

Threshold dynamics and bifurcation of a state-dependent feedback nonlinear control SIR model[☆]

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Abstract

A classic SIR model with nonlinear state-dependent feedback control is proposed and investigated in which integrated control measures, including vaccination, treatment and isolation, are applied once the number of the susceptible population reaches a threshold level. The interventions are density dependent due to limitations on the availability of resources. The existence and global stability of the disease free periodic solution (DFPS) are addressed, and the threshold condition is provided which can be used to define the control reproduction number R_c for the model with state-dependent feedback control. The DFPS may also be globally stable even if the basic reproduction number R_0 of the SIR model is larger than one. To show that the threshold dynamics are determined by the R_c , we employ bifurcation theories of the discrete one-parameter family of maps, which are determined by the Poincaré map of the proposed model, and the main results indicate that under certain conditions a stable or unstable interior periodic solution could be generated through transcritical, pitchfork and backward bifurcations. A biphasic vaccination rate (or threshold level) could result in an inverted U-shape (or U-shape) curve which reveals some important issues related to disease control and vaccine design in bioengineering including vaccine coverage, efficiency and vaccine production. Moreover, the nonlinear

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state-dependent feedback control could result in novel dynamics including various bifurcations.

Keywords: SIR model, Disease free periodic solution, Control reproduction number, Poincaré map, Transcritical and pitchfork bifurcations, Backward bifurcation

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1. Introduction

Although outbreaks of traditional infectious diseases have been prevented or controlled in the recent past, outbreaks of emerging infectious diseases such as SARS, H1N1 influenza, Dengue fever and Ebola, have provided new threats and challenges. Comprehensive prevention and control strategies, including quarantine, isolation, vaccination and treatment, are widely used to reduce the spread of such infectious diseases [1, 2, 3, 4] but evaluating the effectiveness of the mitigating measures is also crucial. To address this problem, mathematical models can play key roles by modelling the control tactics and revealing their effectiveness.

Recently, several mathematical models have been proposed to investigate integrated control impacts [3, 4]. Existing approaches to modelling the impact of integrated control measures have focused on how to include the tactics into models and address their effects on the dynamics and disease control. There are two types of important models to be chosen according to how the control measures are implemented: continuous models with a continuous control strategy [1, 2, 3, 4] and continuous models with a discrete (pulse) control strategy [5, 6, 7]. The two types of models with the continuous vaccination and pulsed vaccination were compared with each other in epidemiological models [5]. The results of that study were confirmed by the pulse vaccination campaigns against measles performed in 1994 in the UK, which revealed that a pulse vaccination strategy had a dramatic impact on the development of the epidemic.

The pulse vaccination strategy was applied at a fixed period T , i.e. at dis-

crete times nT ($n = 1, 2, \dots$) a proportion of the susceptible population was
25 vaccinated and removed into the recovered or removed or vaccinated class instan-
taneously, which action can be formulated by impulsive differential equations
with a fixed moment [8, 9, 10, 11, 12, 13, 14, 7]. The dynamical behaviour,
including the existence and local stability of the disease free periodic solution
has been investigated, which revealed that pulse vaccination was always capa-
30 ble of eradicating the diseases, usually doing better than continuous vaccination.
Numerical bifurcation analyses depict that pulse vaccination can lead to very
complex dynamics including chaotic behaviour. In addition, this type of mod-
elling has been widely used for cancer treatment [15, 16, 17, 18, 19] and HIV
control [20, 21].

35 However, one common assumption of all the above models is that, regard-
less of the size of the susceptible population, the pulse vaccination control is
implemented at fixed periods. The obvious conclusion is that as long as the
pulse period is small enough, the disease can be controlled and eradicated even-
tually. Moreover, from a mathematical point of view, fixed moment control
40 will result in a non-autonomous system which poses a considerable challenge to
theoretical analysis. In particular, it is difficult to define the control reproduc-
tion number R_c and investigate the threshold dynamics of the proposed models
[8, 11, 12, 13, 14, 7]. In order to overcome the above shortcomings, a different
modelling method is proposed in the present paper, i.e. we consider whether
45 or not the integrated control strategy is implemented depends on the number
in the susceptible population rather than having it applied at a fixed period.
This can be modelled by state-dependent impulsive differential equations (so
called impulsive semi-dynamical systems) [22, 23, 24, 25, 26, 27, 28, 29, 30, 31].
Recently, impulsive semi-dynamical systems have been widely used to model
50 biological systems with threshold control strategies, such as biological resource
and pest management programmes, and chemostat cultures in ecological sys-
tems [32, 33, 34, 27, 28, 35, 36, 37].

Therefore, in the present paper we extend the classic SIR infectious model
with a state dependent feedback control strategy. In particular, although state

55 dependent impulsive models triggered by the infectious population size are rea-
sonable, such models do not have a feasible disease-free equilibrium nor can the
disease be completely eradicated, so far as mathematical and epidemiological
points of view are concerned. Therefore, the time when vaccination strategies
are implemented should, perhaps, be dependent on the level of susceptibility in-
60 stead of disease infection rates. So, in this study, we propose a state-dependent
impulsive model describing susceptible population-triggered vaccination and iso-
lation incorporating continuous treatment for the patients. We assume that
there exists a threshold level S_v for the susceptible population such that inte-
grated control measures (pulse vaccination, treatment or isolation strategies) are
65 implemented once the susceptible population number reaches S_v . Furthermore,
the numbers in the susceptible and infected populations that are vaccinated
and treated (or isolated), respectively, depend on their densities. This indi-
cates that the pulse controls are nonlinear due to limitations on the amount of
resources available. Note that linear pulse control has been addressed in the
70 reference [38], from which the existence and stability of a disease free periodic
solution were studied, and the bifurcations related to the key parameters were
also investigated.

The main purpose of this study is to develop analytical techniques and pro-
vide a comprehensive qualitative analysis of the global dynamics by analyzing a
75 planar impulsive SIR semi-dynamic model, and to address the effects of nonlin-
ear feedback pulse control on the dynamics including the bifurcations in compar-
ison with the results obtained in [38]. To achieve these aims, the existence and
global stability of the disease free periodic solution (DFPS), which corresponds
to the disease free equilibrium, are first addressed. The control reproduction
80 number R_c for the model with state-dependent feedback control can be defined
by the Floquet multiplier which ensures the local stability of the DFPS. It is
interesting that the DFPS may also be globally stable even if the basic repro-
duction number R_0 of the classic SIR model is larger than one. In order to
depict the threshold dynamics determined by the R_c , we employ the bifurcation
85 theories of the discrete one-parameter family of maps, which is determined by

the Poincaré map of the planar impulsive SIR semi-dynamic model. We choose the maximal vaccination rate, maximal treatment rate or isolation rate, threshold level S_v and birth rate of the susceptible population to reveal transcritical and pitchfork bifurcations, which can depict the threshold dynamics completely.

90 The main bifurcation results indicate that under certain conditions a stable or an unstable interior periodic solution could be generated through transcritical and pitchfork bifurcations. In particular, the stable DFPS and the interior equilibrium of the SIR model can coexist once an unstable interior periodic solution has bifurcated ($R_c < 1 < R_0$ here), i.e. backward bifurcation occurs. Further-

95 more, we discuss the corresponding biological implications related to the disease control and vaccine design in bioengineering.

2. The model

Let $S(t)$, $I(t)$ and $R(t)$ be the densities of susceptible, infected and removed parts of the population at time t , respectively, and then $N(t) = S(t) + I(t) + R(t)$ denotes the total population. Without loss of generality, we may assume that

100 the total population $N(t)$ is a constant or tends to a constant as t approaches infinity. Therefore, for the classical SIR model we only need to consider the following two dimensional system:

$$\begin{cases} \frac{dS(t)}{dt} = \Lambda - \beta SI - \delta S, \\ \frac{dI(t)}{dt} = \beta SI - qI, \end{cases} \quad (1)$$

where Λ denotes the birth rate, δ is the death rate, γ represents the recovery rate with $q = \gamma + \delta$, and β denotes the transmission rate.

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It is easy to know that the region

$$D = \{(S, I) | S \geq 0, I \geq 0, S + I \leq \Lambda/\delta\}$$

is an invariant set of model (1). Solving $\Lambda - \beta SI - \delta S = 0$ and $\beta SI - qI = 0$ with respect to I yields two isolines as follows:

$$l_1 : S = \frac{q}{\beta}, \quad l_2 : I = \frac{-\delta S + \Lambda}{\beta S} \doteq h(S),$$

where \doteq means definition in this paper.

By defining $R_0 = \frac{\Lambda\beta}{\delta q}$, we have the following results for model (1) which are useful for the coming qualitative analyses [38].

Lemma 1. *If $R_0 \leq 1$ then model (1) has a disease free equilibrium $(K, 0)$ with $K = \frac{\Lambda}{\delta}$ which is a globally stable node; If $R_0 > 1$ then there exists a unique interior equilibrium (S^*, I^*) with $S^* = \frac{q}{\beta}$, $I^* = \frac{\Lambda\beta - \delta q}{\beta q}$, which is a globally stable node (when $\Delta \geq 0$) or focus (when $\Delta < 0$). Further, if $R_0 \in (1, R_1] \cup [R_2, +\infty)$ then the unique endemic equilibrium $P^*(S^*, I^*)$ is a globally stable node; If $R_0 \in (R_1, R_2)$ then $P^*(S^*, I^*)$ is a globally stable focus, where $\Delta = \delta^2 R_0^2 - 4\delta q R_0 + 4\delta q$, $R_1 = \frac{2(\delta + \gamma - \sqrt{\gamma(\delta + \gamma)})}{\delta}$, $R_2 = \frac{2(\delta + \gamma + \sqrt{\gamma(\delta + \gamma)})}{\delta}$ and $R_2 > 2 > R_1 > 1$.*

2.1. The SIR model with state-dependent feedback nonlinear control

In order to consider the saturation phenomenon resulting from the limited resources, we fix the two impulsive functions to be nonlinear continuously differentiable functions $\left[1 - \frac{\eta_1 S(t)}{S(t) + h_1}\right] S(t)$ and $\left[1 - \frac{\eta_2 I(t)}{I(t) + h_2}\right] I(t)$. Here $1 > \eta_1 \geq 0$ represents the maximal vaccination rate and $h_1 \geq 0$ denotes the half saturation constant for the susceptible population. $1 > \eta_2 \geq 0$ represents the maximal treatment or isolation rate and $h_2 \geq 0$ denotes the half saturation constant for the infected population. We assume that the initial density of the susceptible population is less than the threshold vaccination level S_v and the integrated mitigating measures including vaccination, treatment and isolation are conducted once the density of the susceptible population reaches the threshold S_v , when the densities of both the susceptible and infected populations are updated to $\left[1 - \frac{\eta_1 S_v}{S_v + h_1}\right] S_v$ and $\left[1 - \frac{\eta_2 I(t)}{I(t) + h_2}\right] I(t)$, respectively.

Based on the above, we propose the following SIR model with state-dependent

$$\left\{ \begin{array}{l} \frac{dS(t)}{dt} = \Lambda - \beta SI - \delta S, \\ \frac{dI(t)}{dt} = \beta SI - \gamma I - \delta I, \end{array} \right\} \quad S(t) < S_v, \quad (2)$$

$$\left\{ \begin{array}{l} S(t^+) = \left[1 - \frac{\eta_1 S(t)}{S(t)+h_1}\right] S(t), \\ I(t^+) = \left[1 - \frac{\eta_2 I(t)}{I(t)+h_2}\right] I(t), \end{array} \right\} \quad S(t) = S_v.$$

Denoting the following functions $B_1(S) = -\frac{\eta_1 S^2}{S+h_1}$, $B_2(I) = -\frac{\eta_2 I^2}{I+h_2}$ and $f(I) = I + B_2(I) < I$, then by simple calculations we have

$$B_2'(I) = -\frac{\eta_2 I(I+2h_2)}{(I+h_2)^2}, \quad f'(I) = \frac{(1-\eta_2)I^2 + 2h_2(1-\eta_2)I + h_2^2}{(I+h_2)^2}.$$

It is easy to see that $f'(I) > 0$ for all $I > 0$, which indicates that $f(I)$ is a monotonically increasing function.

To prepare for the following definition and analysis of the Poincaré map, we first define two straight lines as follows:

$$l_3 : S = S_v \quad \text{and} \quad l_4 : S = S_u$$

with $S_u \doteq \left[1 - \frac{\eta_1 S_v}{S_v+h_1}\right] S_v$. Given that $0 < S_v < K$ and substituting $S = S_v$ into $h(S)$, yields the intersection point of two lines l_2 and l_3 , denoted by $Q_{S_v} = (S_v, I_{S_v})$ with

$$I_{S_v} = \frac{\Lambda - \delta S_v}{\beta S_v}.$$

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Similarly, we can get the intersection point of the two lines l_2 and l_4 , denoted by $Q_{S_u}^+ = (S_u, I_{S_u})$ with $I_{S_u} = \frac{\Lambda - \delta S_u}{\beta S_u}$.

135 2.2. Existence and stability of the DFPS

Let $I(t) = 0$ and consider the following subsystem

$$\left\{ \begin{array}{l} \frac{dS(t)}{dt} = \Lambda - \delta S, \quad S(t) < S_v, \\ S(t^+) = \left[1 - \frac{\eta_1 S(t)}{S(t)+h_1}\right] S(t), \quad S(t) = S_v. \end{array} \right. \quad (3)$$

Solving equation (3) with initial value $S_0 = S(0^+) = S_u$ we can obtain the following periodic solution

$$S^T(t) = K - (K - S_u) \exp(-\delta t)$$

with period

$$T = \int_{S_u}^{S_v} \frac{1}{\Lambda - \delta S} dS = -\frac{1}{\delta} \ln \left(\frac{K - S_v}{K - S_u} \right).$$

This indicates that a DFPS exists for model (2), denoted by $(S^T(t), 0)$, and for its stability we have the following main results

Theorem 1. *If $R_0 \leq 1$ then the DFPS $(S^T(t), 0)$ of model (2) is globally asymptotically stable.*

Proof 1. *It follows from Lemma A.1 in the Appendix that we have $\phi(S, I) = S - S_v$, $\sigma_1(S, I) = -\frac{\eta_1 S^2}{S+h_1}$ and $\sigma_2(S, I) = -\frac{\eta_2 I^2}{I+h_2}$. By simple calculation one has*

$$\frac{\partial \sigma_2}{\partial I} \frac{\partial \phi}{\partial S} - \frac{\partial \sigma_2}{\partial S} \frac{\partial \phi}{\partial I} + \frac{\partial \phi}{\partial S} = 1 - \frac{\eta_2 I(I+2h_2)}{(h_2+I)^2}, \quad \frac{\partial \sigma_1}{\partial S} \frac{\partial \phi}{\partial I} - \frac{\partial \sigma_1}{\partial I} \frac{\partial \phi}{\partial S} + \frac{\partial \phi}{\partial I} = 0$$

and $\Delta_1 = (1 - \frac{\eta_2 I(I+2h_2)}{(h_2+I)^2})P_+/P = (1 - \frac{\eta_2 I(I+2h_2)}{(h_2+I)^2}) \frac{K-S_u}{K-S_v}$. Moreover,

$$\begin{aligned} \int_0^T \left(\frac{\partial P}{\partial S} + \frac{\partial Q}{\partial I} \right) dt &= \int_0^T (\beta S^T(t) - \delta - q) dt \\ &\doteq J_1 + J_2 + J_3, \end{aligned}$$

where

$$J_1 = -\delta T = \ln \left(\frac{K - S_v}{K - S_u} \right),$$

$$J_2 = \int_0^T \beta S^T(t) dt = \int_{S_u}^{S_v} \frac{\beta S}{\Lambda - \delta S} dS = \frac{\beta}{\delta} \left(K \ln \frac{K - S_u}{K - S_v} + S_u - S_v \right)$$

and

$$J_3 = -qT = \int_{S_u}^{S_v} \frac{-q}{\Lambda - \delta S} dS = \frac{q}{\delta} \ln \left(\frac{K - S_v}{K - S_u} \right).$$

In particular, if $h_2 = 0$ then we have $B'_2(0) = -\eta_2$ with $\Delta_1 = (1 - \eta_2) \frac{K-S_u}{K-S_v}$; If $h_2 \neq 0$ then $B'_2(0) = 0$ with $\Delta_1 = \frac{K-S_u}{K-S_v}$. Therefore, we have $\mu_2 = \exp(J_2 + J_3) > 0$ for $h_2 \neq 0$, and $\mu_2 = (1 - \eta_2) \exp(J_2 + J_3) > 0$ for $h_2 = 0$.

It follows from the monotonicity of the function $\omega_1(x) \doteq \ln(1-x) + x$ and
145 inequalities $0 < S_u < S_v \leq K$ that we have $J_2 > 0$. Obviously, $J_3 < 0$ holds,
and we have

$$\begin{aligned}
J_2 + J_3 &= \int_{S_u}^{S_v} \frac{\beta S - q}{\Lambda - \delta S} dS = -\frac{\beta}{\delta}(S_v - S_u) + \frac{1}{\delta}(\beta K - q) \ln \left(\frac{K - S_u}{K - S_v} \right) \\
&= \frac{q}{\delta} \left[(R_0 - 1) \ln \left(\frac{K - S_u}{K - S_v} \right) - \frac{\beta}{q}(S_v - S_u) \right] \\
&= \frac{q}{\delta} \left[(R_0 - 1) \ln \left(\frac{R_0 - \frac{S_u}{S^*}}{R_0 - \frac{S_v}{S^*}} \right) - \left(\frac{S_v}{S^*} - \frac{S_u}{S^*} \right) \right] \\
&< \frac{q(R_0 - 1)}{\delta} \ln \left(\frac{K - S_u}{K - S_v} \right),
\end{aligned} \tag{4}$$

which indicates that if $R_0 \leq 1$ then we have $\mu_2 < 1$, and consequently the DFPS
is locally stable.

For the global stability, we only need to show that the DFPS $(S^T(t), 0)$ is
150 globally attractive. To do this, we assume, without loss of generality, that the
impulsive point series I_k^+ of any solution starting from the l_4 with $I_k^+ \in [0, I_{S_u}]$
for all $k \geq 0$. It follows from $R_0 \leq 1$ and $S_v < K$ that we have $\frac{dI}{dt} < 0$ for
 $S \leq S_v$. Thus, it follows from the properties of the function $f(I)$ that $\{I_k^+\}$
is a strictly decreasing sequence with $\lim_{k \rightarrow +\infty} I_k^+ = I_0$, as shown in Fig.1(B).
155 Moreover, we claim that $I_0 = 0$ must hold, otherwise it contradicts $\frac{dI}{dt} < 0$ for
 $S \leq S_v$. Therefore, the DFPS $(S^T(t), 0)$ is globally attractive. This completes
the proof.

Remark 1. It follows from the proof of Theorem 1 that $R_0 \leq 1$ implies $\mu_2 < 1$,
which means that the non-existence of the interior equilibrium for uncontrolled
system (1) indicates the existence and global stability of the DFPS $(S^T(t), 0)$ of
controlled system (2). Naturally, we can define the multiplier μ_2 as the control
reproduction number, denoted by R_c with

$$R_c = \exp(J_2 + J_3) > 0 \text{ for } h_2 \neq 0, \text{ and } R_c = (1 - \eta_2) \exp(J_2 + J_3) > 0 \text{ for } h_2 = 0,$$

and interesting questions that arise are as follows: (a) whether or not the thresh-
old dynamical behaviour of model (2) is determined by R_c ? (b) how to determine
160 the dynamics of model (2) when $\mu_2 = R_c < 1 < R_0$ or $R_c > 1$?

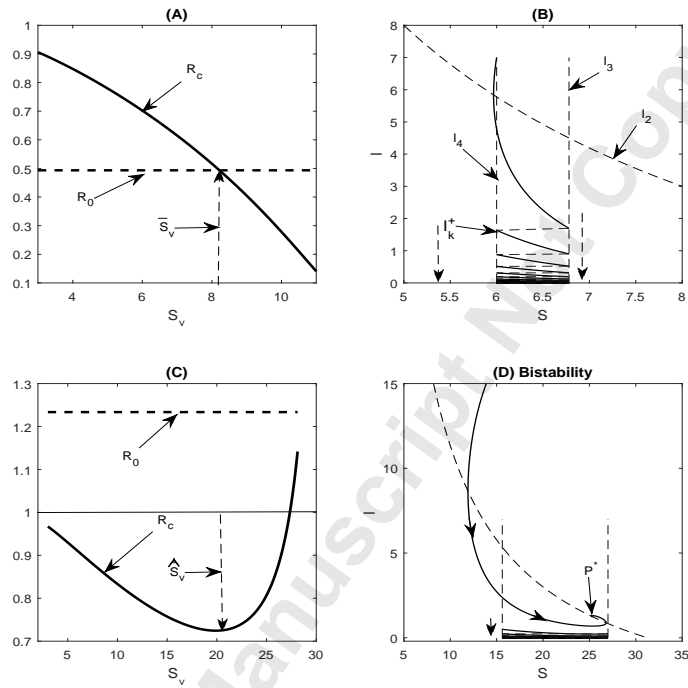


Figure 1: The relationship between R_0 and R_c in (A) and (C), and the global stability of the DFPS and bi-stability in (B) and (D). The parameter values are as follows: $\Lambda = 1$ in (A) and (B) with $S_v = 6.78$ and $\eta_1 = 0.2$, $\Lambda = 2.5$ in (C) and (D) with $S_v = 27$ and $\eta_1 = 0.5$. The other parameter values are as follows: $\beta = 0.015$, $\delta = 0.08$, $\gamma = 0.3$, $h_1 = 5$, $\eta_2 = 0.1$, $h_2 = 3$.

Remark 2. *The relationships between R_0 and R_c have been shown in Fig.1(A) and (C) for $R_0 < 1$ and $R_0 > 1$, respectively. Although $R_0 \leq 1$ indicates $R_c < 1$ (Fig.1(A)), we found that R_c could be larger than R_0 once the threshold S_v is less than the critical value \bar{S}_v . This confirms that the implementation of*
165 *integrated control measures is not conducive to the elimination of the disease when the control threshold level S_v is less than the critical value \bar{S}_v . While for $R_0 > 1$, the relations between the R_0 and R_c could be more complex which will be addressed in the Discussion section. It is interesting to note that the reproduction number R_c is a non-monotonic function of S_v , and there exists a critical*
170 *value, denoted by \hat{S}_v , such that R_c reaches its minimal value. This clarifies that the correct selection of the threshold level S_v is beneficial to the control of the disease. However, the DFPS and interior equilibrium P^* could coexist and bi-stability occurs in this case, which reveals some interesting dynamics related to the transcritical and backward bifurcations for model (2) (see more details in*
175 *the coming sections), as shown in Fig.1(D).*

3. Poincaré map and dynamics of model (2) for $R_0 \leq 1$

Although the global dynamics of model (2) for $R_0 \leq 1$ have been given in Theorem 1, in order to address the threshold dynamics related to R_c and all possible dynamics for $R_0 > 1$ we need to develop new methods, described below.

3.1. The definition of the Poincaré map

Denote $V_{S_v} = \{(S, I) | S = S_v, I \geq 0\}$ and $V_{S_u} = \{(S, I) | S = S_u, I \geq 0\}$. We choose the section V_{S_u} as a Poincaré section. Assume that the point $P_k^+ = (S_u, I_k^+)$ lies in the section V_{S_u} , and the trajectory initiating from P_k^+ will reach the section V_{S_v} in a finite time, denote the intersection point as $P_{k+1} =$
185 (S_v, I_{k+1}) , where I_{k+1} is determined by I_k^+ . Without loss of generality, we can assume that $I_{k+1} \doteq \mathcal{P}(I_k^+)$ is determined by the trajectory of model (1). A single state dependent feedback control action is implemented at point P_{k+1} such that it jumps to point $P_{k+1}^+ = (S_u, I_{k+1}^+)$ with $I_{k+1}^+ = I_{k+1} + B_2(I_{k+1})$ on

V_{S_u} . Therefore, we can define the Poincaré map P_M as

$$I_{k+1}^+ = \mathcal{P}(I_k^+) + B_2(I_{k+1}) \doteq P_M(I_k^+). \quad (5)$$

Now we define the impulsive set M as

$$\mathcal{M} = \{(S, I) \in R_+^2 \mid S = S_v, 0 \leq I \leq I_M\},$$

which is a closed subset of R_+^2 , where $I_M = \mathcal{P}(I_{S_u})$ for $R_0 \leq 1$. Define the continuous function $F : (S_v, I) \in \mathcal{M} \rightarrow (S^+, I^+) = (S_u, f(I)) \in R_+^2$, where $f(I)$ is continuous and increasing in $[0, I_M]$. Thus, the phase set can be defined as follows:

$$\mathcal{N} = F(\mathcal{M}) = \{(S^+, I^+) \in R_+^2 \mid S^+ = S_u, 0 \leq I^+ \leq f(I_M)\}.$$

190 Meanwhile, the Poincaré map P_M can be determined by the impulsive points in the phase set according to the phase portrait. To show this, we define a scalar differential equation in phase space

$$\begin{cases} \frac{dI}{dS} = \frac{I[-q + \beta S]}{\Lambda - \delta S - \beta SI} \doteq G(S, I), \\ I(S_u) = I_0^+. \end{cases} \quad (6)$$

For model (6), we only focus on the region

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$$\Omega = \{(S, I) \mid S > 0, I > 0, I < h(S)\}, \quad (7)$$

in which the function $G(S, I)$ is continuously differentiable. Further we denote $I_0^+ \doteq Y$ with $Y \in \mathcal{N}$ and $Y < I_{S_u}$, i.e. we have $(S_0^+, Y) \in \Omega$. Then we have

$$I(S) = I(S; S_u, Y) = I(S, Y), S_u \leq S \leq S_v$$

and

$$I(S, Y) = Y + \int_{S_u}^S G(s, I(s, Y)) ds.$$

Thus, P_M takes the form in Ω

$$P_M(I_k^+) = I_{k+1}^+ = I(S_v, I_k^+) + B_2(I(S_v, I_k^+)),$$

$$P_M(Y) = I(S_v, Y) + B_2(I(S_v, Y)) = f(I(S_v, Y)).$$

Theorem 2. For $R_0 \leq 1$, the Poincaré map P_M of model (2) satisfies the following properties:

- a) the domain and range of P_M are $[0, I_{S_u}] \cup (I_{S_u}, +\infty)$ and $[0, P_M(I_{S_u})]$ respectively. Moreover, P_M is continuous and concave on the interval $[0, I_{S_u}]$.
- 200 b) P_M has a unique fixed point $I = 0$ which is globally stable, i.e. the DFPS of model (2) is globally stable.

Proof 2. By simple calculations we have

$$\frac{\partial G(S, I)}{\partial I} = \frac{(\Lambda - \delta S)(-q + \beta S)}{(\Lambda - \delta S - \beta SI)^2}, \quad \frac{\partial^2 G(S, I)}{\partial I^2} = \frac{2(\Lambda - \beta S)\beta S(-q + \beta S)}{(\Lambda - \delta S - \beta SI)^3}.$$

It follows from $S_u \leq S_v < K$ and $R_0 \leq 1$ that $\Lambda - \delta S > 0$ and $-q + \beta S < 0$ for $S \leq S_v$, while $\Lambda - \delta S - \beta SI > 0$ for $I < I_{S_u}$ and $\Lambda - \delta S - \beta SI < 0$ for $I > I_{S_u}$. All these results confirm that $\frac{\partial G(S, I)}{\partial I} < 0$ and $\frac{\partial^2 G(S, I)}{\partial I^2} < 0$ for all $I < I_{S_u}$.

According to the theorem of Cauchy and Lipschitz with parameters on the scalar differential equation we have

$$\frac{\partial I(S, Y)}{\partial Y} = \exp\left(\int_{S_u}^S \frac{\partial}{\partial I} G(s, I(s, Y)) ds\right) > 0$$

205 and

$$\frac{\partial^2 I(S, Y)}{\partial Y^2} = \frac{\partial I(S, Y)}{\partial Y} \int_{S_u}^S \frac{\partial^2}{\partial I^2} G(s, I(s, Y)) \frac{\partial I(s, Y)}{\partial Y} ds < 0. \quad (8)$$

Furthermore, it follows from the definition of the function

$$P_M(Y) = I(S_v, Y) \left(1 - \frac{\eta_2 I(S_v, Y)}{h_2 + I(S_v, Y)}\right) = f(I(S_v, Y))$$

that we have

$$\begin{aligned} \frac{\partial P_M(Y)}{\partial Y} &= \left(1 - \frac{\eta_2 I(S_v, Y)(I(S_v, Y) + 2h_2)}{(h_2 + I(S_v, Y))^2}\right) \frac{\partial I(S_v, Y)}{\partial Y} \\ &= \left(1 - \frac{\eta_2 I(S_v, Y)(I(S_v, Y) + 2h_2)}{(h_2 + I(S_v, Y))^2}\right) \exp\left(\int_{S_u}^{S_v} \frac{\partial}{\partial I} G(s, I(s, Y)) ds\right) \\ &= f'(I(S_v, Y)) \exp\left(\int_{S_u}^{S_v} \frac{\partial}{\partial I} G(s, I(s, Y)) ds\right) \end{aligned} \quad (9)$$

and

$$\begin{aligned}
\frac{\partial^2 P_M(Y)}{\partial Y^2} &= \frac{\partial^2 I(S_v, Y)}{\partial Y^2} + \frac{\partial^2 I(S_v, Y)}{\partial Y^2} \frac{\partial B_2(I)}{\partial I} \Big|_{I=I(S_v, Y)} \\
&\quad + \frac{\partial I(S_v, Y)}{\partial Y} \left(\frac{\partial}{\partial Y} \frac{\partial B_2(I)}{\partial I} \Big|_{I=I(S_v, Y)} \right) \\
&= (B_2'(I(S_v, Y)) + 1) \frac{\partial^2 I(S_v, Y)}{\partial Y^2} + \left(\frac{\partial I(S_v, Y)}{\partial Y} \right)^2 B_2''(I(S_v, Y)) \\
&= f'(I(S_v, Y)) \frac{\partial^2 I(S_v, Y)}{\partial Y^2} - \left(\frac{\partial I(S_v, Y)}{\partial Y} \right)^2 \frac{2\eta_2 h_2^2}{(I(S_v, Y) + h_2)^3}.
\end{aligned} \tag{10}$$

Note that if $h_2 = 0$ then $B_2''(0) = 0$, thus one term $-\frac{2\eta_2}{h_2}$ will disappear from the formula for $\frac{\partial^2 P_M(0)}{\partial Y^2}$. Then, it follows from the monotonicity of the function $f(I)$ that if $Y \in (0, I_{s_u}]$ then $\frac{\partial P_M(Y)}{\partial Y} > 0$ and $\frac{\partial^2 P_M(Y)}{\partial Y^2} < 0$, which indicate that $P_M(Y)$ is continuous and concave on the interval $(0, I_{s_u}]$. Further, according to the monotonicity of $P_M(Y)$ on the interval $(0, I_{s_u}]$ we see that $P_M(Y)$ is monotonically decreasing on the interval $(I_{s_u}, +\infty)$. Therefore, it follows from $R_0 \geq 1$ and $S_v < K$ that $\frac{dI}{dt} \leq 0$ for all $S < S_v$, which indicates that $Y \geq \mathcal{P}(Y) > P_M(Y)$ for all $Y \in (0, I_{s_u}] \cup (I_{s_u}, K)$ due to $f(I) < I$. All these results confirm that the Poincaré map P_M only has zero fixed point, i.e. $P_M(0) = 0$, which is globally stable. Consequently, for model (2) there exists a unique DFPS which is globally stable.

Remark 3. It follows from the proofs of Theorems 1 and 2 that the global stability of the DFPS can be proved by using different methods, and the methods shown in Theorem 2 could be widely employed in generalized systems.

4. Bifurcation and reproduction number

Note that if $R_0 > 1$, then for model (1) there exists a unique endemic equilibrium $P^*(S^*, I^*)$. Thus, according to the positions among S_v , S^* and K we consider the following two cases:

- (C1), $S_v \leq S^* < K$, i.e. $R_0 > 1 \geq \frac{\beta S_v}{q}$;
- (C2), $S^* < S_v < K$, i.e. $1 < \frac{\beta S_v}{q} < R_0$.

For case (C1), any solution initiating from the line l_4 will experience infinitely many impulsive effects. Moreover, it follows from $\frac{\beta S - q}{\Lambda - \delta S} < 0$ for all $S \in [S_u, S_v]$ due to $S_v \leq S^*$ that we have $J_2 + J_3 < 0$, i.e. $R_c < 1$. This indicates that the DFPS is locally stable for case (C1).

Moreover, for case (C1), the Poincaré map P_M is well defined, which satisfies all properties shown in Theorem 2 by using similar methods. Thus, for case (C1) we have the following main results:

Theorem 3. *If $R_0 > 1 \geq \frac{\beta S_v}{q}$ then the DFPS $(S^T(t), 0)$ of model (2) is globally stable.*

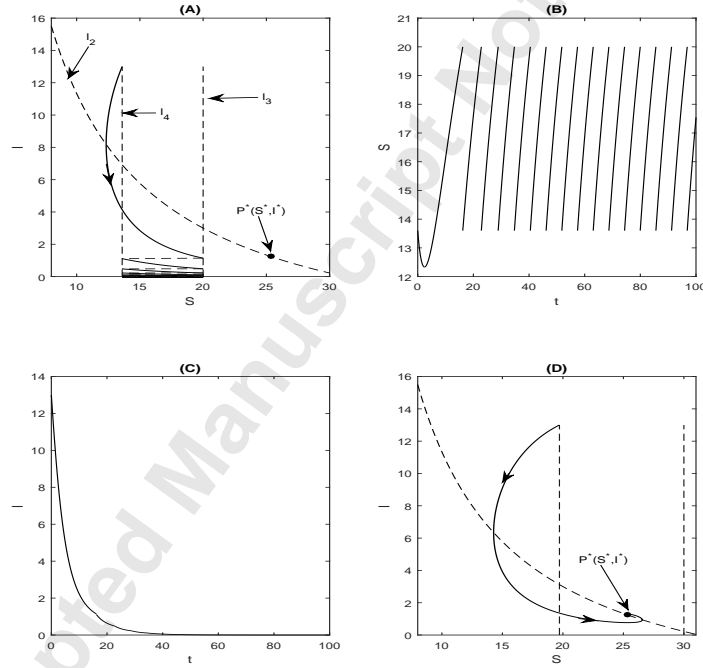


Figure 2: Illustration of the global stability of the DFPS for case (C1), i.e. $R_0 > 1 \geq \frac{\beta S_v}{q}$. The parameter values are as follows: $\Lambda = 2.5$, $\beta = 0.015$, $\delta = 0.08$, $\gamma = 0.3$, $\eta_1 = 0.4$, $h_1 = 5$, $\eta_2 = 0.1$, $h_2 = 3$ and $S_v = 20$ in (A)-(C), $S_v = 30$ in (D).

Remark 4. Under the conditions of Theorem 3, we see that the uncontrolled ODE model will be stable in the endemic state $P^*(S^*, I^*)$, as shown in Fig.2(D). If so, we only need to correctly choose the threshold of the susceptible population S_v , i.e. $S_v < S^*$, then the disease can be successfully controlled such that it quickly declines towards extinction, as shown in Fig.2(A-C).

For case (C2), since the sign of $J_2 + J_3$ can vary, the DFPS could be unstable in this case. Thus, interesting dynamics may occur as parameter values vary. For convenience, we only need to assume that the Poincaré map P_M is well defined in the domain $U(0^+) = [0, \epsilon)$ for $\epsilon > 0$ small enough in the following, as shown in Fig.1(D), from which we can see that the P_M is only well defined in a small interval $U(0^+) = [0, \epsilon)$. Based on this assumption, we address the bifurcations related to the DFPS and discuss the threshold dynamics determined by the R_0 and R_c in the following.

4.1. Transcritical and pitchfork bifurcations for η_1

In this subsection we choose η_1 as the bifurcation parameter and focus on $h_2 \neq 0$ first. Thus, we consider $J_2 + J_3$ as a function of η_1 , i.e.

$$J_{12}(\eta_1) \doteq J_2 + J_3 = -\frac{\beta}{\delta}(S_v - S_u) + \frac{1}{\delta}(\beta K - q) \ln \left(\frac{K - S_u}{K - S_v} \right), \quad R_c(\eta_1) = \exp(J_{12}(\eta_1))$$

with $S_u = S_v + B_1(S_v, \eta_1)$. By simple calculation we have

$$\frac{\partial S_u}{\partial \eta_1} = \frac{\partial B_1(S_v, \eta_1)}{\partial \eta_1} = -\frac{S_v^2}{S_v + h_1} < 0$$

and

$$\frac{dR_c(\eta_1)}{d\eta_1} = \exp(J_{12}(\eta_1)) \frac{\partial B_1(S_v, \eta_1)}{\partial \eta_1} \left(-\frac{\beta S_u - q}{\Lambda - \delta S_u} \right).$$

Solving $\frac{dR_c(\eta_1)}{d\eta_1} = 0$ with respect to η_1 yields a unique root, denoted by $\bar{\eta}_1$, which is equivalent to the unique root of the equation $S_u = S^*$, i.e.

$$\bar{\eta}_1 = \left(1 - \frac{S^*}{S_v} \right) \left(1 + \frac{h_1}{S_v} \right) > 0.$$

In order to ensure that $\bar{\eta}_1$ (i.e. $0 < \bar{\eta}_1 \leq 1$), we need $\frac{S_v h_1}{S_v + h_1} \leq S^* < S_v$.

Note that if $\eta_1 \in (0, \bar{\eta}_1)$ then we have $S_u > S^*$ and $\frac{dR_c(\eta_1)}{d\eta_1} > 0$; if $\eta_1 \in (\bar{\eta}_1, 1)$ then we have $S_u < S^*$ and $\frac{dR_c(\eta_1)}{d\eta_1} < 0$. Moreover, it follows from $R_c(0) = 1$ and $R_c(\bar{\eta}_1) = \exp(\int_{S^*}^{S_v} \frac{\beta s - q}{\Lambda - \delta s} ds) > 1$ that $R_c(\eta_1) > 1$ for $S_u > S^*$ (i.e. $\eta_1 \in (0, \bar{\eta}_1)$).
 255 Thus, the DFPS is unstable and bifurcation does not occur at all. If $S_u < S^*$ (i.e. $\eta_1 \in (\bar{\eta}_1, 1)$), then the bifurcation could occur provided that there exists an $\eta_1^* \in (\bar{\eta}_1, 1)$ with $R_c(\eta_1^*) = 1$, which means that we need $R_c(1) < 1$. Further, according to the monotonicity of $R_c(\eta_1)$ we conclude that η_1^* is unique. All these results confirm that if $0 < \eta_1 < \eta_1^*$ then the periodic solution $(S^T(t), 0)$
 260 is unstable; if $1 > \eta_1 > \eta_1^*$ then the periodic solution $(S^T(t), 0)$ is stable. This shows that the possible bifurcation could occur at $\eta_1 = \eta_1^*$.

Furthermore, if $\eta_1 = 1$ then $S_u = \frac{h_1 S_v}{h_1 + S_v}$. It follows from $\frac{K - S_u}{K - S_v} > 1$ that

$$\ln \left(\frac{K - S_u}{K - S_v} \right) < \frac{\frac{K - S_u}{K - S_v} - 1}{\sqrt{\frac{K - S_u}{K - S_v}}} = \frac{S_v - S_u}{\sqrt{(K - S_u)(K - S_v)}}.$$

Therefore, we have

$$\begin{aligned} J_{12}(1) &= \frac{\beta}{\delta} \left[-(S_v - S_u) + (K - S^*) \ln \left(\frac{K - S_u}{K - S_v} \right) \right] \\ &< \frac{\beta}{\delta} (S_v - S_u) \left[-1 + \frac{(K - S^*)}{\sqrt{(K - S_u)(K - S_v)}} \right]. \end{aligned}$$

That is, in order to ensure that $J_{12}(1) < 0$ (i.e. $R_c(1) < 1$), we only need $K - S^* < \sqrt{(K - S_v)(K - S_u)}$, i.e. $\frac{h_1 S_v}{h_1 + S_v} < K - \frac{(K - S^*)^2}{K - S_v}$ with $K - \frac{(K - S^*)^2}{K - S_v} < S^*$. According to

$$0 < \frac{h_1 S_v}{h_1 + S_v} < K - \frac{(K - S^*)^2}{K - S_v} = S^* \left[R_0 - \frac{(R_0 - 1)^2}{R_0 - \frac{S_v}{S^*}} \right]$$

we have $S^* > K - \sqrt{K(K - S_v)}$ (i.e. $R_0 > \frac{1}{2 - \frac{S_v}{S^*}} > 1$). Based on the above discussion, we have the following main results related to the bifurcation parameter η_1 and reproduction number R_c .

Theorem 4. If $1 < \frac{S_v}{S^*} < R_0$, $R_c(1) < 1$ and $M \doteq \frac{\partial^2 I(S_v, 0)}{\partial Y^2} < \frac{2\eta_2}{h_2}$, then
 265 $P_M(Y, \eta_1)$ and the transcritical bifurcation occurs at $\eta_1 = \eta_1^*$. That is, a stable positive fixed point of the $P_M(Y, \eta_1)$ appears when the parameter η_1 changes through η_1^* from right to left. Correspondingly, system (2) has a stable positive

periodic solution if $\eta_1 \in (\eta_1^* - \epsilon, \eta_1^*)$ with $\epsilon > 0$ small enough. However, if
 270 $M > \frac{2\eta_2}{h_2}$, an unstable positive fixed point of the $P_M(Y, \eta_1)$ appears when the
 parameter η_1 changes through η_1^* from left to right. Correspondingly, system (2)
 has an unstable positive periodic solution if $\eta_1 \in (\eta_1^*, \eta_1^* + \epsilon)$ with $\epsilon > 0$ small
 enough.

Proof 3. Since $P_M(Y, \eta_1)$ is well defined in the domain $U(0^+) = [0, \epsilon)$, we
 275 see that it is continuous and differentiable. In order to prove Theorem 4, we
 only need to verify that P_M satisfies the four conditions of Lemma A.2 in the
 Appendix.

Letting $I(S; S_u, Y) = I(S, Y)$, we have $P_M(Y, \eta_1) = I(S_v, Y)$ and $P_M(0, \eta_1) =$
 $I(S_v, 0) = 0$, which indicates that the first condition of Lemma A.2 holds true.
 According to the inequality (9) we have

$$\frac{\partial P_M(0, \eta_1)}{\partial Y} = \exp\left(\int_{S_u}^{S_v} \frac{\beta S - q}{\Lambda - \delta S} dS\right) = R_c(\eta_1).$$

Thus, $\frac{\partial P_M(0, \eta_1^*)}{\partial Y} = R_c(\eta_1^*) = 1$ and the second condition of Lemma A.2 follows.

By simple calculations we have

$$\begin{aligned} \frac{\partial^2 P_M(0, \eta_1)}{\partial Y \partial \eta_1} &= \frac{\partial S_u}{\partial \eta_1} \frac{\partial}{\partial S_u} \left(\frac{\partial P_M(0, \eta_1)}{\partial Y} \right) = \frac{\partial S_u}{\partial \eta_1} \frac{\partial}{\partial S_u} \left(\frac{\partial I(S_v, Y)}{\partial Y} \right) \\ &= \left(\frac{\partial I(S_v, Y)}{\partial Y} \right) \frac{\partial S_u}{\partial \eta_1} \frac{\partial}{\partial S_u} \int_{S_u}^{S_v} \frac{\partial G(s, I(s, 0))}{\partial Y} ds = \frac{dR_c(\eta_1)}{d\eta_1}, \end{aligned} \quad (11)$$

280 which indicates that $\frac{\partial^2 P_M(0, \eta_1)}{\partial Y \partial \eta_1} = \frac{dR_c(\eta_1^*)}{d\eta_1} < 0$, and the third condition of Lemma
 A.2 follows.

Further, it follows from inequality (10) that

$$\frac{\partial^2 P_M(0, \eta_1)}{\partial Y^2} = \exp(J_{12}) \int_{S_u}^{S_v} \frac{\partial^2}{\partial I^2} (G(s, I(s, 0))) \frac{\partial I(s, 0)}{\partial Y} ds - \frac{2\eta_2}{h_2} \exp(2J_{12}). \quad (12)$$

Denote

$$\begin{aligned} l_1(s) &= \int_{S_u}^s \frac{\beta s - q}{\Lambda - \delta s} ds, \\ k_1(s) &= \frac{\partial I(s, 0)}{\partial Y} = \exp\left(\int_{S_u}^s \frac{\beta s - q}{\Lambda - \delta s} ds\right) = \exp l_1(s), \end{aligned}$$

and it is easy to know that $k_1(S_u) = 1$ and $k_1(S_v) = R_c(\eta_1)$. Taking the derivative of $l_1(s)$ with respect to s yields

$$l_1'(s) = \frac{\beta s - q}{\Lambda - \delta s}.$$

Letting

$$l_2(s) = \frac{\partial^2}{\partial I^2} G(s, I(s, 0)) = \frac{2\beta s(-q + \beta s)}{(\Lambda - \delta s)^2}, \quad k_2(s) = \frac{l_2(s)}{l_1'(s)} = \frac{2\beta s}{\Lambda - \delta s} = \frac{2}{h(s)},$$

and $S_{u\eta_1^*} = (1 - \frac{\eta_1^* S_v}{S_v + h_1}) S_v$, then we have

$$\frac{\partial^2 P_M(0, \eta_1^*)}{\partial Y^2} = \frac{\partial^2 I(S_v, 0)}{\partial Y^2} - \frac{2\eta_2}{h_2}, \quad (13)$$

where

$$\begin{aligned} M &\doteq \frac{\partial^2 I(S_v, 0)}{\partial Y^2} = \int_{S_{u\eta_1^*}}^{S_v} \frac{\partial^2}{\partial I^2} (G(s, I(s, 0))) \frac{\partial I(s, 0)}{\partial Y} ds \\ &= \int_{S_{u\eta_1^*}}^{S_v} l_2(s) k_1(s) ds \\ &= \int_{S_{u\eta_1^*}}^{S_v} \frac{l_2(s)}{l_1'(s)} l_1'(s) \exp l_1(s) ds \\ &= \int_{S_{u\eta_1^*}}^{S_v} k_2(s) d(k_1(s)). \end{aligned} \quad (14)$$

285 Note that the function $k_1(s)$ is monotonically decreasing on the interval $[S_u, S^*]$ and monotonically increasing on the interval $[S^*, S_v]$, which indicates that if $\eta_1 = \eta_1^*$ then $k_1(S_{u\eta_1^*}) = R_c(\eta_1^*) = 1$. Thus, $k_1(S^*) < k_1(s) \leq 1$ for all $s \in [S_{u\eta_1^*}, S_v]$. According to $k_2'(s) = \frac{2\Lambda\beta}{(\Lambda - \beta s)^2} > 0$ for any $s \in [S_{u\eta_1^*}, S_v]$, $k_2(s)$ is a monotonically increasing function. Moreover, we have

$$\begin{aligned} \int_{S_{u\eta_1^*}}^{S_v} k_2(s) d(k_1(s)) &= k_1(s)k_2(s)|_{S_{u\eta_1^*}}^{S_v} - \int_{S_{u\eta_1^*}}^{S_v} k_1(s)k_2'(s) ds \\ &= k_2(S_v) - k_2(S_{u\eta_1^*}) - \int_{S_{u\eta_1^*}}^{S_v} k_1(s)k_2'(s) ds \\ &= \int_{S_{u\eta_1^*}}^{S_v} (1 - k_1(s))k_2'(s) ds, \end{aligned} \quad (15)$$

290 which means that $0 < M < (1 - k_1(S^*))(k_2(S_v) - k_2(S_{u\eta_1^*}))$.

Therefore, if $1 \leq \frac{S_v}{S^*} < R_0$, $R_c(1) < 1$ and $M \neq \frac{2\eta_2}{h_2}$, then the transcritical bifurcation occurs at $\eta_1 = \eta_1^*$. Furthermore, if $M > \frac{2\eta_2}{h_2}$, then $P_M(Y, \eta_1)$ generates an unstable fixed point when η_1 passes η_1^* from left to right. That is, for

$\eta_1 \in (\eta_1^*, \eta_1^* + \epsilon)$ with $\epsilon > 0$ small enough, an unstable positive periodic solution
 295 exists for model (2), as shown in Fig.1(D). While, if $M < \frac{2\eta_2}{h_2}$, then $P_M(Y, \eta_1)$
 exists with a positive stable fixed point when η_1 goes through η_1^* from right to
 left. That is, for $\eta_1 \in (\eta_1^* - \epsilon, \eta_1^*)$ with $\epsilon > 0$ small enough, a stable positive
 periodic solution exists for model (2).

Corollary 1. If $1 < \frac{\beta S_v}{q} < R_0$, $R_c(1) < 1$ and $M > \frac{2\eta_2}{h_2}$, system (2) undergoes
 300 the backward bifurcation at $\eta_1 \in (\eta_1^*, \eta_1^* + \epsilon)$ with $\epsilon > 0$ small enough.

Note that the condition $R_c(1) < 1$ can be replaced by the inequality $\frac{h_1 S_v}{h_1 + S_v} <$
 $K - \frac{(K - S^*)^2}{K - S_v}$. It follows from the conditions of Corollary 1 that the positive
 equilibrium P^* exists which is stable for model (2) due to $S_v > S^*$. Moreover,
 the DFPS $(S^T(t), 0)$ is stable for all $\eta_1 \in (\eta_1^*, 1)$, i.e. we have $R_c(\eta_1) < 1$ for all
 305 $\eta_1 \in (\eta_1^*, 1)$. Therefore, the stable DFPS $(S^T(t), 0)$ and stable equilibrium P^*
 can coexist, as shown in Fig.1(D), and there exists an unstable order-1 periodic
 solution which is bifurcated from $(S^T(t), 0)$ once the parameter η_1 increases and
 exceeds the critical value η_1^* . Obviously, the transcritical bifurcation generates a
 backward bifurcation, which is a novel result generated by the state-dependent
 310 pulse vaccination model (2).

In the following, we address the special case, i.e. $M = \frac{2\eta_2}{h_2}$. For this special
 case we only need to calculate $\frac{\partial^3 P_M(0, \eta_1^*)}{\partial Y^3}$, i.e.

$$\begin{aligned}
 \frac{\partial^3 P_M(0, \eta_1^*)}{\partial Y^3} &= \frac{\partial^3 I(S_v, 0)}{\partial Y^3} + 3 \int_{S_{u\eta_1^*}}^{S_v} k_2(s) d(\exp l_1(s)) B_2''(0) + B_2'''(0) \\
 &= \frac{\partial^3 I(S_v, 0)}{\partial Y^3} - \frac{6\eta_2}{h_2} M + \frac{6\eta_2}{h_2^2} \\
 &= \frac{\partial^3 I(S_v, 0)}{\partial Y^3} - \frac{12\eta_2^2}{h_2^2} + \frac{6\eta_2}{h_2^2},
 \end{aligned} \tag{16}$$

where

$$\frac{\partial^3 I(S_v, 0)}{\partial Y^3} = \frac{4\eta_2^2}{h_2^2} + \frac{\partial}{\partial Y} \left(\int_{S_{u\eta_1^*}}^{S_v} \frac{\partial^2}{\partial I^2} (G(s, I(s, 0))) \frac{\partial I(s, 0)}{\partial Y} ds \right) \tag{17}$$

and

$$\frac{\partial}{\partial Y} \left(\int_{S_{u\eta_1^*}}^{S_v} \frac{\partial^2}{\partial I^2} (G(s, I(s, 0))) \frac{\partial I(s, 0)}{\partial Y} ds \right) = \int_{S_{u\eta_1^*}}^{S_v} l_2(s) k_1(s) \left[\frac{3}{2} k_1(s) k_2(s) + l_1(s) \right] ds. \tag{18}$$

315 Therefore, according to Lemma A.3 we have the following main results for this special case:

Theorem 5. *If $1 < \frac{S_v}{S^*} < R_0$, $R_c(1) < 1$, $M = \frac{2\eta_2}{h_2}$ and $\frac{\partial^3 P_M(0, \eta_1^*)}{\partial Y^3} \neq 0$ then the Poincaré map of system (2) occurs with a pitchfork bifurcation at η_1^* . Furthermore, if $\frac{\partial^3 P_M(0, \eta_1^*)}{\partial Y^3} < 0$, then the Poincaré map (2) occurs with a subcritical pitchfork bifurcation such that it appears as a stable positive fixed point; if 320 $\frac{\partial^3 P_M(0, \eta_1^*)}{\partial Y^3} > 0$, then the Poincaré map (2) occurs with a supercritical pitchfork bifurcation such that it appears as an unstable positive fixed point.*

Note that the formula of R_c depends on the h_2 , in particular if $h_2 = 0$ then we have $R_c(\eta_1) = (1 - \eta_2) \exp(J_{12}(\eta_1))$, thus $R_c(0) = 1 - \eta_2$, $R_c(\bar{\eta}_1) > 1 - \eta_2$. It follows from the monotonicity of $J_{12}(\eta_1)$ that $R_c(\eta_1)$ is monotonically increasing on the interval $[0, \bar{\eta}_1]$ and monotonically decreasing on the interval $(\bar{\eta}_1, 1)$. In order to ensure that η_1^* exists and satisfies $R_c(\eta_1^*) = 1$, we need $R_c(\bar{\eta}_1) > 1$ which indicates that $\eta_1^* \in (0, \bar{\eta}_1)$. Further, if we have $R_c(1) < 1$ then there exists a unique $\eta_1^{**} \in (\bar{\eta}_1, 1)$ such that $R_c(\eta_1^{**}) = 1$. Therefore, if both the η_1^* and η_1^{**} exist, then we have

$$\begin{aligned} \frac{\partial^2 P_M(0, \eta_1^*)}{\partial Y \partial \eta_1} &= \frac{dR_c(\eta_1^*)}{d\eta_1} > 0, & \frac{\partial^2 P_M(0, \eta_1^{**})}{\partial Y \partial \eta_1} &= \frac{dR_c(\eta_1^{**})}{d\eta_1} < 0, \\ \frac{\partial^2 P_M(0, \eta_1^*)}{\partial Y^2} &= M > 0, & \frac{\partial^2 P_M(0, \eta_1^{**})}{\partial Y^2} &= M > 0, \end{aligned}$$

and consequently we have the following main result:

Corollary 2. *If $h_2 = 0$, $1 < \frac{S_v}{S^*} < R_0$, $R_c(\bar{\eta}_1) > 1$ and $R_c(1) < 1$, then 325 $P_M(Y, \eta_1)$ occurs with the transcritical bifurcation at $\eta_1 = \eta_1^*$ and $\eta_1 = \eta_1^{**}$. That is, an unstable positive fixed point of the $P_M(Y, \eta_1)$ appears when the parameter η_1 changes through η_1^* from right to left or through η_1^{**} from left to right. Correspondingly, system (2) has an unstable positive periodic solution if $\eta_1 \in (\eta_1^* - \epsilon, \eta_1^*)$ or $\eta_1 \in (\eta_1^{**}, \eta_1^{**} + \epsilon)$ with $\epsilon > 0$ small enough.*

Note that when $h_2 = 0$, we could choose η_2 as a bifurcation parameter. If so, we have

$$R_c(\eta_2) = (1 - \eta_2) \exp(J_1 + J_2).$$

330 Letting $R_c(\eta_2) = 1$ and solving it one has $\eta_2^* = 1 - \exp(-J_1 - J_2)$ such that $R_c(\eta_2^*) = 1$ which requires $J_1 + J_2 > 0$ to ensure that η_2^* is well defined. Moreover, $S_u > S^*$ implies $J_1 + J_2 > 0$ holds true.

It follows from (4) that

$$J_2 + J_3 = \frac{q}{\delta} \left[(R_0 - 1) \ln \left(\frac{R_0 - \frac{S_u}{S^*}}{R_0 - \frac{S_v}{S^*}} \right) - \left(\frac{S_v}{S^*} - \frac{S_u}{S^*} \right) \right].$$

Denote the function $\omega_4(x) \doteq (R_0 - 1) \ln(R_0 - x) + x$ with $\omega_4'(x) = \frac{1-x}{R_0-x}$. Thus, $\omega_4(x)$ is monotonically increasing on the interval $[0, 1)$, and decreasing on the interval $[1, R_0)$. Moreover, $x = R_0$ is an asymptote of the function $\omega_4(x)$, and both $\frac{S_u}{S^*}$ and $\frac{S_v}{S^*} \in [0, R_0)$, then we have

$$J_2 + J_3 = \frac{q}{\delta} \left[\omega_4\left(\frac{S_u}{S^*}\right) - \omega_4\left(\frac{S_v}{S^*}\right) \right].$$

It follows from the monotonicity of the function $\omega_4(x)$ that if $S_u > S^*$ then $J_2 + J_3 > 0$. Moreover, $\omega_4\left(\frac{S_v}{S^*}\right) < \omega_4(0) = (R_0 - 1) \ln(R_0)$ also indicates that $J_2 + J_3 > 0$. Solving the inequality $\omega_4\left(\frac{S_v}{S^*}\right) < (R_0 - 1) \ln(R_0)$ one has $\frac{S_v}{S^*} \geq (R_0 - 1) \text{LambertW}\left(-\frac{R_0}{R_0-1} \exp\left(-\frac{R_0}{R_0-1}\right)\right) + R_0 \doteq \zeta$, and it is easy to see that $\zeta \in (1, R_0)$, where the definition and properties of the Lambert W function can be found in references [27, 29, 31]. Therefore,

$$\frac{\partial P_M(0, \eta_2)}{\partial Y} = R_c(\eta_2), \quad \frac{\partial^2 P_M(0, \eta_2)}{\partial Y \partial \eta_2} = \frac{dR_c(\eta_2)}{d\eta_2} = -\exp(J_1 + J_2) < 0$$

and

$$\frac{\partial^2 P_M(0, \eta_2^*)}{\partial Y^2} = M > 0.$$

By methods similar to those above we can evaluate the conditions of Lemma A.2, to give the following main results:

335 **Corollary 3.** *If $h_2 = 0$, $1 < \frac{\beta S_v}{q} < R_0$ and $J_2 + J_3 > 0$, then $P_M(Y, \eta_2)$ occurs with the transcritical bifurcation at $\eta_2 = \eta_2^*$. That is, an unstable positive fixed point of the $P_M(Y, \eta_2)$ appears when the parameter η_2 changes through η_2^* from left to right. Correspondingly, system (2) has an unstable positive periodic solution if $\eta_2 \in (\eta_2^*, \eta_2^* + \epsilon)$ with $\epsilon > 0$ small enough. Particular-*
 340 *ly, the condition of $J_2 + J_3 > 0$ can be strengthened to be $1 \leq \frac{S_u}{S^*}$ or $\frac{S_v}{S^*} \geq (R_0 - 1) \text{LambertW}\left(-\frac{R_0}{R_0-1} \exp\left(-\frac{R_0}{R_0-1}\right)\right) + R_0$.*

4.2. Transcritical bifurcation for threshold level S_v

In this subsection, we choose the threshold level S_v as a bifurcation parameter, which can help us to evaluate how to determine the number in the population to be vaccinated such that the disease could be eradicated. To do this, we consider the control reproduction number R_c as a function of S_v , i.e. we have $R_c(S_v) = \exp(J_{12}(S_v))$ and

$$\begin{aligned} J_{12}(S_v) \doteq J_2 + J_3 &= \int_{S_u}^{S_v} \frac{\beta s - q}{\Lambda - \delta s} ds \\ &= -\frac{\beta}{\delta}(S_v - S_u) + \frac{1}{\delta}(\beta K - q) \ln \left(\frac{K - S_u}{K - S_v} \right). \end{aligned}$$

By simple calculations we have

$$\frac{dR_c(S_v)}{dS_v} = \exp(J_{12}(S_v)) \frac{dJ_{12}(S_v)}{dS_v}$$

and

$$\frac{dJ_{12}(S_v)}{dS_v} = \frac{\partial}{\partial S_v} \int_{S_u}^{S_v} \frac{\beta s - q}{\Lambda - \delta s} ds,$$

where $S_u = S_v + B_1(S_v)$. Letting $\omega_3(x) \doteq \frac{\beta x - q}{\Lambda - \delta x} = \frac{\beta(x - S^*)}{\delta(K - x)}$, one has

$$\frac{dJ_{12}(S_v)}{dS_v} = \omega_3(S_v) - (1 + I_1'(S_v))\omega_3(S_u). \quad (19)$$

If $S_u \leq S^*$, then $\omega_3(S_u) \leq 0$. It follows from $1 + I_1'(S_v) > 0$ that $\frac{dJ_{12}(S_v)}{dS_v} > 0$; If $S_u > S^*$, then $\omega_3(S_u) > 0$. Taking the derivative of $\omega_3(x)$ with respect to x yields

$$\omega_3'(x) = \frac{q}{\delta} \frac{R_0 - 1}{(K - x)^2} > 0.$$

Thus, we have

$$\begin{aligned} \omega_3(S_v) - (1 + I_1'(S_v))\omega_3(S_u) &> \omega_3(S_u) - (1 + I_1'(S_v))\omega_3(S_u) \\ &= (-I_1'(S_v))\omega_3(S_u) > 0. \end{aligned} \quad (20)$$

In conclusion, no matter what the position between S_u and S^* is, we always have $\frac{dJ_{12}(S_v)}{dS_v} > 0$, i.e. $J_{12}(S_v)$ is a monotonically increasing function of S_v . Moreover, we have

$$\lim_{S_v \rightarrow S^*} J_{12}(S_v) = \lim_{S_v \rightarrow S^*} \int_{S_u}^{S_v} \omega_3(x) dx = \int_{S^* + B_1(S^*)}^{S^*} \omega_3(x) dx < 0$$

and

$$\lim_{S_v \rightarrow K} J_{12}(S_v) = \lim_{S_v \rightarrow K} \frac{q}{\delta} [(R_0 - 1) \ln \left(\frac{K - S_u}{K - S_v} \right) - \frac{\beta}{q} (S_v - S_u)] = +\infty.$$

It follows from the continuity of the function $J_{12}(S_v)$ that there exists a unique $S_v^* \in (S^*, K)$ such that $J_{12}(S_v^*) = 0$, i.e. there is a unique $S_v^* \in (S^*, K)$ such that $R_c(S_v^*) = 1$. Further, by simple calculations we have

$$\frac{\partial P_M(0, S_v)}{\partial Y} = R_c(S_v)$$

and

$$\frac{\partial^2 P_M(0, S_v)}{\partial Y \partial S_v} = \frac{dR_c(S_v)}{dS_v} > 0.$$

345 By employing similar methods to those shown in Theorem 4 we can address the signs of $\frac{\partial^2 P_M(0, S_v^*)}{\partial^2 Y}$ and $\frac{\partial^3 P_M(0, S_v^*)}{\partial^3 Y}$. Therefore, we have the following main results:

Theorem 6. *If $1 < \frac{S_v}{S^*} < R_0$ and $M < \frac{2\eta_2}{h_2}$, then $P_M(Y, S_v)$ occurs with the transcritical bifurcation at $S_v = S_v^*$. That is, a stable positive fixed point of*
 350 *the $P_M(Y, S_v)$ appears when the parameter S_v changes through S_v^* from left to right. Correspondingly, system (2) has a stable positive periodic solution if $S_v \in (S_v^*, S_v^* + \epsilon)$ with $\epsilon > 0$ small enough. Moreover, if $M > \frac{2\eta_2}{h_2}$, an unstable positive fixed point of the $P_M(Y, S_v)$ appears when the parameter S_v changes through S_v^* from right to left. Correspondingly, system (2) has an unstable*
 355 *positive periodic solution if $S_v \in (S_v^* - \epsilon, S_v^*)$ with $\epsilon > 0$ small enough.*

Theorem 7. *If $1 < \frac{S_v}{S^*} < R_0$, $M = \frac{2\eta_2}{h_2}$ and $\frac{\partial^3 P_M(0, S_v^*)}{\partial Y^3} \neq 0$, then the Poincaré map of system (2) occurs with a pitchfork bifurcation at S_v^* . Furthermore, if $\frac{\partial^3 P_M(0, S_v^*)}{\partial Y^3} < 0$, then the Poincaré map (2) occurs with a supercritical pitchfork bifurcation such that it appears as a stable positive fixed point; if $\frac{\partial^3 P_M(0, S_v^*)}{\partial Y^3} > 0$,*
 360 *then the Poincaré map (2) occurs with a subcritical pitchfork bifurcation such that it appears as an unstable positive fixed point.*

Similarly, for the special case $h_2 = 0$ we have

$$R_c(S_v) = (1 - \eta_2) \exp(J_{12}(S_v))$$

and according to the properties of the function $J_{12}(S_v)$ we have $\frac{\partial^2 P_M(0, S_v)}{\partial Y \partial S_v} = \frac{dR_c(S_v)}{dS_v} > 0$, which indicates that there exists a unique S_v^* such that $J_{12}(S_v^*) = \ln(\frac{1}{1-\eta_2}) > 0$, i.e. $R_c(S_v^*) = 1$. It follows from $\frac{\partial^2 P_M(0, S_v^*)}{\partial Y^2} = M > 0$ and Lemma

365 A.2 that

Corollary 4. *If $h_2 = 0$, $1 < \frac{\beta S_v}{q} < R_0$, then $P_M(Y, S_v)$ occurs with the trans-critical bifurcation at $S_v = S_v^*$, i.e. an unstable positive fixed point of the $P_M(Y, S_v)$ appears when the parameter S_v changes through S_v^* from right to left. Correspondingly, system (2) has an unstable positive periodic solution if*

370 $S_v \in (S_v^* - \epsilon, S_v^*)$ with $\epsilon > 0$ small enough.

4.3. Trans-critical bifurcation for Λ

In this subsection we choose Λ as a bifurcation parameter, i.e. we denote $R_c(\Lambda) = \exp(J_{12}(\Lambda)) = P_M(0, \Lambda)$ and

$$\begin{aligned} J_{12}(\Lambda) \doteq J_2 + J_3 &= \int_{S_u}^{S_v} \frac{\beta s - q}{\Lambda - \delta s} ds \\ &= -\frac{\beta}{\delta} (S_v - S_u) + \frac{1}{\delta} (-q + \beta(\frac{\Lambda}{\delta})) \ln(\frac{-\delta S_u + \Lambda}{-\delta S_v + \Lambda}). \end{aligned}$$

with

$$\lim_{\frac{\Lambda}{\delta} \rightarrow +\infty} J_{12}(\Lambda) = 0, \quad \lim_{\frac{\Lambda}{\delta} \rightarrow S_v} J_{12}(\Lambda) = +\infty.$$

By calculations we have

$$\begin{aligned} \frac{dR_c(\Lambda)}{d\Lambda} &= \frac{\partial P_M(0, \Lambda)}{\partial Y} = R_c(\Lambda) \frac{dJ_{12}(\Lambda)}{d\Lambda}, \\ \frac{dJ_{12}(\Lambda)}{d\Lambda} &= -\int_{S_u}^{S_v} \frac{\beta s - q}{(\Lambda - \delta s)^2} ds \\ &= -(S_v - S_u) \frac{\frac{\beta \Lambda}{\delta} - q}{(\Lambda - \delta S_v)(\Lambda - \delta S_u)} + \frac{\beta}{\delta^2} \ln(\frac{-\delta S_u + \Lambda}{-\delta S_v + \Lambda}) \end{aligned}$$

and $\lim_{\Lambda \rightarrow +\infty} \frac{dJ_{12}(\Lambda)}{d\Lambda} = 0$. Moreover, we have

$$\begin{aligned} \frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} &= \int_{S_u}^{S_v} \frac{2(\beta s - q)}{(\Lambda - \delta s)^3} ds \\ &= -\frac{\beta(S_u - S_v)[(S_u + S_v - 2S^*)\Lambda - (2S_u S_v - S^*(S_u + S_v))\delta]}{(-S_v \delta + \Lambda)^2 (-S_u \delta + \Lambda)^2}. \end{aligned}$$

If $S^* = \frac{S_v + S_u}{2}$, then we have $\frac{2}{\frac{1}{S_v} + \frac{1}{S_u}} < S^*$ and

$$\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} = \int_{S_u}^{S_v} \frac{2(\beta s - q)}{(\Lambda - \delta s)^3} ds = \frac{\beta(S_v - S_u)[-(\frac{2}{\frac{1}{S_v} + \frac{1}{S_u}} - S^*)(S_u + S_v)\delta]}{(-S_v \delta + \Lambda)^2 (-S_u \delta + \Lambda)^2} > 0,$$

which indicates that $\frac{dJ_{12}(\Lambda)}{d\Lambda}$ is monotonically increasing on the interval $(S_v\delta, +\infty)$.

Moreover, it follows from $\lim_{\Lambda \rightarrow +\infty} \frac{dJ_{12}(\Lambda)}{d\Lambda} = 0$ that $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$ for all $\Lambda \in (S_v\delta, +\infty)$, i.e. $J_{12}(\Lambda)$ is monotonically decreasing on the interval $(S_v\delta, +\infty)$.

375 All these results confirm that $J_{12}(\Lambda) > 0$ (i.e. $R_c(\Lambda) > 1$) for all $\Lambda \in (\delta S_v, +\infty)$,

which means that the DFPS is unstable and no bifurcation occurs with respect to Λ . If $S^* \neq \frac{S_v+S_u}{2}$, then solving $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} = 0$ yields $\tilde{\Lambda} = \delta \left(\frac{\frac{2S_v S_u}{S_v+S_u} - S^*}{(\frac{S_u+S_v}{2} - S^*) \frac{2}{S_v+S_u}} \right)$.

For the bifurcation related to the parameter Λ , we consider the following cases:

(i) If $\frac{S_v+S_u}{2} > \frac{2}{\frac{1}{S_v} + \frac{1}{S_u}} > S^*$ then we have $\tilde{\Lambda} > 0$.

380 (a) For this case we first consider $\tilde{\Lambda} > \delta S_v$, and we have $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} < 0$

for $\Lambda \in (\delta S_v, \tilde{\Lambda}]$; $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} \geq 0$ for all $\Lambda \in [\tilde{\Lambda}, +\infty)$. Correspondingly, the function $\frac{dJ_{12}(\Lambda)}{d\Lambda}$ is monotonically decreasing on the interval

$(\delta S_v, \tilde{\Lambda}]$ and monotonically increasing on the interval $[\tilde{\Lambda}, +\infty)$.

Moreover, $\lim_{\Lambda \rightarrow +\infty} \frac{dJ_{12}(\Lambda)}{d\Lambda} = 0$, thus we have $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$ for all

385 $\Lambda \in [\tilde{\Lambda}, +\infty)$. Now we claim that $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$ for all $\Lambda \in (\delta S_v, \tilde{\Lambda}]$.

Otherwise, we assume that there exists a unique $\hat{\Lambda} \in (\delta S_v, \tilde{\Lambda})$ such that $\frac{dJ_{12}(\hat{\Lambda})}{d\Lambda} = 0$, then we have $\frac{dJ_{12}(\Lambda)}{d\Lambda} \geq 0$ for all $\Lambda \in (\delta S_v, \hat{\Lambda}]$ and $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$

for all $\Lambda \in (\hat{\Lambda}, +\infty)$. Consequently, $J_{12}(\Lambda)$ is monotonically increasing

on the interval $(\delta S_v, \hat{\Lambda}]$ and decreasing on the interval $(\hat{\Lambda}, +\infty)$,

390 which contradicts $\lim_{\Lambda \rightarrow +\infty} J_{12}(\Lambda) = 0$ and $\lim_{\Lambda \rightarrow S_v} J_{12}(\Lambda) = +\infty$.

Therefore, $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$ for all $\Lambda \in (\delta S_v, +\infty)$ and $J_{12}(\Lambda)$ is monotonically decreasing on $(S_v\delta, +\infty)$, with $J_{12}(\Lambda) > 0$ for all $\Lambda \in (\delta S_v, +\infty)$.

All these results confirm that the DFPS is unstable and the bifurcation does not occur at all for this case.

395 (b) When $\tilde{\Lambda} \leq \delta S_v$, from which we have $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} > 0$ for all $\Lambda \in (\delta S_v, +\infty)$.

Thus, $\frac{dJ_{12}(\Lambda)}{d\Lambda}$ is monotonically increasing on the interval $(\delta S_v, +\infty)$

with $\lim_{\Lambda \rightarrow +\infty} \frac{dJ_{12}(\Lambda)}{d\Lambda} = 0$, which means that $\frac{dJ_{12}(\Lambda)}{d\Lambda} < 0$ for all

$\Lambda \in (S_v\delta, +\infty)$ and $J_{12}(\Lambda)$ is monotonically decreasing on $(S_v\delta, +\infty)$.

According to $\lim_{\Lambda \rightarrow +\infty} J_{12}(\Lambda) = 0$ we have $J_{12}(\Lambda) > 0$ for all $\Lambda \in$

400 $(\delta S_v, +\infty)$ and then $R_c(\Lambda) > 1$ holds true, and again the DFPS is

unstable and the bifurcation does not occur at all for this case.

(ii) If $\frac{S_v+S_u}{2} > S^* \geq \frac{2}{\frac{1}{S_v}+\frac{1}{S_u}}$ then $\tilde{\Lambda} < 0$. For this case we have $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} > 0$ for all $\Lambda \in (\delta S_v, +\infty)$. By using methods similar to those in (b) we can show that $J_{12}(\Lambda) > 0$ for all $\Lambda \in (\delta S_v, +\infty)$ and $R_c(\Lambda) > 1$, which indicates that no bifurcation occurs at all in such case.

(iii) If $S^* > \frac{S_v+S_u}{2} > \frac{2}{\frac{1}{S_v}+\frac{1}{S_u}}$ then $\tilde{\Lambda} > 0$. Now we claim that $\tilde{\Lambda} > \delta S_v$. Otherwise, we assume that $\tilde{\Lambda} \leq \delta S_v$, then we have $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} < 0$ for all $\Lambda \in (\delta S_v, +\infty)$ and $\frac{dJ_{12}(\Lambda)}{d\Lambda}$ is monotonically decreasing on $(\delta S_v, +\infty)$. Moreover, we have $\lim_{\frac{\Delta}{\delta} \rightarrow +\infty} J_{12}(\Lambda) = 0$, thus $\frac{dJ_{12}(\Lambda)}{d\Lambda} > 0$ for all $\Lambda \in (\delta S_v, +\infty)$, i.e. $J_{12}(\Lambda)$ is monotonically increasing on $(S_v\delta, +\infty)$, which contradicts $\lim_{\frac{\Delta}{\delta} \rightarrow +\infty} J_{12}(\Lambda) = 0$ and $\lim_{\frac{\Delta}{\delta} \rightarrow S_v} J_{12}(\Lambda) = +\infty$. Thus, $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} \geq 0$ for all $\Lambda \in (\delta S_v, \tilde{\Lambda}]$, and $\frac{dJ_{12}^2(\Lambda)}{d\Lambda^2} < 0$ for all $\Lambda \in (\tilde{\Lambda}, +\infty)$. Moreover, according to $\lim_{\Lambda \rightarrow +\infty} \frac{dJ_{12}(\Lambda)}{d\Lambda} = 0$ we have $\frac{dJ_{12}(\Lambda)}{d\Lambda} > 0$ for all $\Lambda \in [\tilde{\Lambda}, +\infty)$. It is easy to know that there exists a unique $\hat{\Lambda} \in (\delta S_v, \tilde{\Lambda})$ such that $\frac{dJ_{12}(\hat{\Lambda})}{d\Lambda} = 0$, and $\frac{dJ_{12}(\Lambda)}{d\Lambda} \leq 0$ for all $\Lambda \in (\delta S_v, \hat{\Lambda}]$; $\frac{dJ_{12}(\Lambda)}{d\Lambda} > 0$ for all $\Lambda \in (\hat{\Lambda}, +\infty)$. According to $\lim_{\frac{\Delta}{\delta} \rightarrow +\infty} J_{12}(\Lambda) = 0$ and $\lim_{\frac{\Delta}{\delta} \rightarrow S_v} J_{12}(\Lambda) = +\infty$ we know that there exists a unique $\Lambda^* \in (\delta S_v, \hat{\Lambda})$ such that $J_{12}(\Lambda^*) = 0$, i.e. $R_c(\Lambda^*) = 1$ with $\frac{dJ_{12}(\Lambda^*)}{d\Lambda} < 0$ and $\frac{dR_c(\Lambda^*)}{d\Lambda} < 0$. By employing methods similar to those above, we can show that $P_M(Y, \Lambda)$ will occur with the bifurcation as $\Lambda = \Lambda^*$ for this case, which depends on the magnitude of M , i.e. if $M \neq \frac{2\eta_2}{h_2}$ then a transcritical bifurcation occurs; if $M = \frac{2\eta_2}{h_2}$ and $\frac{\partial^3 P_M(0, \Lambda^*)}{\partial Y^3} \neq 0$ then a pitchfork bifurcation occurs.

In summary, if $\frac{S_v+S_u}{2} \geq S^*$ then $J_{12}(\Lambda)$ is monotonically decreasing on the interval $(S_v\delta, +\infty)$; if $\frac{S_v+S_u}{2} < S^*$ then $J_{12}(\Lambda)$ is monotonically decreasing on the interval $(\delta S_v, \hat{\Lambda}]$ and increasing on the interval $(\hat{\Lambda}, +\infty)$. Moreover, we have the following main results:

Theorem 8. Assume that $h_2 \neq 0$, $1 < \frac{S_v}{S^*} < R_0$. If $\frac{S_v+S_u}{2} \geq S^*$ then $J_{12}(\Lambda)$ is monotonically decreasing on the interval $(S_v\delta, +\infty)$ and $R_c(\Lambda) > 1$ for all Λ , which means that the DFPS $(S^T(t), 0)$ is unstable. If $\frac{S_v+S_u}{2} < S^*$ then $J_{12}(\Lambda)$ is monotonically decreasing on the interval $(\delta S_v, \hat{\Lambda}]$ and increasing on the interval $(\hat{\Lambda}, +\infty)$. There exists a unique $\Lambda^* \in (S_v\delta, +\infty)$ such that $R_c(\Lambda^*) = 1$ and

$P_M(Y, \Lambda)$ occurs with a bifurcation at $\Lambda = \Lambda^*$. Moreover, if $M \neq \frac{2\eta_2}{h_2}$ then a transcritical bifurcation occurs; if $M = \frac{2\eta_2}{h_2}$ and $\frac{\partial^3 P_M(0, \Lambda^*)}{\partial Y^3} \neq 0$, then a pitchfork bifurcation occurs.

435 In particular, if $h_2 = 0$, we have $R_c(\Lambda) = (1 - \eta_2) \exp(J_{12}(\Lambda))$ with $\lim_{K \rightarrow +\infty} J_{12}(\Lambda) = 0$ and $\lim_{K \rightarrow S_v} J_{12}(\Lambda) = +\infty$, which indicate that there exists a unique $\Lambda^* \in (S_v \delta, +\infty)$ such that $J_{12}(\Lambda^*) = \ln \frac{1}{1 - \eta_2} > 0$, i.e. $R_c(\Lambda^*) = 1$ with $\frac{dJ_{12}(\Lambda^*)}{d\Lambda} < 0$. Based on the monotonicity of $J_{12}(\Lambda)$ we conclude that no matter what the relationship between $\frac{S_v + S_u}{2}$ and S^* , critical value Λ^* exists and is unique. Moreover, according to $\frac{\partial^2 P_M(0, \Lambda^*)}{\partial Y^2} = M > 0$ we have the following main results:

Corollary 5. *If $h_2 = 0$, $1 < \frac{S_u}{S^*} < R_0$, then $P_M(Y, \Lambda)$ occurs with the transcritical bifurcation at $\Lambda = \Lambda^*$, which indicates that an unstable positive fixed point of the $P_M(Y, \Lambda)$ appears when the parameter Λ changes through Λ^* from left to right. Correspondingly, system (2) has an unstable positive periodic solution if*

445 $\Lambda \in (\Lambda^*, \Lambda^* + \epsilon)$ with $\epsilon > 0$ small enough.

Note that if $h_2 = 0$ and $P_M(Y)$ occurs with bifurcations with respect to parameters $\eta_1, \eta_2, S_v, \Lambda$, then we must have $\frac{\partial^2 P_M(0, \eta_1^*)}{\partial Y^2} = \frac{\partial^2 P_M(0, \eta_2^*)}{\partial Y^2} = \frac{\partial^2 P_M(0, S_v^*)}{\partial Y^2} = \frac{\partial^2 P_M(0, \Lambda^*)}{\partial Y^2} = M > 0$. Thus, we conclude that:

Theorem 9. *If $h_2 = 0$ and $P_M(Y)$ occurs with a bifurcation with respect to the parameters $\eta_1, \eta_2, S_v, \Lambda$, then it must be a transcritical bifurcation and generates an unstable interior periodic solution, i.e. a backward bifurcation occurs and bistability appears when the DFPS of model (2) and the interior equilibrium $P^*(S^*, I^*)$ can coexist.*

5. Discussion

455 The basic or control reproduction number plays a key role in analyzing dynamics of epidemic models, but how to define and calculate it is challenging due to the complexity of the various control measures involved in the models. In particular, most control measures are implemented instantaneously, which

can be modelled by impulsive differential equations with fixed or unfixed mo-
460 ments. Infectious disease models with pulse vaccination or treatment strategies
have been widely studied recently [5, 19, 8, 10, 15, 20, 13, 7], and most of the
models assume that the pulse vaccination or treatment tactics occur at a fixed
period (i.e. fixed moment) resulting in non-autonomous periodic systems. If
so, we cannot employ the theories of dynamic systems, especially the theories
465 of impulsive dynamical systems, to study the dynamic behaviour of the model,
and then to determine the threshold dynamic behaviour and bifurcation phe-
nomenon of the proposed model. Therefore, in order to overcome the above
shortcomings, in the present paper we have extended the classic SIR model of
infections by involving state-dependent feedback control guided by a threshold
470 size of the susceptible population, aiming to address the threshold dynamics
and determine the threshold condition through the bifurcation theories of the
discrete one-parameter family of maps.

The existence and global stability of the DFPS, which corresponds to the
disease free equilibrium of the model without control actions, have been inves-
475 tigated in detail, and the main results show that $R_0 \leq 1$ indicates $R_c < 1$,
which reveals that the non-existence of the interior equilibrium for the classical
SIR model (1) implies the existence and global stability of the DFPS $(S^T(t), 0)$.
Moreover, if $R_0 > 1$, we conclude that the disease can still be completely erad-
icated provided that a proper choice of the threshold susceptible population
480 size S_v is made, i.e. if we choose the threshold level $S_v < S^*$ then the DFPS
 $(S^T(t), 0)$ could be globally stable even if $R_0 > 1$ for the SIR model (1). All
these results confirm that state-dependent feedback control can be effectively
used for mitigating and eradicating infectious diseases [8, 12, 11, 13, 14, 7].

The control reproduction number R_c for model (2) could be defined based
485 on the threshold condition for the stability of the DFPS $(S^T(t), 0)$. Further,
the bifurcation analyses of the discrete one-parameter family of maps, which is
determined by the Poincaré map of the proposed model (2), related to all in-
teresting parameters of model (2) confirm that R_c can determine the threshold
dynamics of model (2). In particular, the super-critical or sub-critical trans-

490 critical and pitchfork bifurcations related to the maximal vaccination rate η_1 ,
treatment or isolation rate η_2 , threshold size S_v and model parameter Λ have
been shown when we assume that the Poincaré map P_M is well defined in the
neighborhood of $U(0^+)$.

In fact, the Poincaré map $P_M(Y)$ is well defined for all small Y shown in
495 Fig.2(D). Moreover, it can be confirmed from the properties of the phase portrait
of model (1). It follows from $S_v > S^*$ that there exists a unique trajectory Γ_2 of
model (1) which tangents to the line $S = S_v$ (i.e. l_3) at the point $A = (S_v, I_A)$,
and intersects with the line $l_4(S = S_u)$ at the point $C(S_u, I_c)$ with $I_A = \frac{-\delta S_v + \Lambda}{\beta S_v}$.
Therefore, $P_M(Y)$ is well defined on the domain $[0, I_c]$ with range $[0, f(I_A)]$.
500 However, since the sign of $\frac{\partial^2 P_M(Y)}{\partial Y^2}$ on the interval $[0, I_c]$ varies as parameters
change, we cannot determine the convexity and concavity of $P_M(Y)$, which
presents a major challenge when addressing the existence and stability of the
order-1 periodic solution by using the properties of the Poincaré map $P_M(Y)$.

The existence of threshold parameter values of $\eta_1^*, \eta_2^*, S_v^*, \Lambda^*$ for transcritical
505 bifurcations and backward bifurcations are shown in Fig.3, which further con-
firm that the four interesting parameters chosen for bifurcation analyses can be
well defined. Meanwhile, the unstable interior order-1 periodic solution could
bifurcate from the DFPS through the transcritical bifurcation, i.e. the back-
ward bifurcation occurs, and consequently the stable DFPS of model (2) and
510 equilibrium P^* of model (1) could coexist. Note that it follows from the main
Theorems in Section 4 that the necessary conditions for the occurrences of the
backward bifurcations are $R_c < 1 < R_0$ and $S_v > S^*$, which show that the
disease will have outbreaks when the control measures are not involved (i.e.
the global stability of P^* of model (1)). Once the control measures are in-
515 volved in the model, we conclude that the disease can be controlled even if the
threshold level is relatively large ($S_v > S^*$ here). Furthermore, the necessary
condition $R_c < 1 < R_0$ for the occurrence of a backward bifurcation implies that
there must exist a threshold value \bar{R}_c such that the DFPS is globally stable for
 $R_c < \bar{R}_c$, and the stable DFPS of model (2) and equilibrium P^* of model (1)
520 could coexist $\bar{R}_c < R_c < 1 < R_0$. An interesting question is how to analytically

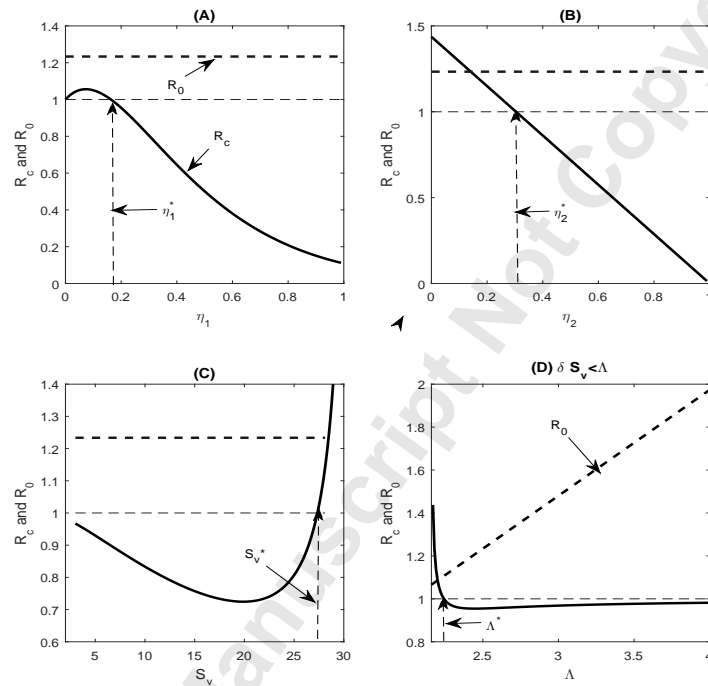


Figure 3: The existence of threshold parameter values for transcritical bifurcations for $\eta_1^*, \eta_2^*, S_v^*, \Lambda^*$. The base line parameter values are as follows: $\Lambda = 2.5$, $\beta = 0.015$, $\delta = 0.08$, $\gamma = 0.3$, $h_1 = 5$, $\eta_1 = 0.2$, $\eta_2 = 0.1$, $h_2 = 3$ and $S_v = 27$ in (A), (D) and 29 in (B) with $h_2 = 0$.

determine the threshold value \bar{R}_c , a challenge for future work.

Moreover, the relations between R_c and R_0 for the four interesting parameters shown in Fig.1(A) and (C) and Fig.3 reveal that no matter whether the R_0 is greater than or less than 1, there is a certain parameter space that makes R_c greater than R_0 . The results shown in Fig.1(A) confirm that the implementation of integrated control measures is not conducive to the elimination of disease when the control threshold level S_v is less than the critical value \bar{S}_v , and the results shown in Fig.1(C) show the importance of selecting the threshold level S_v for infectious disease control. Moreover, the results presented in Fig.3 provide an important way of thinking about how to control infectious diseases. For example, if $-\frac{\beta}{\delta}(S_v - S^*) + \frac{1}{\delta}(\beta K - q) \ln\left(\frac{K - S^*}{K - S_v}\right) > \ln R_0$, i.e. $R_c(\bar{\eta}_1) > R_0$, then $R_c(\eta_1) > R_0$ for $\eta_1 \in U(\bar{\eta}_1, \delta)$ (i.e. the δ domain of $\bar{\eta}_1$). Numerical simulations reveal that this phenomenon occurs only when the maximal vaccination η_1 is very small, as shown in Fig.3(A). Note that the biphasic vaccination rate results in an inverted U-shape curve for η_1 shown in Fig.3(A), i.e. too low a vaccination rate increases the R_c and a relatively high vaccination rate decreases the R_c . All these results confirm that increasing vaccination coverage is very important in controlling infectious diseases, especially when the implementation of the vaccination strategy depends on the size of the susceptible population.

Similarly, the biphasic threshold level S_v results in a U-shape curve shown in Fig.3(C), i.e. too small or too large a threshold level S_v is not beneficial for disease control. Moreover, R_c could be larger than R_0 provided that the S_v is large enough, which can be confirmed as follows: it follows from $\lim_{S_v \rightarrow S^*} J_{12}(S_v) < 0$, $\lim_{S_v \rightarrow K} J_{12}(S_v) = +\infty$ and the continuity of $J_{12}(S_v)$ that there exists a $S_{v0} \in (S^*, K)$ such that $J_{12}(S_{v0}) = \ln R_0 > 0$ (i.e. $R_c(S_{v0}) = R_0$), and $R_c(S_{v0}) > R_0$ for all $S_v \in (S_{v0}, K)$. Moreover, the line $S_v = K$ is an asymptote of $R_c(S_v)$. Thus, we conclude that the U-shape curve related to the S_v reveals the importance of the correct selection of the threshold level to ensure the best integrated disease control effect. Note that biphasic dose response curves have been reported in many areas recently, including cancer treatment and pest control [39, 40, 41], but this is the first time that we have found that biphasic

vaccination and threshold size responses occur in an infectious disease model with state-dependent feedback control. In order to eradicate infectious diseases, the size of the susceptible population (i.e. the critical level S_v here) plays a key role, which provides important ideas and guidance for designing vaccine output and coverage according to population size and vaccine effectiveness.

The results shown in Fig.3(B) clarify that only high rates of effective treatment or isolation could successfully mitigate or eradicate the infectious disease. The birth rate Λ could also influence the R_c significantly, as shown in Fig.3(D). For the parameter values given in Fig.3(D) we have $\frac{S_v+S_u}{2} = 24.72 < S^*$, where $S^* = 25.33$, $S_u = 22.44$ and the results correspond to Theorem 8. In particular, if $\eta_1 \leq \frac{2(S_v+h_1)(1-\frac{S^*}{S_v})}{S_v} \doteq 0.147$, then $R_c(\Lambda)$ is monotonically decreasing on the interval $[\delta S_v, \infty)$ and tends to 1; If $\eta_1 > \frac{2(S_v+h_1)(1-\frac{S^*}{S_v})}{S_v} \doteq 0.147$, then $R_c(\Lambda)$ is monotonically decreasing first and then increasing on the interval $[\delta S_v, \infty)$ which will tend to 1 eventually, and this is the case shown in Fig.3(D). The relations between η_1 and $\frac{2(S_v+h_1)(1-\frac{S^*}{S_v})}{S_v}$ clearly reveal that how to design the vaccination campaign (i.e. choosing the vaccination rate η_1) should be based on the threshold size S_v and stable population level S^* without control measures. Also, the results shown in Fig.3(D) demonstrate that for a relatively large birth rate Λ we must choose a high vaccination rate η_1 such that $R_c < 1$. In conclusion, in order to effectively control the outbreak of infectious diseases, we should take effective, timely, measures that are stronger than usual, including vaccination, treatment and isolation, which should be adopted in relation to monitored population births and growth.

The properties of the function $f(I) = 1 - \frac{\eta_1 I(t)}{I(t)+h_2}$ can significantly affect the monotonicity and concavity of the Poincaré map P_M , which will result in the complexity of the P_M , and consequently various different methods have been developed in the present paper to show the global stability of DFPS by considering nonlinear impulsive perturbations have in comparison with the results obtained in [38]. Moreover, it is interesting to note that the control reproduction number $R_c = (1 - \eta_2) \exp(J_2 + J_3)$ for $h_2 = 0$, which indicates that the nonlinear pulse perturbation can significantly affect R_c and consequently influences the

bifurcation of $P_M(Y)$. In particular, $P_M(Y)$ could only occur with a transcritical bifurcation when $h_2 = 0$, and the pitchfork bifurcation could occur when
585 $h_2 > 0$. Further, the Poincare map $P_M(Y, \eta_1)$ can occur with a transcritical bifurcation at $\eta_1 = \eta_1^*$ when $h_2 > 0$, i.e. there exists a unique bifurcation value η_1^* when the nonlinear pulse is considered, while there may exist two bifurcation values once $h_2 = 0$.

Therefore, we conclude that the nonlinear impulsive perturbations addressed
590 in the present paper are not only more practical and can produce rich dynamic behaviour, but also required new analytical techniques and methods to investigate their global dynamic behaviour. The idealized hypothesis proposed allowed us to simplify our model and thus conduct a thorough theoretical analysis, but undeniably it led to limitations to the application of our model to real events.
595 Thus, we leave the interesting question of how to apply our new methods and techniques to a more general model for our future research.

Appendix A:

The following lemma shows the local stability of the T -periodic solution of the plane impulsive semi-dynamical system.

600 **Lemma A.1.** *The T -periodic solution $(x, y) = (\xi(t), \eta(t))$ of the system*

$$\left\{ \begin{array}{l} \frac{dx(t)}{dt} = P(x, y), \\ \frac{dy(t)}{dt} = Q(x, y), \\ \Delta x = \sigma_1(x, y), \\ \Delta y = \sigma_2(x, y), \end{array} \right\} \begin{array}{l} \text{if } \phi(x, y) \neq 0, \\ \\ \text{if } \phi(x, y) = 0, \end{array} \quad (21)$$

is orbitally asymptotically stable if the Floquet multiplier μ_2 satisfies the condition $|\mu_2| < 1$, where

$$\mu_2 = \prod_{k=1}^q \Delta_k \exp \left[\int_0^T \left(\frac{\partial P}{\partial x}(\xi(t), \eta(t)) + \frac{\partial Q}{\partial y}(\xi(t), \eta(t)) \right) dt \right] \quad (22)$$

with

$$\Delta_k = \frac{P_+ \left(\frac{\partial \sigma_2}{\partial y} \frac{\partial \phi}{\partial x} - \frac{\partial \sigma_2}{\partial x} \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial x} \right) + Q_+ \left(\frac{\partial \sigma_1}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial \sigma_1}{\partial y} \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \right)}{P \frac{\partial \phi}{\partial x} + Q \frac{\partial \phi}{\partial y}},$$

and P , Q , $\frac{\partial \sigma_1}{\partial x}$, $\frac{\partial \sigma_1}{\partial y}$, $\frac{\partial \sigma_2}{\partial x}$, $\frac{\partial \sigma_2}{\partial y}$, $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are calculated at the point
605 $(\xi(t), \eta(t))$. $P_+ = P(\xi(t_k^+), \eta(t_k^+))$ and $Q_+ = Q(\xi(t_k^+), \eta(t_k^+))$. Here $\phi(x, y)$ is a sufficiently smooth function such that $\text{grad}\phi(x, y) \neq 0$, and $t_k (k \in \mathbb{N})$ is the time of the k -th jump.

The following two lemmas show the transcritical and pitchfork bifurcations of the discrete one-parameter family of maps, which can be used to address the
610 stability and bifurcation of the Poincaré map determined by the impulsive point series of the impulsive semi-dynamical system.

Lemma A.2. *(Transcritical bifurcation) Let $G : U \times I \rightarrow \mathbb{R}$ define a one-parameter family of maps, where G is C^r with $r \geq 2$, and U, I are open intervals of the real line containing 0. Assume that*

615 (1) $G(0, \alpha) = 0$ for all α ; (2) $\frac{\partial G}{\partial x}(0, 0) = 1$;

(3) $\frac{\partial^2 G}{\partial x \partial \alpha}(0, 0) > 0$; (4) $\frac{\partial^2 G}{\partial^2 x}(0, 0) > 0$.

Then there are $\alpha_1 < 0 < \alpha_2$ and $\varepsilon > 0$ such that

(i) If $\alpha_1 < \alpha < 0$, then $G_\alpha = G(\cdot, \alpha)$ has two fixed points, 0 and $x_{1\alpha} > 0$ in $(-\varepsilon, \varepsilon)$, then the origin is asymptotically stable and the other fixed point is

620 unstable.

(ii) If $0 < \alpha < \alpha_2$, then G_α has two fixed points, 0 and $x_{1\alpha} < 0$ in $(-\varepsilon, \varepsilon)$.

The origin is unstable, the other fixed point is asymptotically stable.

Note that the case $\frac{\partial^2 G}{\partial x \partial \alpha}(0, 0) < 0$ is handled by making the change of parameter $\alpha \rightarrow -\alpha$. If the inequality (4) is reversed (i.e. $\frac{\partial^2 G}{\partial^2 x}(0, 0) < 0$), then

625 (i) If $\alpha_1 < \alpha < 0$, then G_α has two fixed points, 0 and $x_{1\alpha} < 0$ in $(-\varepsilon, \varepsilon)$.

The origin is asymptotically stable, the other fixed point is unstable.

(ii) If $0 < \alpha < \alpha_2$, then G_α has two fixed points, 0 and $x_{1\alpha} > 0$ in $(-\varepsilon, \varepsilon)$.

The origin is unstable, the other fixed point is asymptotically stable.

Lemma A.3. (Supercritical pitchfork bifurcation) Let $G : U \times I \rightarrow \mathbb{R}$ be as in

630 Lemma A.2, except that G is C^r with $r \geq 3$ and $\frac{\partial^2 G}{\partial x^2}(0, 0) = 0$. Further, if

$\frac{\partial^3 G}{\partial x^3}(0, 0) < 0$ then there are $\alpha_1 < 0 < \alpha_2$ and $\varepsilon > 0$ such that

(i) If $\alpha_1 < \alpha \leq 0$, then $G_\alpha = G(\cdot, \alpha)$ has a unique fixed point, $x = 0$, in $(-\varepsilon, \varepsilon)$. It is asymptotically stable.

(ii) If $0 < \alpha < \alpha_2$, then G_α has three fixed points in $(-\varepsilon, \varepsilon)$. The origin is

635 an unstable fixed point, the two others, $x_{1\alpha} < 0 < x_{2\alpha}$, are asymptotically stable.

Note that the case $\frac{\partial^2 G}{\partial x \partial \alpha}(0, 0) < 0$ is identical to the above after the change

of parameter $\alpha \rightarrow -\alpha$. If $\frac{\partial^3 G}{\partial x^3}(0, 0) > 0$, it is a so-called subcritical pitchfork bifurcation. Then for $\alpha < 0$, there are three fixed points near the origin, but

640 only $x = 0$ is asymptotically stable. For $\alpha \geq 0$, the origin is the unique fixed point near $x = 0$, and it is unstable.

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