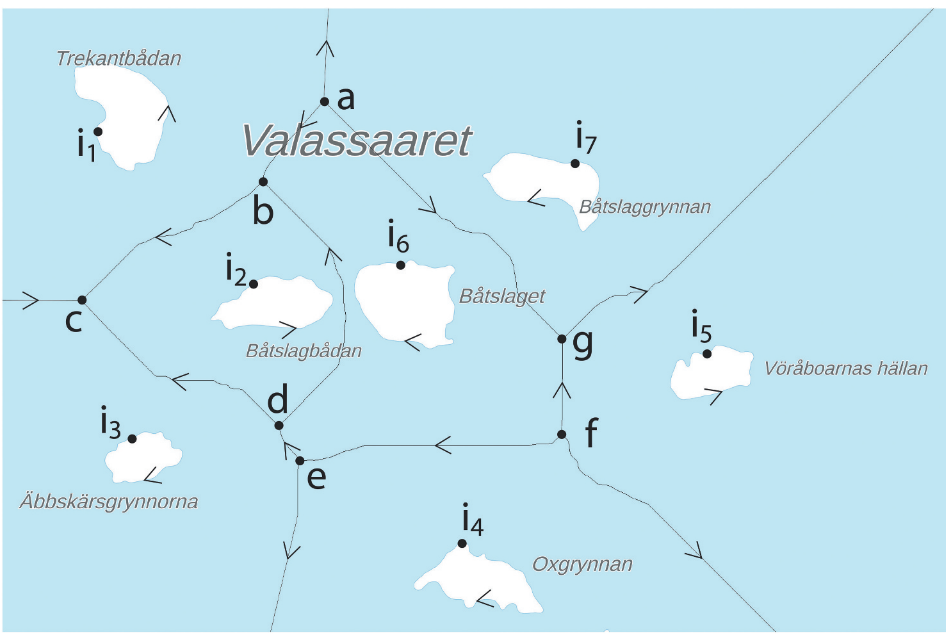
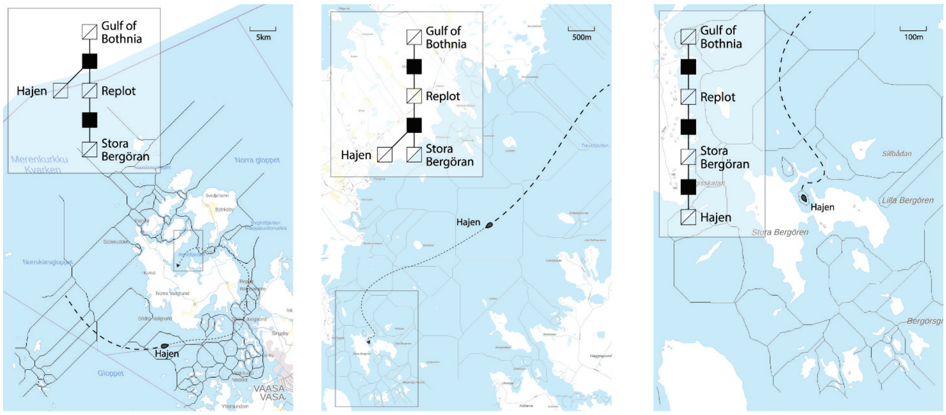


Figure 25. Returning to the questions posed in Section 1 for seven islands from Figure 1, showing graph embedding of island boundaries i_1–i_7 and partial RVD boundaries a–g. (Data source: National Land Survey of Finland under a Creative Commons 4.0 license.).



a. Gulf of Bothnia b. Bay within Replot c. Mooring off Stora Bergöran

Figure 26. The voyage of Hayen from the Gulf of Bothnia to mooring at Stora Bergöran. (Data source: National Land Survey of Finland under a Creative Commons 4.0 license.).

surrounded by the Gulf of Bothnia, and 'arriving' in the mooring cove at Stora Bergöran. The formal foundations of the R-maptree presented in this paper provide both solid and fertile ground for this future work.

Data and codes availability statement

The data and codes that support the findings of this study are available as Duckham, M. and Worboys, M. (2021): Maptree Software, RMIT University at https://rmit.figshare.com/articles/software/Maptree_Software/13585031.

Disclosure statement

No potential conflict of interest was reported by the authors.

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Appendix A. Complete set of R-maptrees for two to seven regions

Appendix A1. R-maptrees for two regions (one adjacency submatrix)

$$\begin{vmatrix}1\\1\end{vmatrix}$$

Appendix A2. R-maptrees for three regions (two adjacency submatrices)

$$\begin{vmatrix}1\\1\\1\end{vmatrix} \qquad \begin{vmatrix}1&1\\1&0\\0&1\end{vmatrix}$$

Appendix A3. R-maptrees for four regions (four adjacency submatrices)

$$\begin{vmatrix}1\\1\\1\\1\end{vmatrix}\begin{vmatrix}1&1\\1&0\\1&0\\0&1\end{vmatrix}\begin{vmatrix}1&1&1\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}\begin{vmatrix}1&1&0\\1&0&1\\0&1&0\\0&0&1\end{vmatrix}$$

Appendix A4. R-maptrees for five regions (nine adjacency submatrices)

$$\begin{vmatrix}1\\1\\1\\1\\1\end{vmatrix}\begin{vmatrix}1&1\\1&0\\1&0\\0&1\\0&1\end{vmatrix}\begin{vmatrix}1&1\\1&0\\1&0\\1&0\\0&1\end{vmatrix}\begin{vmatrix}1&1&1\\1&0&0\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}\begin{vmatrix}1&1&0\\0&1&1\\1&0&0\\1&0&0\\0&0&1\end{vmatrix}\begin{vmatrix}1&1&0\\1&0&1\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}\begin{vmatrix}1&1&1&1\\1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{vmatrix}\begin{vmatrix}1&1&1&0\\1&0&0&1\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}\begin{vmatrix}1&1&0&0\\1&0&1&0\\0&0&1&1\\0&1&0&0\\0&0&0&1\end{vmatrix}$$

Appendix A5. R-maptrees for 6 regions (23 adjacency submatrices)

$$
\begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}
$$

$$
\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}
$$

$$
\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
\quad
\begin{vmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}
$$

Appendix A6. R-maptrees for 7 regions (72 adjacency submatrices)

$\begin{vmatrix}1\\1\\1\\1\\1\\1\\1\end{vmatrix}$	$\begin{vmatrix}1&1\\1&0\\1&0\\1&0\\1&0\\1&0\\0&1\end{vmatrix}$	$\begin{vmatrix}1&1\\1&0\\1&0\\1&0\\1&0\\0&1\\0&1\end{vmatrix}$	$\begin{vmatrix}1&1\\1&0\\1&0\\1&0\\0&1\\0&1\\0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0\\0&1&1\\1&0&0\\1&0&0\\1&0&0\\1&0&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1\\1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0\\1&0&1\\1&0&0\\0&1&0\\0&1&0\\0&0&1\\0&0&1\end{vmatrix}$
$\begin{vmatrix}1&0&1\\0&1&1\\1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&1&0\end{vmatrix}$	$\begin{vmatrix}1&1&1\\1&0&0\\1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1\\1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0\\0&1&1\\1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0\\1&0&1\\1&0&0\\1&0&0\\1&0&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0\\1&0&1\\1&0&0\\1&0&0\\0&1&0\\0&1&0\\0&0&1\end{vmatrix}$	$\begin{vmatrix}0&1&1&1\\1&1&0&0\\1&0&0&0\\1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$
$\begin{vmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\end{vmatrix}$	$\begin{vmatrix}1&0&1&0\\0&1&1&0\\0&1&0&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&1&0\\1&0&0&1\\0&1&0&1\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\end{vmatrix}$	$\begin{vmatrix}1&1&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&0\\0&0&1&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&1&1\\0&1&1&0\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&0\\1&0&0&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$
$\begin{vmatrix}1&1&0&0\\1&0&1&0\\0&0&1&1\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&0\\0&1&0&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0\\1&1&1&0\\0&0&1&1\\1&0&0&0\\1&0&0&0\\1&0&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0\\1&0&1&0\\0&0&1&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&1\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$
$\begin{vmatrix}1&1&1&0\\1&0&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&0&1&1\\1&1&0&0\\1&0&0&0\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&0&0\\0&1&0&1&0\\0&0&0&1&1\\1&0&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&1&0\\0&1&1&0&0\\0&0&0&1&1\\1&0&0&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&1&0&0\\1&0&0&1&0\\0&0&0&1&1\\1&0&0&0&1\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\end{vmatrix}$	$\begin{vmatrix}1&1&0&0&0\\1&0&1&0&0\\1&0&0&1&0\\0&1&0&0&1\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{vmatrix}$

(*Continued*)

(Continued).

$\begin{vmatrix} 1&1&0&0&0 \\ 0&1&0&0&1 \\ 0&0&1&1&0 \\ 0&0&1&0&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 1&0&0&1&0 \\ 0&0&1&0&1 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 0&1&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 0&0&1&1&0 \\ 0&0&0&1&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 1&0&0&1&0 \\ 0&0&0&1&1 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 0&1&1&0&0 \\ 0&1&0&1&1 \\ 1&0&0&1&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0 \\ 1&0&0&1&0 \\ 1&0&0&0&1 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$
$\begin{vmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 0&1&0&0&1 \\ 0&0&1&1&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 0&1&1&1&0 \\ 1&0&0&0&1 \\ 0&1&0&0&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{vmatrix}$	$\begin{vmatrix} 0&1&1&1&0 \\ 1&1&0&0&0 \\ 1&0&0&0&1 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 0&1&1&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0 \\ 1&0&1&1&0 \\ 0&0&1&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&1&0 \\ 0&1&0&0&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0 \\ 1&0&0&1&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$
$\begin{vmatrix} 1&1&1&1&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 0&1&1&1&1 \\ 1&1&0&0&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&1&0 \\ 1&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0 \\ 0&1&0&1&1 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&1&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&0&1&1 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&1&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$
$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&1&1&0 \\ 1&0&1&0&0&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&1&1&0 \\ 0&0&1&0&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&1&0&0&1 \\ 0&0&0&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&1&0 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&1&1 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&1&0&0 \\ 1&0&0&0&1&0 \\ 0&0&0&0&1&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0&0 \\ 1&0&0&0&0&1 \\ 0&0&0&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&0&0 \\ 0&0&0&1&1&0 \\ 0&0&0&0&1&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$
$\begin{vmatrix} 1&1&1&0&0&0 \\ 1&0&0&0&1&0 \\ 0&0&0&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&0&0&1&0&1 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0&0 \\ 0&0&0&1&1&1 \\ 1&0&0&1&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&0&0&0&0 \\ 1&0&1&0&0&0 \\ 0&1&0&0&1&0 \\ 0&0&1&1&0&0 \\ 0&0&0&1&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&0&0 \\ 0&1&0&0&0&1 \\ 0&0&0&1&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$	$\begin{vmatrix} 1&1&1&1&1&1 \\ 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{vmatrix}$

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(Continued).

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad \begin{vmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$