# Generalized Markov Branching Models

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A thesis submitted in partial fulfilment of the requirements of the University of Greenwich for the Degree of Doctor of Philosophy

October 2005

# DECLARATION

I certify that this work has not been accepted in substance for any degree, and is not concurrently submitted for any degree other than that of Doctor of Philosophy (PhD) of the University of Greenwich. I also declare that this work is the result of my own research work except where otherwise stated.

## ACKNOWLEDGEMENTS

I would like to thank my supervisors, Dr Anyue Chen and Dr Nadarajah Ramesh for their meticulous guidance and strong support throughout my studies, research work and daily life. I could not have completed my thesis without their supervision and help.

I am grateful to Professor Keith Rennolls for his help when I studied in the University of Greenwich. For the financial support, I thank the University of Greenwich for awarding me the bursary for this research.

I would like to express many thanks to Professor Eric Renshaw for his considerable suggestions during the writing of this thesis.

I also take this opportunity to thank Professor Kevin Parrott and Mr Tony Mann for their help when it was most needed.

Finally, I would like to thank my wife, Ying Tang, for her support and belief in me.

# ABSTRACT

In this thesis, we first considered a modified Markov branching process incorporating both state-independent immigration and resurrection. After establishing the criteria for regularity and uniqueness, explicit expressions for the extinction probability and mean extinction time are presented. The criteria for recurrence and ergodicity are also established. In addition, an explicit expression for the equilibrium distribution is presented.

We then moved on to investigate the basic properties of an extended Markov branching model, the weighted Markov branching process. The regularity and uniqueness criteria of such general structures are first established. There after closed expressions for the mean extinction time and conditional mean extinction time are presented. The explosion behaviour and the mean explosion time are also investigated. In particular, the Harris regularity criterion for ordinary Markov branching process is extended to a more general case of non-linear Markov branching process.

Finally, we studied a new Markov branching model, the weighted collision branching process, and considered two special cases of this process. For this weighted collision branching process, some conditions of regularity and uniqueness are obtained and the extinction behaviour and explosion behaviour of the process are investigated. For the two special cases, a collision branching process and a general collision branching process with 2 parameters, the regularity and uniqueness criteria of the process are established and explicit expressions for extinction probability vector, mean extinction times, conditional mean extinction times and mean explosion time are all obtained.

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# Chapter 1. Introduction

In this first chapter we introduce the background and basic properties of continuous-time Markov chains and Markov branching processes upon which our later models are built. We also outline the structure of the thesis at the end of this chapter.

## 1.1. Background

There is no field of science or engineering that the concepts of randomness and probability do not touch. Even everyday life cannot be comprehended fully without some familiarity with them. As Laplace commented more than 180 years ago, "... there is no science more worthy of our meditations, and no more useful one could be incorporated in the system of public instruction." The Markov process is a clear and lucid presentation of the most fundamental models of random phenomena. This process is named after A. A. Markov who introduced this extremely important concept in 1907, with emphasis on the case of finite number of states. The denumerable case was launched by A. N. Kolmogorov in 1936, and more general cases have been considered by many mathematicians since then. Looking back on the course of development of Markov processes, considerable achievement has been made in the past century. Even now, the theory of Markov processes is still thriving and has been extensively used in applied probability, statistics and many other branches of sciences.

The continuous-time Markov chain (henceforth referred to as the Markov chain, for simplicity) is one of the most important classical fields of Markov processes and has a vast range of applications. The literature in the fields of science, engineering, finance and humanities have plenty of examples of stochastic processes that have been modelled by Markov chains, with varying degrees of success. The first systematic study of Markov chains was made by A. N. Kolmogorov (1931). In his study, he found that the probability law governing the evolution of a process occurs as the solution of either of two systems of differential equations, now called the Kolmogorov backward and forward equations, respectively. These investigations continued into the 1940s. During this period, more and more mathematicians engaged in the study of Markov chains. Over the past forty years, Markov chains have sufficiently shown their power in many areas of science and technology with real applications to queueing theory, demography, and epidemiology.

One of the most important subclasses of Markov chains is the Markov branching process with denumerable state space. Originally evolved in the 19th century from analysis of the survival of family names, the subject of Markov branching processes has had an obvious impact on population dynamics. With the development of computer science, new applications have been found in several new areas such as algorithms, data structures, combinatorics, and molecular biology particularly in molecular DNA sequencing.

As is well-known, the basic property of a Markov branching process is the branching property, i.e., different particles act independently when they give offspring. This branching property is very important in the study of Markov branching processes and it makes this class of processes a very fruitful subject of Markov chains.

However, in many practical applications, this independence property is unlikely to be appropriate. Indeed, in such cases, the branching events are affected by the interaction of two or more particles rather than by the particles individually. In other words, particles evolve dependently in the system.

In view of this, the main aim of the thesis is to consider interacting branching systems and to investigate their basic properties.

## 1.2. Basic Concepts of Continuous-time Markov Chains

In this section, we recall the basic concepts and general properties of continuous-time Markov chains. Recall that a stochastic process is a family of random variables indexed by the time parameter, either discrete or continuous time.

**Definition 1.2.1.** A continuous-time stochastic process $\{X(t); t \geq 0\}$, defined on a probability space $(\Omega, \mathcal{F}, P)$, with values in a countable set $\mathbf{E}$ (to be called the state space of the process), is called a continuous-time Markov chain if for any finite set $0 \leq t_1 < t_2 < \cdots < t_{n-1} < s$ and $t \geq 0$ of "times", and corresponding set $i_1, i_2, \cdots, i_{n-1}, i, j$ of states in $\mathbf{E}$ such that if $P(X(s) = i, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1) > 0$, we have

$$\begin{aligned} & P(X(t+s) = j | X(s) = i, X(t_{n-1}) = i_{n-1}, \cdots, X(t_1) = i_1) \\ = \ & P(X(t+s) = j | X(s) = i). \end{aligned} \tag{1.2.1}$$

Equation (1.2.1) is called the *Markov property.* If for all $s, t \geq 0$ and all $i, j \in \mathbf{E}$, the conditional probability $P(X(t+s) = j | X(s) = i)$ appearing on the right-hand side of (1.2.1) depends only on $t$ and not on $s$, then we say that the process $\{X(t); t \geq 0\}$ is homogeneous. In this case, we can define

$$p_{ij}(t) =: P(X(t+s) = j | X(s) = i), \quad i, j \in \mathbf{E}, \ t \geq 0$$

and $(p_{ij}(t); i, j \in \mathbf{E}, t \geq 0)$ is called the transition function of the process.

It is well-known that the finite-dimensional probability distributions of the process $\{X(t); t \geq 0\}$ are all expressible in terms of the transition function $(p_{ij}(t); i, j \in \mathbf{E}, t \geq 0)$ and the initial probability distribution $p_i = P(X(0) = i)$ $(i \in \mathbf{E})$ of $X(0)$. Moreover, by general theory of Markov chains, a continuous-time Markov chain is uniquely determined by its transition function and a given initial probability distribution. Therefore, we may concentrate on discussing the transition function and sometimes call it a "process". Now, we give the general definition of transition function:

**Definition 1.2.2.** Let $\mathbf{E}$ be a countable set, to be called the state space. A family of functions $(p_{ij}(t); i, j \in \mathbf{E}, t \geq 0)$ is called a transition function on $\mathbf{E}$ if

(i) $p_{ij}(t) \geq 0$ for all $t \geq 0$ and $i, j \in \mathbf{E}$; and

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 1, & if\ i = j; \\ 0, & if\ i \neq j. \end{cases}$$

(ii) For all $t \geq 0,\ i \in \mathbf{E}$,

$$\sum_{j \in \mathbf{E}} p_{ij}(t) \leq 1. \tag{1.2.2}$$

(iii) For all $s, t \geq 0$ and $i, j \in \mathbf{E}$,

$$p_{ij}(s+t) = \sum_{k \in \mathbf{E}} p_{ik}(s) p_{kj}(t). \tag{1.2.3}$$

(iv) $\lim_{t \to 0} p_{ii}(t) = 1$ for all $i \in \mathbf{E}$.

A transition function is called honest if $\sum_{j \in \mathbf{E}} p_{ij}(t) = 1$ for all $t \geq 0,\ i \in \mathbf{E}$, and dishonest otherwise. The equation (1.2.3) is called Chapman-Kolmogorov equation, or the semigroup property.

The basic conclusion of the following theorem is a significant result in the study of Markov processes.

**Theorem 1.2.1.** Let $(p_{ij}(t); i, j \in \mathbf{E}, t \geq 0)$ be a transition function. Then for all $i, j \in \mathbf{E}$, $q_{ij} = p'_{ij}(0)$ exists and $0 \leq q_{ij} < +\infty$ for all $i \neq j$ and $-\infty \leq q_{ii} \leq 0$. Furthermore, $\sum_{j \in \mathbf{E}} q_{ij} \leq 0$ for all $i \in \mathbf{E}$.

The matrix $Q = (q_{ij}; i, j \in \mathbf{E})$ is called the density matrix of $(p_{ij}(t); i, j \in \mathbf{E})$. A state $i$ is said to be stable if $q_i =: -q_{ii} < +\infty$ and instantaneous if $q_i = +\infty$. The matrix $Q$ is called stable if all states $i \in \mathbf{E}$ are stable.

Based on the above result, we now give the following fundamental definition of a stable, conservative $Q$ matrix.

**Definition 1.2.3.** A matrix $Q = (q_{ij}; i, j \in \mathbf{E})$ is called a stable $q$-matrix if $q_{ij} \geq 0\ (i \neq j)$ and $\sum_{j \neq i} q_{ij} \leq -q_{ii} < +\infty\ (i \in \mathbf{E})$. The matrix $Q$ is called conservative if furthermore $\sum_{j \neq i} q_{ij} = -q_{ii}$ for all $i \in \mathbf{E}$. A state $i$ is called absorbing if $q_i = 0$.

Since the $q$-matrix associated with the models considered in this thesis is always stable and conservative, we shall frequently omit these two adjectives unless it is necessary to mention the difference.

Let $(r_{ij}(\lambda); i, j \in \mathbf{E}, \lambda > 0)$ be the Laplace transform of a transition function $(p_{ij}(t); i, j \in \mathbf{E})$. Then by the property of Laplace transform, the conditions (i)–(iv) in Definition 1.2.2 become

$$r_{ij}(\lambda) \geq 0, \qquad \lambda \sum_{k \in \mathbf{E}} r_{ik}(\lambda) \leq 1,$$

$$r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu) \sum_{k \in \mathbf{E}} r_{ik}(\lambda) r_{kj}(\mu)$$

and

$$\lim_{\lambda \to \infty} \lambda r_{ii}(\lambda) = 1.$$

Therefore, we now give the following definition which is parallel with the transition function.

**Definition 1.2.4.** A family of functions $(r_{ij}(\lambda); i, j \in \mathbf{E}, \lambda > 0)$ is called a resolvent function if

$$r_{ij}(\lambda) \geq 0, \tag{1.2.4}$$

$$\lambda \sum_{k \in \mathbf{E}} r_{ik}(\lambda) \leq 1, \tag{1.2.5}$$

$$r_{ij}(\lambda) - r_{ij}(\mu) + (\lambda - \mu) \sum_{k \in \mathbf{E}} r_{ik}(\lambda) r_{kj}(\mu), \tag{1.2.6}$$

$$\lim_{\lambda \to \infty} \lambda r_{ii}(\lambda) = 1 \tag{1.2.7}$$

The resolvent function $(r_{ij}(\lambda);\ i, j \in \mathbf{E})$ is called honest if equality holds in (1.2.5). Equation (1.2.6) is called the resolvent equation.

Let $(p_{ij}(t); i, j \in \mathbf{E})$ be a transition function and let $r_{ij}(\lambda)$ be the Laplace transform of $p_{ij}(t)$. Then $(r_{ij}(\lambda); i, j \in \mathbf{E})$ is a resolvent function and is honest if $(p_{ij}(t); i, j \in \mathbf{E})$ is. Conversely, let $(r_{ij}(\lambda); i, j \in \mathbf{E})$ be a resolvent function then there is a unique transition function $(p_{ij}(t); i, j \in \mathbf{E})$ such that $(r_{ij}(\lambda); i, j \in \mathbf{E})$ is its Laplace transform and $(p_{ij}(t); i, j \in \mathbf{E})$ is honest if $(r_{ij}(\lambda); i, j \in \mathbf{E})$ is.

The importance of resolvent functions lies in the fact that there exists a one to one correspondence between the resolvent function and transition function. Therefore, sometimes we even call a resolvent function a "process". The following theorem, the proof of which can be found in the second part of Chung (1967) or in Anderson (1991), shows the basic properties of a transition function.

**Theorem 1.2.2.** Let $(p_{ij}(t); i, j \in \mathbf{E}, t \geq 0)$ be a transition function.

(i) Suppose $i$ is a stable state, then $p'_{ij}(t)$ exists and is finite and continuous on $[0, \infty)$ for all $j \in \mathbf{E}$. Furthermore,

$$p'_{ij}(s + t) = \sum_{k \in \mathbf{E}} p'_{ik}(s) p_{kj}(t), \quad s > 0,\ t \geq 0,$$

$$\sum_{j\in\mathbf{E}} |p'_{ij}(t)| \le 2q_i$$

and

$$\sum_{j\in\mathbf{E}} p'_{ij}(t) + d'_i(t) = 0, \qquad t > 0,$$

where $d_i(t) = 1 - \sum_{j\in\mathbf{E}} p_{ij}(t)$.

(ii) Suppose $j$ is a stable state, then $p'_{ij}(t)$ exists and is finite and continuous on $[0,\infty)$ for all $i \in \mathbf{E}$. Furthermore,

$$p'_{ij}(s+t) = \sum_{j\in\mathbf{E}} p_{ik}(s)p'_{kj}(t), \qquad s \ge 0,\ t > 0.$$

It can be proved easily that for any stable state $i \in \mathbf{E}$, we have the following backward inequality:

$$p'_{ij}(t) \ge \sum_{k\in\mathbf{E}} q_{ik}p_{kj}(t), \qquad t \ge 0,\ j \in \mathbf{E} \tag{1.2.8}$$

and the forward inequality:

$$p'_{ij}(t) \ge \sum_{k\in\mathbf{E}} p_{ik}(t)q_{kj}, \qquad t \ge 0,\ j \in \mathbf{E}. \tag{1.2.9}$$

The equation

$$p'_{ij}(t) = \sum_{k\in\mathbf{E}} q_{ik}p_{kj}(t), \qquad t \ge 0,\ j \in \mathbf{E} \tag{1.2.10}$$

is called the Kolmogorov backward equation and the equation

$$p'_{ij}(t) = \sum_{k\in\mathbf{E}} p_{ik}(t)q_{kj}, \qquad t \ge 0,\ j \in \mathbf{E} \tag{1.2.11}$$

is called the Kolmogorov forward equation.

The following are the resolvent versions of the Kolmogorov backward and forward equations respectively.

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k\in\mathbf{E}} q_{ik}r_{kj}(\lambda), \qquad \lambda > 0,\ j \in \mathbf{E} \tag{1.2.12}$$

$$\lambda r_{ij}(\lambda) = \delta_{ij} + \sum_{k\in\mathbf{E}} r_{ik}(\lambda)q_{kj}, \qquad \lambda > 0,\ j \in \mathbf{E}. \tag{1.2.13}$$

In general, "forward equation" approach is more convenient and thus could get more results (in physics, "forward equation" is refered as "Master equation"). Therefore, we will mainly use Kolmogorov forward equation in this thesis.

A general transition function (or resolvent function) may satisfy neither the Kolmogorov backward equation nor the Kolmogorov forward equation. However, it can be proved that for any stable $q$-matrix, there exists a transition function (or resolvent function) satisfying both Kolmogorov backward and forward equations, as shown in the later Theorem 1.3.1.

The general theory says that the Kolmogorov backward equation holds for any stable state $i$ satisfying $\sum_{j\in\mathbf{E}} q_{ij} = 0$. Hence, if $Q$ is stable and conservative then the Kolmogorov backward equation holds for all $i \in \mathbf{E}$.

All the results in this section can be found **in many books on** continuous-time Markov chains, for example, Anderson (1991) or Yang (1990).

## 1.3. Classical Problems on Markov chains

Suppose $Q$ is the $q$-matrix of a continuous-time Markov chain determined by its transition function. It is clear that $Q$ describes the infinitesimal characteristics of the process. So we often call $(p_{ij}(t); i, j \in \mathbf{E})$ a $Q$-function or $Q$-process and call its Laplace transform $(\phi_{ij}(\lambda); i, j \in \mathbf{Z}_+)$ a $Q$-resolvent. In most cases, it is much easier to obtain $Q = (q_{ij}; i, j \in \mathbf{E})$ rather than $(p_{ij}(t); i, j \in \mathbf{E})$ itself. Therefore, in nearly all the cases, particularly in applications, the starting point for study is the $q$-matrix. Therefore, for a given $q$-matrix $Q = (q_{ij}; i, j \in \mathbf{E})$, the following fundamental and classical problems arise.

(i)(**Existence Problem**). Under what conditions does there exist a $Q$-function $(p_{ij}(t))$?

(ii)(**Uniqueness Problem**). If there exists a $Q$-function, then under what conditions will it be unique?

(iii)(**Property**). How do we study the properties of the $Q$-process in terms of the given $q$-matrix?

The following is the standard existence theorem of $Q$-functions for a given $Q = (q_{ij})$, which is due to Feller(1940).

**Theorem 1.3.1.** Let $Q = (q_{ij}; i, j \in \mathbf{E})$ be a stable but not necessarily conservative $q$-matrix. Then there exists a (possibly dishonest) $Q$-function $(p_{ij}(t); i, j \in \mathbf{E})$ satisfying both the Kolmogorov backward and forward equations. Moreover, this $Q$-function $(p_{ij}(t); i, j \in \mathbf{E})$ is minimal in the sense that for any other $Q$-function $(\tilde{p}_{ij}(t); i, j \in \mathbf{E})$, $\tilde{p}_{ij}(t) \geq p_{ij}(t)$ $(i, j \in \mathbf{E}, t \geq 0)$. (This minimal solution is often called the Feller minimal $Q$-function and the Laplace transform of the Feller minimal $Q$-function is called the Feller minimal resolvent function).

The Feller minimal $Q$-resolvent function $(\phi_{ij}(\lambda); i, j \in \mathbf{E})$ can be obtained either by the backward integral recursion

$$\begin{cases} \phi_{ij}^{(0)}(\lambda) = \frac{\delta_{ij}}{\lambda+q_i}, \\ \phi_{ij}^{(n+1)}(\lambda) = \frac{\delta_{ij}}{\lambda+q_i} + \sum_{k\neq i} \frac{q_{ik}}{\lambda+q_i} \cdot \phi_{kj}^{(n)}(\lambda), \quad n \geq 0 \end{cases} \tag{1.3.1}$$

with $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ as $n \to \infty$ for all $i, j \in \mathbf{E}$, or by the forward integral recursion

$$\begin{cases} \phi_{ij}^{(0)}(\lambda) = \frac{\delta_{ij}}{\lambda+q_j}, \\ \phi_{ij}^{(n+1)}(\lambda) = \frac{\delta_{ij}}{\lambda+q_j} + \sum_{k\neq j} \phi_{ik}^{(n)}(\lambda) \cdot \frac{q_{kj}}{\lambda+q_j}, \quad n \geq 0 \end{cases} \tag{1.3.2}$$

with $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ as $n \to \infty$ for all $i, j \in \mathbf{E}$, where "$\uparrow$" means "increasing".

Now we recall the conditions for the uniqueness of $Q$-functions.

**Theorem 1.3.2.** The following statements are equivalent.

(i) The Feller minimal $Q$-function is the unique solution of the Kolmogorov backward equation.

(ii) The equation $Qx = \lambda x, \ 0 \leq x \leq 1$; i.e.,

$$\sum_{j\in\mathbf{E}} q_{ij}x_j = \lambda x_i, \quad 0 \leq x_i \leq 1, \ i \in \mathbf{E} \tag{1.3.3}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

(iii) The inequality $Qx \geq \lambda x, \ 0 \leq x \leq 1$; i.e.,

$$\sum_{j\neq i} q_{ij}x_j = (\lambda + q_i)x_i, \quad 0 \leq x_i \leq 1, \ i \in \mathbf{E} \tag{1.3.4}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

(iv) The equation $Qx = \lambda x, \ -1 \leq x \leq 1$; i.e.,

$$\sum_{j \in \mathbf{E}} q_{ij} x_j = \lambda x_i, \quad -1 \leq x_i \leq 1, \ i \in \mathbf{E} \tag{1.3.5}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

If $Q$ is conservative, then the Feller minimal $Q$-function is the unique $Q$-function if and only if one of the above statements holds.

**Definition 1.3.1.** A conservative $q$-matrix which satisfies any one of conditions (i)–(iv) in Theorem 1.3.2 is said to be regular. The corresponding $Q$-function is then also said to be regular. In this case, the Feller minimal $Q$-function is honest.

Condition (ii) for regularity in Theorem 1.3.2 is the one most commonly encountered in the literature. There is a similar test for uniqueness of solutions to the Kolmogorov forward equation.

**Theorem 1.3.3.** Suppose that the Feller minimal $Q$-function (where $Q$ is not necessarily conservative) is dishonest. Then it is the unique $Q$-function satisfying the Kolmogorov forward equation if and only if the equation

$$\sum_{i \in \mathbf{E}} y_i q_{ij} = \lambda y_j, \quad y_j \geq 0, \ j \in \mathbf{E}, \ \sum_{j \in \mathbf{E}} y_j < +\infty \tag{1.3.6}$$

has no nontrivial solution, for some (and therefore for all) $\lambda > 0$.

The uniqueness problem is one of the most important problems for $Q$-processes. Doob and Reuter solved this problem for the conservative case while the nonconservative case was solved by Zhenting Hou in 1974.

All the above results can be found in Anderson (1991) or Yang (1990).

If a $q$-matrix $Q$ is regular then further important and interesting properties of the corresponding process are about the problems regarding absorbtion, recurrence and ergodicity. We shall now define these terms in the following definitions.

**Definition 1.3.2.** Given $i, j \in \mathbf{E}$, we say that $j$ can be reached from $i$, and write $i \hookrightarrow j$, if $p_{ij}(t) > 0$ for some (and therefore for all) $t > 0$. We say that $i$ and $j$ communicate, and write $i \leftrightarrow j$, if $i$ and $j$ can be reached from each other.

Using the inequality $p_{ik}(t+s) \geq p_{ij}(t)p_{jk}(s)$, it is very easy to show that $\leftrightarrow$ is an equivalence relation on the state space $\mathbf{E}$, and thus can partition $\mathbf{E}$ into disjoint equivalence classes called the communicating classes. We call $(p_{ij}(t); i, j \in \mathbf{E})$ irreducible if the entire state space $\mathbf{E}$ forms the only communicating class.

**Definition 1.3.3.** Let $(p_{ij}(t); i, j \in \mathbf{E})$ be a regular transition function.

(i) A state $i$ is called absorbing if $q_i = 0$.

(ii) A state $i$ is called transient if $\int_0^\infty p_{ii}(t)dt < +\infty$ and recurrent if $\int_0^\infty p_{ii}(t)dt = +\infty$. The process $(p_{ij}(t))$ is called recurrent if all the states are recurrent.

(iii) A state $i$ is called positive recurrent if $\lim_{t\to\infty} p_{ii}(t) > 0$. The process $((p_{ij}(t); i, j \in \mathbf{E})$ is called positive recurrent if all the states are positive recurrent.

It is well-known that if $i \leftrightarrow j$ then $i$ is recurrent if and only if $j$ is recurrent. The same relationship holds for positive recurrence. Therefore, if $((p_{ij}(t); i, j \in \mathbf{E})$ is an irreducible transition function, then the process is recurrent (positive recurrent) if and only if any particular state is recurrent (positive recurrent). It can be further proved that for an irreducible transition function $((p_{ij}(t); i, j \in \mathbf{E})$, it is positive recurrent if and only if there exists a probability measure $(\pi_j; j \in \mathbf{E})$ such that

$$\lim_{t\to\infty} p_{ij}(t) = \pi_j, \quad i, j \in \mathbf{E}.$$

An irreducible and positive recurrent transition function is often called ergodic.

**Definition 1.3.4.** Let $(p_{ij}(t); i, j \in \mathbf{E})$ be an irreducible and regular transition function and $(\pi_j; j \in \mathbf{E})$ be a probability measure.

(i) $(p_{ij}(t); i,j \in \mathbf{E})$ is called exponentially ergodic if there exists $\beta > 0$ such that for any $i,j \in \mathbf{E}$, $|p_{ij}(t) - \pi_j| = O(e^{-\beta t})$ as $t \to \infty$.

(ii) $(p_{ij}(t); i,j \in \mathbf{E})$ is called strongly ergodic (or uniformly ergodic) if

$$\sup_{i \in \mathbf{E}} |p_{ij}(t) - \pi_j| \to 0 \quad as\ t \to \infty.$$

Fo a more detailed account of the criteria for recurrence, ergodicity, exponentially ergodic and strongly ergodic, one may refer to, for example, Anderson (1991) and Chen (1992).

## 1.4. Markov branching processes

A (one-dimensional) continuous-time Markov branching process (MBP) is a special class of continuous-time Markov chain with the state space $\mathbf{Z}_+ = \{0, 1, \cdots\}$ and the $q$-matrix $Q = (q_{ij}; i,j \in \mathbf{E})$ having the form

$$q_{ij} = \begin{cases} ib_{j-i+1}, & if\ i \geq 1 \\ 0, & \quad otherwise \end{cases} \tag{1.4.1}$$

where

$$b_k \geq 0\ (k \neq 1), -b_1 = \sum_{k \neq 1} b_k < +\infty. \tag{1.4.2}$$

Due to their importance in probability theory and applications, Markov branching processes form one of the most important topics in Markov chain theory. The study of Markov branching processes has a long history, which, as might be expected, is closely interwoven with applications in the physical and biological sciences. There are several specialized books devoted to this subject (see Harris (1963), Athreya and Ney (1972) and Asmussen and Hering (1983), for instance).

Like the one-dimensional case, the multi-dimensional Markov branching process is also important and has a wide range of applications. However, we only consider the one-dimensional case throughout this thesis and omit the term "one-dimensional" from now on.

The following theorems deal with the most important properties of Markov branching processes; see Anderson (1991), Athreya and Ney

(1972) or Asmussen and Hering (1983) for more detail.

**Theorem 1.4.1.** Let $Q$ be given in (1.4.1)–(1.4.2) and $(p_{ij}(t))$ be the corresponding Feller minimal $Q$-function. Then the following holds.

(i) $(p_{ij}(t))$ is the unique solution of the Kolmogorov forward equation.

(ii) For all $i \geq 1, j \geq 0$ and $t \geq 0$, we have

$$p_{ij}(t) = \sum_{r_1+r_2+\cdots+r_i=j} p_{1r_1}(t) \cdots p_{1r_i}(t) \tag{1.4.3}$$

and hence

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j = (\sum_{j=0}^{\infty} p_{1j}(t)s^j)^i, \quad t \geq 0, \ |s| \leq 1. \tag{1.4.4}$$

Property (1.4.3) is the most important property of Markov branching processes. It is often called the branching property.

The following theorem deals with the regularity criteria for Markov branching processes.

**Theorem 1.4.2.**(Harris Criterion) Let $Q$ be given in (1.4.1)–(1.4.2). Denote $B(s) = \sum_{j=0}^{\infty} b_j s^j$. Then $Q$ is regular (i.e., the Feller minimal $Q$-function $(p_{ij}(t))$ is honest) if and only if one of the following condition holds.

(i) $B'(1) < +\infty$.

(ii) $B'(1) = +\infty$ and for some (therefore for all) $\varepsilon \in (q, 1)$,

$$\int_{\varepsilon}^{1} \frac{ds}{B(s)} = -\infty \tag{1.4.5}$$

where $q$ is the smallest positive root of $B(s) = 0$.

It is clear that a Markov branching process has the property that all positive states are transient. Such instability runs contrary to the behaviour of many natural populations (for a description of such population models, see the excellent reference of Renshaw (1991)). Therefore, there has been an increasing effort to generalise Markov branching process to more general models (Athreya and Ney (1996)).

## 1.5. Outline of the Thesis

In this thesis, the models considered in the references cited earlier will be generalised and, in addition, some new mathematical models for complex branching systems, as mentioned in the previous section, will be constructed and studied.

This thesis concentrates mainly on the theoretical study of such mathematical models. Until now, we have not made any attempt to perform realistic simulation. However, we realise the importance of applications and simulations of the theoretical results. Such important problems will be considered in future.

Chapter Two is devoted to studying the impact on the Markov branching process caused by state-independent immigration and resurrection. We will consider the regularity of the process, its extinction behaviour if there is no resurrection and its recurrence behaviour if there is resurrection.

In Chapter Three, we study the regularity, uniqueness and hitting times of weighted Markov branching models.

Chapter Four concentrates on the study of the collision branching processes. The extinction behaviour and hitting times of the process are discussed in detail.

In Chapter five, we shall consider the general collision branching processes which contains our collision branching process as a special case. The regularity, uniqueness, extinction behaviour and explosion behaviour of the process will be studied in detail.

The most general collision branching models, weighted collision branching processes, will be constructed and their regularity, uniqueness and hitting times will be considered in Chapter Six.

The main conclusions of this thesis will be briefly summarised and some related problems will be pointed out in the final Chapter Seven.

Throughout this thesis, we always let $\mathbf{Z}_+$ denote the set of all nonnegative integers. All the $q$-matrices discussed in this thesis are assumed to be stable and conservative from now on.

I have tried to be fairly exhaustive in citing references closely related to the material presented in this thesis. Some other papers which are not cited in the text but judged to be important and interesting are included in the References.

# Chapter 2. Markov Branching Processes with Immigration and Resurrection

We shall study a modified Markov branching process incorporating both immigration and resurrection in this chapter.

## 2.1. Introduction

The recent developments regarding Markov branching processes can be seen in Athreya and Jagers (1996). Within the Markov branching framework both state-independent and state-dependent immigration have important roles to play. For the former, Aksland (1975, 1977) considered a modified birth-death process where the state-independent immigration is imposed on a simple birth-death underlying structure, see also Adke (1969). A detailed and systemic analysis of this model, using the partial differential equation approach, can be found in Section 3.2 of Anderson (1991). Renshaw (1972, 1973) considered birth and death processes with migration. On the other hand, the latter (state-dependent immigration) can be traced to Foster (1971) and Pakes (1971) who considered a discrete branching process with immigration occurring only when the process occupies state 0. Yamazato (1975) investigated the continuous-time version in which once the process reaches 0 it remains there for an exponentially distributed time and then jumps to some positive state with a given probability law. This latter model is also called a Markov branching process with resurrection. See also the discussion in Pakes and Tavaré (1981). Chen and Renshaw (1990, 1993b, 1995, 2000) considered a new type of resurrection, namely the instantaneous resurrection. For further discussion, see Pakes (1993). More recently, Chen (1997) considered a more general branching process with or without resurrection which may be viewed as an extension of Yamazato's model. For further discussion of this latter model, see Chen (2002a,b).

We now combine the state-independent immigration and resurrection to generalise Yamazato's model to include further immigration effect when the process is in some positive state.

**Definition 2.1.1** A $q$-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called a branching $q$-matrix with immigration and resurrection (henceforth referred to as a BIR $q$-matrix), if

$$q_{ij} = \begin{cases} h_j, & if\ i = 0, j \geq 0 \\ ib_{j-i+1} + a_{j-i}, & if\ \ i \geq 1, j \geq i \\ ib_0, & if\ \ i \geq 1, j = i-1 \\ 0, & otherwise \end{cases} \tag{2.1.1}$$

where

$$\begin{cases} h_j \geq 0\ (j \neq 0),\ 0 \leq -h_0 = \Sigma_{j=1}^{\infty} h_j < \infty \\ a_j \geq 0\ (j \neq 0),\ 0 \leq -a_0 = \Sigma_{j=1}^{\infty} a_j < \infty \\ b_j \geq 0\ (j \neq 1),\ 0 < -b_1 = \Sigma_{j \neq 1} b_j < \infty. \end{cases} \tag{2.1.2}$$

The corresponding continuous-time Markov chain is called a Markov branching process with immigration and resurrection (henceforth referred to as a MBPIR). Hence, the basic known conditions for our process are three sequences $\{b_n; n \geq 0\}$, $\{a_n; n \geq 0\}$ and $\{h_n;\ n \geq 0\}$.

For convenience, let $b := -b_1$, $a := -a_0$, and $h := -h_0$ and hence $b > 0$, $a \geq 0$ and $h \geq 0$. Note that $a = 0$ if and only if $a_j = 0$ for all $j \geq 0$. Similarly, $h = 0$ if and only if $h_j = 0$ for all $j \geq 0$.

It is worth pointing out that if $a_j = h_j$ $(j \geq 0)$, as in the BDI process, the techniques in the theory of partial differential equations can be applied. However, if the two sequences $\{a_j;\ j \geq 0\}$ and $\{h_j;\ j \geq 0\}$ are not identical, this approach fails since no partial differential equation can be constructed and we have to find some new techniques and methods. In this chapter, we shall not assume that $\{a_j;\ j \geq 0\}$ and $\{h_j;\ j \geq 0\}$ are identical. Indeed, one of the main questions investigated in this chapter is the case that $a > 0$ together with $h = 0$ and hence $\{a_j;\ j \geq 0\}$ and $\{h_j;\ j \geq 0\}$ are not identical. Fortunately, the new techniques required have been discovered in this chapter. Therefore, in addition to the new results, our approaches used have methodological significance and may be applicable in some other models.

Note that if $b_0 = 0$, then the property of the corresponding process is easily obtained and therefore, in order to avoid discussing this trivial case, we shall assume $b_0 > 0$ throughout this chapter. Note also that if $a = 0$, then the property of the corresponding process is well-discussed and understood. For this reason, we shall also assume that $a > 0$. However, all of our conclusions and methods work perfectly well if $a = 0$.

## 2.2. Preliminary and Regularity

To begin with the study, we introduce the generating functions $B(s)$, $A(s)$ and $H(s)$ of the three known sequences in (2.1.1)–(2.1.2) as

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \quad A(s) = \sum_{j=0}^{\infty} a_j s^j, \quad H(s) = \sum_{j=0}^{\infty} h_j s^j.$$

It is clear that $B(s)$, $A(s)$ and $H(s)$ are all well-defined and finite at least on $[-1, 1]$ with $B(1) = 0$ and therefore 1 is a root of $B(s) = 0$. Also let

$$m_b := \sum_{j=1}^{\infty} j b_{j+1}, \quad m_a := \sum_{j=1}^{\infty} j a_j, \quad m_h := \sum_{j=1}^{\infty} j h_j$$

denote the mean birth, immigration and resurrection rates of the process respectively. These, together with the (mean) death rate $b_0$, are the most important quantities of the process.

The following simple yet interesting lemma has important application later. Recall that a root is called simple if it has multiplicity 1.

**Lemma 2.2.1.** (i) If $a > 0$ then $A(s) < 0$ for all $s \in [-1, 1)$ and $\lim_{s \uparrow 1} A(s) = A(1) = 0$, while if $a = 0$ then $A(s) \equiv 0$. Similar property holds for $H(s)$.

(ii) $B(s)$ is a convex function on $[0, 1]$. If $m_b \leq b_0$ then $B(s) > 0$ for all $s \in [0, 1)$ and $B(s) = 0$ has exactly one root 1 on $[0, 1]$. Furthermore, if $m_b < b_0$ then 1 is a simple root while if $m_b = b_0$ then 1 is a root of multiplicity 2. If $b_0 < m_b \leq +\infty$, then $B(s) = 0$ has exactly two roots $q$ and 1 on $[0, 1]$ with $0 < q < 1$ such that $B(s) > 0$ for $s \in (0, q)$ and $B(s) < 0$ for $s \in (q, 1)$. Both $q$ and 1 are simple roots of $B(s) = 0$.

(iii) For any $k > 0$, $kB(s) + sA(s)$ is convex on $[0, 1]$ and thus has at most two zeros on $[0, 1]$. If $k(m_b - b_0) + m_a \le 0$ then $kB(s) + sA(s) > 0$ for all $s \in [0, 1)$ and $kB(s) + sA(s) = 0$ has only one root 1 on $[0, 1]$ which is simple or of multiplicity 2 depending on whether $k(m_b - b_0) + m_a < 0$ or $k(m_b - b_0) + m_a = 0$.

If $k(m_b - b_0) + m_a > 0$, then $kB(s) + sA(s) = 0$ has exactly two roots $s_k$ and 1 on $[0, 1]$ with $0 < s_k < 1$ such that $kB(s)+sA(s) > 0$ for $s \in (0, s_k)$ and $kB(s) + sA(s) < 0$ for $s \in (s_k, 1)$. Moreover, both $s_k$ and 1 are simple roots.

(iv) If $m_a = +\infty$ and $b_0 \ge m_b$ or if $0 < m_a < +\infty$ and $b_0 = m_b$, then for any $k > 0$, the equation $kB(s) + sA(s) = 0$ has exactly one root $s_k \in (0, 1)$ such that $s_k$ is increasing with respect to $k$ and that $\lim_{k\to\infty} s_k = 1$.

(v) Suppose that $0 < m_a < +\infty$ and $b_0 > m_b$. If $0 < k < \frac{m_a}{b_0 - m_b}$ then the equation $kB(s) + sA(s) = 0$ has exactly one root $s_k \in (0, 1)$ such that $s_k$ is increasing with respect to $k$ and that $s_k \uparrow 1$ as $k \to \frac{m_a}{b_0 - m_b}$ while if $k \ge \frac{m_a}{b_0 - m_b}$ then $kB(s) + sA(s) > 0$ for all $s \in (0, 1)$ and thus 1 is the only root of $kB(s) + sA(s) = 0$ on $[0, 1]$.

**Proof.** First, if $a = -a_0 > 0$ then $A(s) = -\sum_{j=1}^{\infty} a_j(1 - s^j) \le -a_1(1 - s) < 0$ for all $s \in [-1, 1)$. $\lim_{s\uparrow 1} A(s) = A(1) = 0$ follows from the definition of $A(s)$. If $a = 0$ then $a_j = 0$ for all $j \ge 0$, thus $A(s) \equiv 0$. A same argument can yield the statement regarding $H(s)$. (i) is proved.

We now prove (ii). Note that $B''(s) = \sum_{j=2}^{\infty} j(j-1)b_j s^{j-2} \ge 0$ for $s \in [0, 1]$, so $B(s)$ is convex on $[0, 1]$. It follows from the definition of $B(s)$ that $B(0) = b_0 > 0$, $B(1) = 0$ and

$$B'(s) = \sum_{j=1}^{\infty} jb_j s^{j-1} = \sum_{j=2}^{\infty} b_j \cdot (js^{j-1} - 1) - b_0$$

which is a increasing function on $[0,1]$.

If $m_b < b_0$, then $B'(s) \le \Sigma_{j=2}^{\infty}(j-1)b_j - b_0 = m_b - b_0 < 0$ for all $s \in [0,1]$ and hence $B(s) = 0$ has exactly one root 1 on $[0,1]$, this root is simple. If $m_b = b_0$, then $B'(s) < \Sigma_{j=2}^{\infty}(j-1)b_j - b_0 = m_b - b_0 = 0$ for all $s \in [0,1)$ and $B'(1) = m_b - b_0 = 0$, therefore $B(s) = 0$ has exactly one root 1 on $[0,1]$ and this root is of multiplicity 2.

If $b_0 < m_b \le +\infty$, then $B'(0) = b_1 < 0$ and $B'(1) = m_b - b_0 > 0$. So, the increasing function $B'(s)$ has exactly one zero point $\zeta$ in $(0,1)$. By the convexity of $B(s)$, we know that $B(s)$ is decreasing on $[0,\zeta)$ and increasing on $(\zeta,1]$. Thus, $B(s) = 0$ has exactly two roots $q$ and 1 on $[0,1]$ with $0 < q < 1$ such that $B(s) > 0$ for $s \in (0,q)$ and $B(s) < 0$ for $s \in (q,1)$. Both $q$ and 1 are simple roots of $B(s) = 0$. Therefore (ii) is proved. The proof of (iii) is same as that of (ii).

Next prove (iv). Denote $F_k(s) := kB(s)+sA(s)$ for any $k > 0$. Under either assumption in (iv), we always have $k(m_b - b_0) + m_a > 0$ $(k > 0)$. Therefore, by (iii), $kB(s) + sA(s) = 0$ has exactly one root $s_k \in (0,1)$ for any $k > 0$. Now, view this root $s_k$ as a function of $k > 0$ and we prove that $s_k$ is increasing with respect to $k > 0$. Indeed, suppose that $0 < k_1 < k_2$ and $s_{k_1}$, $s_{k_2}$ are the corresponding roots. Then $F_{k_1}(s_{k_1}) = F_{k_2}(s_{k_2}) = 0$ and $F_{k_2}(s_{k_1}) = k_2B(s_{k_1}) + s_{k_1}A(s_{k_1}) = (k_2 - k_1)B(s_{k_1}) > 0$ since $B(s) > 0$ for all $s \in [0,1)$. However, $F_{k_2}(s) > 0$ for $s \in (0, s_{k_2})$ and $F_{k_2}(s) < 0$ for $s \in (s_{k_2}, 1)$. Hence $F_{k_2}(s_{k_1}) > 0$ implies $s_{k_1} < s_{k_2}$. This proves the increase property of $s_k$ on $k$. It then follows that the limit $\tilde{s} := \lim_{k\to\infty} s_k$ exists and $\tilde{s} \le 1$. We now claim that $\tilde{s} = 1$. Indeed, if $\tilde{s} < 1$, then by (iii) we have $F_k(\tilde{s}) < 0$ for all $k > 0$. On the other hand, noting $B(\tilde{s}) > 0$ we have

$$F_k(\tilde{s}) = kB(\tilde{s}) + \tilde{s}A(\tilde{s}) \uparrow +\infty \quad as\ k \to \infty$$

which contradicts $F_k(\tilde{s}) < 0$ for all $k > 0$. Hence $\tilde{s} = 1$, which completes the proof of (iv). Finally, the proof of (v) can be given exactly the same as in the proof of (iv) together with the simple fact that $k(m_b - b_0) + m_a \le 0$ if and only if $k \ge m_a/(b_0 - m_b)$. □

**Remark 2.2.1.** The conclusions (iii)−(v) in Lemma 2.2.1 hold for any real number $k > 0$. However, in the latter application $k$ will be mainly taken to be a positive integer. Note also that the conclusions (iv)–(v) in Lemma 2.2.1 hold similarly for the case $b_0 < m_b \leq +\infty$. The only difference is that for this latter case, the unique root $s_k \in (0,1)$ of the equation $kB(s) + sA(s) = 0$ will increasingly tend to $q < 1$ rather than to 1. Indeed, If $b_0 < m_b \leq +\infty$, then $F_k'(1) = k(m_b - b_0) + m_a > 0$ and $B(s) = 0$ has a root $q$ in $(0,1)$. By (iii) in Lemma 2.2.1, $F_k(s) = 0$ has a root $s_k \in (0,1)$. Note that $kB(s_k)+s_kA(s_k) = F_k(s_k) = 0$ and $A(s_k) < 0$, it follows that $B(s_k) > 0$ and hence $0 < s_k < q$. Secondly, suppose that $0 < k_1 < k_2$ and $s_{k_1}$, $s_{k_2}(\in (0,q))$ are the corresponding roots, i.e., $F_{k_1}(s_{k_1}) = F_{k_2}(s_{k_2}) = 0$. Then $F_{k_2}(s_{k_1}) = (k_2 - k_1)B(s_{k_1}) > 0$ since $B(s) > 0$ for all $s \in [0,q)$, hence $s_{k_1} < s_{k_2}$. Now denote $\tilde{s} := \lim_{k\to\infty} s_k$. If $\tilde{s} < q$ then by (iii) in Lemma 2.2.1 and the fact $s_k < \tilde{s}$ for all $k > 0$, we have $F_k(\tilde{s}) < 0$. On the other hand, $F_k(\tilde{s}) = kB(\tilde{s}) + \tilde{s}A(\tilde{s}) \uparrow +\infty$ since $B(\tilde{s}) > 0$, which contradicts $F_k(\tilde{s}) < 0$. Thus, $\tilde{s} \geq q$ and hence $\tilde{s} = q$ since we already have $\tilde{s} \leq q$.

In the sequel, we always let $q(< 1)$ denote the smallest nonnegative root of $B(s) = 0$ for the case $b_0 < m_b \leq +\infty$.

**Lemma 2.2.2.** Let $(p_{ij}(t); i,j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a BIR $q$-matrix. Then for any $i \geq 0$ and $s \in [0,1)$,

$$\begin{aligned}\sum_{j=0}^{\infty} p_{ij}'(t)s^j &= H(s)\cdot p_{i0}(t) + A(s)\cdot\sum_{j=1}^{\infty} p_{ij}(t)s^j \\ &\quad + B(s)\cdot\sum_{k=1}^{\infty} p_{ik}(t)\cdot ks^{k-1}\end{aligned} \tag{2.2.1}$$

or equivalently if in resolvent version,

$$\begin{aligned}\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j - s^i &= H(s)\cdot\phi_{i0}(\lambda) + A(s)\cdot\sum_{j=1}^{\infty}\phi_{ij}(\lambda)s^j \\ &\quad + B(s)\cdot\sum_{k=1}^{\infty}\phi_{ik}(\lambda)\cdot ks^{k-1}.\end{aligned} \tag{2.2.2}$$

**Proof.** From the Kolmogorov forward equations we obtain that

$$p_{ij}'(t) = p_{i0}(t)h_j + \sum_{k=1}^{j} p_{ik}(t)\cdot(kb_{j-k+1} + a_{j-k}) + b_0 p_{ij+1}(t)\cdot(j+1).$$

Multiplying $s^j$ on both sides of the above equality and then summing over $j$ yields that for $s \in [0,1)$,

$$\begin{aligned}\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= H(s)\cdot p_{i0}(t) + \sum_{j=0}^{\infty}\left(\sum_{k=1}^{j+1} p_{ik}(t)kb_{j-k+1}\right)s^j \\ &\quad + \sum_{j=1}^{\infty}\left(\sum_{k=1}^{j} p_{ik}(t)a_{j-k}\right)s^j \\ &= H(s)\cdot p_{i0}(t) + \sum_{k=1}^{\infty} p_{ik}(t)ks^{k-1}\cdot \sum_{j=k-1}^{\infty} b_{j-k+1}s^{j-k+1} \\ &\quad + \sum_{k=1}^{\infty} p_{ik}(t)s^k \cdot \sum_{j=k}^{\infty} a_{j-k}s^{j-k} \\ &= H(s)\cdot p_{i0}(t) + A(s)\cdot \sum_{j=1}^{\infty} p_{ij}(t)s^j \\ &\quad + B(s)\cdot \sum_{k=1}^{\infty} p_{ik}(t)\cdot ks^{k-1}.\end{aligned}$$

So we have proved (2.2.1). Finally, (2.2.2) is just the Laplace transform version of (2.2.1). □

Since the BIR $q$-matrix $Q$ is conservative and totally stable, general theory (see Anderson (1991) or Yang (1990)) dictates that there always exists an MBPIR, namely the (possibly dishonest) Feller minimal process. Therefore the existence problem does not occur. However, the regularity and uniqueness problems do arise, which will be the main topic of this section. Recall that a conservative $q$-matrix $Q$ is called regular if the corresponding Feller minimal $Q$-process is honest.

**Theorem 2.2.3.** A BIR $q$-matrix $Q$ is regular if and only if one of the following conditions holds.

(i) $m_b < +\infty$.

(ii) $m_b = +\infty$ and $\int_\varepsilon^1 \frac{1}{-B(s)}ds = +\infty$ for some (equivalently, for all) $\varepsilon \in (q,1)$, where $q(<1)$ is the smallest nonnegative root of $B(s) = 0$, guaranteed by the condition $m_b = +\infty$.

**Proof.** If $b_0 \geq m_b$ then (ii) of Lemma 2.2.1 implies that $B(s) \geq 0$ for all $s \in [0,1]$. Thus by (2.2.2) we have that for all $0 \leq s < 1$,

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j - s^i \geq H(s)\cdot\phi_{i0}(\lambda) + A(s)\cdot\sum_{j=1}^{\infty}\phi_{ij}(\lambda)s^j.$$

Now letting $s \uparrow 1$ in the above inequality and using (i) of Lemma 2.2.1 immediately yields $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \geq 1$, which implies that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = 1$ since the converse inequality always holds. Hence the Feller minimal $Q$-process is honest and thus $Q$ is regular.

We now assume $b_0 < m_b \leq +\infty$. Note that $b_0 < m_b < +\infty$ implies $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ for some (equivalently, for all) $\varepsilon \in (q, 1)$. Therefore we only need to prove that $Q$ is regular if and only if $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ for some (equivalently, for all) $\varepsilon \in (q, 1)$.

Suppose that $\int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ for some (equivalently, for all) $\varepsilon \in (q, 1)$ but $Q$ is not regular, i.e., $\rho =: 1 - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) > 0$ for some $i \geq 0$ and a fixed $\lambda > 0$. Then there exists an $\varepsilon \in (q, 1)$ such that

$$s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j \geq \rho/2 \quad \textit{and} \quad |H(s) + A(s)| \leq \lambda\rho/4, \qquad s \in (\varepsilon, 1].$$

Note that $B(s) < 0$ for $s \in (\varepsilon, 1)$, it follows from (2.2.2) that for all $s \in (\varepsilon, 1)$,

$$\begin{aligned} &\sum_{j=1}^{\infty} \phi_{ij}(\lambda) \cdot j s^{j-1} \\ = \; &\frac{s^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j + H(s) \cdot \phi_{i0}(\lambda) + A(s) \cdot \sum_{j=1}^{\infty} \phi_{ij}(\lambda) s^j}{-B(s)} \\ \geq \; &\frac{\rho}{-4B(s)}. \end{aligned}$$

Therefore, $\sum_{j=1}^{\infty} \phi_{ij}(\lambda) \cdot (1 - \varepsilon^j) \geq \frac{\rho}{4} \int_\varepsilon^1 \frac{1}{-B(s)} ds = +\infty$ which is a contradiction.

Conversely, suppose that $Q$ is regular but $\int_\varepsilon^1 \frac{1}{-B(s)} ds < +\infty$ for some $\varepsilon \in (q, 1)$. Hence the Feller minimal $Q$-process is honest, i.e., $\sum_{j=0}^{\infty} p_{ij}(t) = 1$ $(i \geq 0)$. It follows from (2.2.1) that for any $i \geq 1$ and $s \in (\varepsilon, 1)$,

$$\sum_{j=1}^{\infty} p_{ij}(t) \cdot j s^{j-1} + \frac{H(s)}{B(s)} \cdot p_{i0}(t) = \frac{\sum_{j=0}^{\infty} p'_{ij}(t) s^j - A(s) \cdot \sum_{j=1}^{\infty} p_{ij}(t) s^j}{B(s)}.$$

Since $\frac{A(s)}{B(s)} \geq 0$ for $s \in (\varepsilon, 1)$, we have

$$\sum_{j=1}^{\infty} p_{ij}(t) \cdot j s^{j-1} + \frac{H(s)}{B(s)} \cdot p_{i0}(t) \leq \frac{\sum_{j=0}^{\infty} p'_{ij}(t) s^j}{B(s)}$$

and hence

$$\sum_{j=1}^{\infty} p_{ij}(t) \cdot (1-\varepsilon^j) + p_{i0}(t) \cdot \int_{\varepsilon}^{1} \frac{H(s)}{B(s)} ds \le \int_{\varepsilon}^{1} \frac{\sum_{j=0}^{\infty} p'_{ij}(t) s^j}{B(s)} ds. \qquad (2.2.3)$$

Noting the facts that $B(s) < 0$ in $(q, 1)$ and $\sum_{j=0}^{\infty} |p'_{ij}(t)| \le 2q_i$ for any $i \ge 1$ (see Chung (1967) or Yang (1990), for example) we have

$$\int_{\varepsilon}^{1} \left| \frac{\sum_{j=0}^{\infty} p'_{ij}(t) s^j}{B(s)} \right| ds \le 2q_i \int_{\varepsilon}^{1} \frac{ds}{-B(s)} < \infty.$$

Therefore we may let $t \uparrow \infty$ in (2.2.3) and apply the Dominated Convergence Theorem to obtain

$$\lim_{t\to\infty} \left[ \sum_{j=1}^{\infty} p_{ij}(t) \cdot (1-\varepsilon^j) + p_{i0}(t) \cdot \int_{\varepsilon}^{1} \frac{H(s)}{B(s)} ds \right] \le 0, \quad i \ge 1. \qquad (2.2.4)$$

Here in obtaining (2.2.4) we have used the fact that $\lim_{t\to\infty} p'_{ij}(t) = 0$ for any $i, j \ge 0$ (see Chung (1967)). Since both $\sum_{j=1}^{\infty} p_{ij}(t) \cdot (1-\varepsilon^j)$ and $p_{i0}(t) \cdot \int_{\varepsilon}^{1} \frac{H(s)}{B(s)} ds$ are nonnegative, (2.2.4) shows that both of them must tend to 0 as $t \to \infty$, which, by using the fact that $1-\varepsilon < 1-\varepsilon^j < 1$ for all $j \ge 1$, is equivalent to

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} p_{ij}(t) = 0 = \lim_{t\to\infty} p_{i0}(t) \cdot \int_{\varepsilon}^{1} \frac{H(s)}{B(s)} ds, \quad i \ge 1, \ s \in (\varepsilon, 1). \qquad (2.2.5)$$

By the first equality in (2.2.5) and the honesty of the $Q$-process, we obtain

$$\lim_{t\to\infty} p_{i0}(t) = 1, \quad i \ge 1, \qquad (2.2.6)$$

which, in turn, by using the last equality in (2.2.5), clearly implies that $H(s) = 0$ for all $s \in (\varepsilon, 1)$. Therefore, by (i) of Lemma 2.2.1 we must have $H(s) \equiv 0$ for all $s \in [0, 1]$. Now, letting $s = q < 1$ in (2.2.1) and noting $B(q) = 0$, integrating with respect to $t$ from 0 to $\infty$ and again using the first equality in (2.2.5) immediately yield

$$\lim_{t\to\infty} p_{i0}(t) = q^i + A(q) \cdot \sum_{j=1}^{\infty} \left(\int_0^{\infty} p_{ij}(u) du\right) \cdot q^j \le q^i < 1, \quad i \ge 1$$

which contradicts (2.2.6). The proof is thus complete. □

¿From the general theory of continuous time Markov chains we know that if a $q$-matrix $Q$ is regular then there exists only one $Q$-function, the Feller minimal $Q$-function which satisfies the Kolmogorov forward equations. However, if a conservative $q$-matrix $Q$ is not regular, then there exist infinitely many $Q$-functions (even infinitely many honest $Q$-functions). The following conclusion shows that even if a BIR $q$-matrix $Q$ is not regular, there still exists only one $Q$-function (i.e., the Feller minimal $Q$-function) which can satisfy the Kolmogorov forward equations.

**Theorem 2.2.4.** There always exists only one MBPIR which satisfies the Kolmogorov forward equations.

**Proof.** By Theorem 2.2.3, we only need to consider the case $m_b = +\infty$. For this purpose, it is sufficient to show that (see Theorem 1.3.3) the equation

$$\begin{cases} \mathbf{Y}(\lambda I - Q) = 0, \\ \mathbf{0} \leq \mathbf{Y}, \mathbf{Y1} < +\infty \end{cases} \tag{2.2.7}$$

has only a trivial solution for some (and therefore for all) $\lambda > 0$ where $\mathbf{1}$ denotes the column vector on $\mathbf{Z}_+$ whose components are all equal to 1.

Suppose that $\mathbf{Y} = (y_i; i \geq 0)$ is a solution of (2.2.7) for $\lambda = 1$. Then (2.2.7) can be rewritten as

$$\begin{cases} y_0 = y_0 h_0 + y_1 b_0 \\ y_n = y_0 h_n + \sum_{j=1}^{n} y_j \cdot (j b_{n-j+1} + a_{n-j}) + y_{n+1} \cdot (n+1) b_0, \quad n \geq 1. \end{cases}$$

Multiplying both sides of the above equation by $s^n$, summing over $n \geq 0$ then yields that for any $|s| < 1$,

$$\sum_{n=0}^{\infty} y_n s^n = y_0 \cdot H(s) + A(s) \cdot \sum_{n=1}^{\infty} y_n s^n + B(s) \cdot \sum_{n=1}^{\infty} y_n \cdot n s^{n-1}.$$

i.e., for any $|s| < 1$,

$$y_0(1 - H(s)) + (1 - A(s)) \cdot \sum_{n=1}^{\infty} y_n s^n = B(s) \sum_{n=1}^{\infty} y_n \cdot n s^{n-1}. \tag{2.2.8}$$

Since $m_b = +\infty$, by Lemma 2.2.1(ii), $B(s) = 0$ has a root $q \in [0, 1)$ and $B(s) < 0$ in $(q, 1)$. Therefore, by comparing both sides of (2.2.8) in $(q, 1)$ and noting that the left hand-side of (2.2.8) is certainly nonnegative, we obtain that $y_n = 0$ $(n \geq 0)$. The proof is complete. □

From Theorems 2.2.3 and 2.2.4, we see that the regularity and uniqueness criteria for MBPIRs are the same as that for Markov branching process without immigration. In other words, immigration does not have any effect on the regularity and uniqueness property.

From now on, we shall assume that the BIR $q$-matrix is regular and thus the corresponding MBPIR is honest.

## 2.3. Absorbing Property

Throughout this section, we shall always assume that $h = 0$ and thus the state 0 is an absorbing state. For this process, the most interesting problem is the property regarding the extinction probability and extinction time. Note that if we further have $a = 0$, then we recover the ordinary Markov branching process for which the extinction property is well-known. In particular, we know that, when $a = 0$ the extinction probability is 1 if and only if the death rate is greater than or equal to the mean birth rate, i.e., $b_0 \geq m_b$. On the other hand, when $a > 0$ and thus immigration occurs, then, intuitively speaking, the immigration will reduce the possibility of extinction. Therefore, our main interest is to examine the depth of the effect caused by such immigration. For this purpose, let $(X(t); t \geq 0)$ be the MBPIR determined by a regular BIR $q$-matrix $Q$, and let $\tau_0 = \inf\{t > 0;\ X(t) = 0\}$ be the extinction time and let $a_{i0} = P_i(\tau_0 < \infty)$ $(i \geq 1)$ be the corresponding extinction probability when the process starts at state $i \geq 1$, where $P_i$ is the probability distribution under the condition $X(0 = i)$. It is clear that $a_{i0} = \lim_{t\to\infty} p_{i0}(t)$, where $P(t) = (p_{ij}(t); i, j \geq 0)$ is the transition function of the corresponding absorbing process. Note that each $p_{i0}(t)$ $(i \geq 1)$ is an increasing function of $t \geq 0$ and thus the limit $a_{i0}$ does exist. Before stating our main result in this section, we first provide some useful lemmas.

**Lemma 2.3.1.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function where $Q$ is an absorbing (regular) BIR $q$-matrix. Then for any $i \geq 0$,

$$\int_0^\infty p_{ik}(t)dt < \infty, \qquad k \geq 1 \tag{2.3.1}$$

and thus

$$\lim_{t\to\infty} p_{ik}(t) = 0, \qquad i \geq 0,\ k \geq 1. \tag{2.3.2}$$

Moreover, for any $i \geq 0$ and $s \in [0, 1)$, we have

$$\sum_{k=1}^{\infty} \left( \int_0^{\infty} p_{ik}(t)dt \right) \cdot s^k < \infty. \tag{2.3.3}$$

**Proof.** For any fixed $i \geq 0$, it follows from the Kolmogorov forward equations that

$$p_{i0}(t) = \delta_{i0} + b_0 \int_0^t p_{i1}(u)du$$

which clearly implies that $\int_0^{\infty} p_{i1}(t)dt < \infty$. Suppose $\int_0^{\infty} p_{ik}(t)dt < \infty$ for $k \leq j$. From Kolmogorov forward equations we can see that

$$p_{ij}(t) - \delta_{ij} = \sum_{k=1}^{j} (kb_{j-k+1} + a_{j-k}) \cdot \int_0^{\infty} p_{ik}(t)dt + (j+1)b_0 \int_0^{\infty} p_{ij+1}(t)dt$$

and hence $\int_0^{\infty} p_{ij+1}(t)dt < \infty$. Therefore, (2.3.1) follows from the mathematical induction principle. (2.3.2) immediately follows from (2.3.1). Finally we turn to (2.3.3). For this purpose, we shall consider two different cases separately.

First, consider the case $b_0 < m_b \leq +\infty$. By (ii) of Lemma 2.2.1, $B(s) = 0$ has a root $q \in (0, 1)$ such that $B(s) > 0$ in $(0, q)$ and $B(s) < 0$ in $(q, 1)$. Note also that $A(s) \leq 0$ for all $s \in [0, 1]$ and thus for all $s \in (q, 1)$ we have $B(s) + sA(s) < 0$. Therefore using (2.2.1) and noting that $H(s) \equiv 0$ we obtain that for all $s \in (q, 1)$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j \leq (sA(s) + B(s)) \cdot \sum_{j=1}^{\infty} p_{ij}(t)s^{j-1}.$$

The above inequality together with (2.3.2) yields that for all $s \in (q, 1)$,

$$\sum_{j=1}^{\infty} \left( \int_0^{\infty} p_{ij}(t)dt \right) \cdot s^j \leq \frac{s(a_{i0} - s^i)}{B(s) + sA(s)} < \infty.$$

Next consider the case $b_0 \geq m_b$. If either $0 < m_a < +\infty$ together with $b_0 = m_b$ or $m_a = +\infty$, then from (iii)$-$(iv) of Lemma 2.2.1, we know that for any $k \geq 1$, $kB(s) + sA(s) = 0$ has a root $s_k \in (0, 1)$ such that $s_k \uparrow 1$ as $k \to \infty$. Thus for any $\bar{s} \in [0, 1)$, we may find a positive integer $k$ such that $s_k > \bar{s}$ and thus $kB(\bar{s}) + \bar{s}A(\bar{s}) > 0$. Therefore, it follows from (2.2.1) that for any $t \geq 0$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)\bar{s}^j = A(\bar{s}) \cdot \sum_{j=1}^{\infty} p_{ij}(t)\bar{s}^j + B(\bar{s}) \sum_{j=1}^{\infty} p_{ij}(t) \cdot j\bar{s}^{j-1}$$

$$\geq A(\bar{s}) \cdot \sum_{j=1}^{\infty} p_{ij}(t)\bar{s}^j + B(\bar{s}) \sum_{j=k+1}^{\infty} p_{ij}(t) \cdot j\bar{s}^{j-1}$$

$$\geq [kB(\bar{s}) + \bar{s}A(\bar{s})] \cdot \sum_{j=k+1}^{\infty} p_{ij}(t) \cdot \bar{s}^{j-1} + A(\bar{s}) \cdot \sum_{j=1}^{k} p_{ij}(t)\bar{s}^j.$$

Integrating the above inequality with $t$ then yields that for any $t \geq 0$, we have

$$\sum_{j=0}^{\infty} p_{ij}(t)\bar{s}^j - \bar{s}^i \geq [kB(\bar{s}) + \bar{s}A(\bar{s})] \cdot \sum_{j=k+1}^{\infty} (\int_0^t p_{ij}(u)du) \cdot \bar{s}^{j-1}$$

$$+A(\bar{s}) \cdot \sum_{j=1}^{k} (\int_0^t p_{ij}(u)du)\bar{s}^j.$$

Letting $t \uparrow \infty$ and noting (2.3.1) immediately yields that

$$\sum_{j=k+1}^{\infty} (\int_0^{\infty} p_{ij}(u)du) \cdot \bar{s}^{j-1} < \infty$$

which clearly implies (2.3.3). If $b_0 > m_b$ and $0 < m_a < +\infty$, then by (v) of Lemma 2.2.2, there exists $k \geq 1$ such that $k(m_b - b_0) + m_a < 0$. By (iii) of Lemma 2.2.1, we again have $kB(s) + sA(s) > 0$ for all $s \in (0, 1)$. Thus (2.3.3) can be similarly proven. □

**Lemma 2.3.2.** Suppose that $c_{ij} \geq 0$ and $b_i \geq 0$, $i, j \in \mathbf{Z}_+$. Then the minimal solution, denoted by $(x_i^*; i \in \mathbf{Z}_+)$, to the equation

$$x_i = \sum_{k=0}^{\infty} c_{ik}x_k + b_i, \quad x_i \geq 0,\ i \in \mathbf{Z}_+$$

always exists and is indeed unique. Further, it can be obtained by the following procedure: Let

$$x_i^{(0)} = 0,\ \forall\, i \in \mathbf{Z}_+,$$

$$x_i^{(n+1)} = \sum_{k=0}^{\infty} c_{ik}x_k^{(n)} + b_i, \quad \forall\, i \in \mathbf{Z}_+$$

for $n \geq 0$. Then $(x^{(n)}; i \in \mathbf{Z}_+)$ increases to $(x_i^*; i \in \mathbf{Z}_+)$.

The above Lemma 2.3.2 is just the Theorem 2.2 in Chen (1992), a full proof can be found there.

The following lemma, a direct consequence of Lemma 2.3.2, can be applied to any absorbing denumerable Markov processes not only for our MBPIR.

**Lemma 2.3.3.** Suppose that the conservative $q$-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ is regular and $(p_{ij}(t);\ i, j \in \mathbf{Z}_+)$ is the corresponding $Q$-function. Further assume that $q_{00} = 0$ and thus 0 is an absorbing state. Denote $x_i^* = \lim_{t\to\infty} p_{i0}(t),\ i \geq 1$. Then $(x_i^*; i \geq 1)$ is the minimal solution of the equation:

$$\sum_{j=1}^{\infty} q_{ij}x_j + q_{i0} = 0, \quad 0 \leq x_i \leq 1,\ i \geq 1.$$

**Proof.** Let $X(t)$ be the corresponding Markov chain and $\sigma_n$ denote the $n$'th transition time of $X(t)$. And also let $(\pi_{ij};\ i, j \in \mathbf{Z}_+)$ denote the transition probabilities of the embedding chain $\{X(\sigma_n); n \geq 0\}$. We may then rewrite the above equation as

$$x_i = \sum_{j=1}^{\infty} \pi_{ij}x_j + \pi_{i0}, \quad 0 \leq x_i \leq 1, i \geq 1.$$

Let $\tau_0$ denote the first hitting time of state 0. Then for any $i \geq 1$,

$$P_i(\tau_0 = \sigma_1) = P_i(X(\sigma_1) = 0) = \pi_{i0}$$

and by the Markov property of $X(\sigma_n)$,

$$P_i(\tau_0 = \sigma_{n+1}) = \sum_{j=1}^{\infty} \pi_{ij} \cdot P_j(\tau_0 = \sigma_n), \quad n \geq 1.$$

where $P_i$ is the probability distribution of the process $(p_{ij}(t))$ starting at state $i \geq 0$. Therefore,

$$\sum_{m=1}^{n+1} P_i(\tau_0 = \sigma_m) = \sum_{j=1}^{\infty} \pi_{ij} \cdot (\sum_{m=1}^{n} P_j(\tau_0 = \sigma_m)), \quad n \geq 1.$$

Noting that $x_i^* = \sum_{n=1}^{\infty} P_i(\tau_0 = \sigma_n)$ $(i \geq 1)$, we immediately see that the assertion follows directly from Lemma 2.3.2. □

Now we can give the main result regarding the extinction property of the process.

**Theorem 2.3.4.** For any $i \geq 1$, $a_{i0} = 1$ if and only if $b_0 \geq m_b$ and $J = +\infty$ where

$$J := \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy. \tag{2.3.4}$$

More specifically,

(i) If $b_0 \geq m_b$ and $J = +\infty$, then $a_{i0} = 1$ $(i \geq 1)$.

(ii) If $b_0 \geq m_b$ and $J < +\infty$, then

$$a_{i0} = \frac{\int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy}{\int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy} < 1, \quad i \geq 1. \tag{2.3.5}$$

(iii) If $0 < b_0 < m_b \leq +\infty$ and thus the equation $B(s) = 0$ possesses a smallest nonnegative root $q \in (0,1)$, then

$$a_{i0} = \frac{\int_0^q \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy}{\int_0^q \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy} < q^i < 1, \quad i \geq 1. \tag{2.3.6}$$

**Proof.** It follows from (2.2.1) that

$$B(s) \cdot \sum_{k=1}^{\infty} p_{ik}(t) \cdot ks^{k-1} + A(s) \cdot \sum_{j=1}^{\infty} p_{ij}(t)s^j = \sum_{j=0}^{\infty} p'_{ij}(t)s^j$$

since $H(s) \equiv 0$ in the present case. Integrating with respect to $t$ on $[0,u)$ yields that

$$B(s) \cdot \sum_{k=1}^{\infty} \Big(\int_0^u p_{ik}(t)dt\Big) \cdot ks^{k-1} + A(s) \cdot \sum_{j=1}^{\infty} \Big(\int_0^u p_{ij}(t)dt\Big)s^j$$
$$= \sum_{j=0}^{\infty} p_{ij}(u)s^j - s^i.$$

Letting $u \to \infty$ and using (2.3.2)–(2.3.3) in Lemma 2.3.1 yield that

$$B(s)F_i'(s) + A(s)F_i(s) = a_{i0} - s^i, \quad s \in [0,1), i \geq 1 \tag{2.3.7}$$

where $F_i(s) = \sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(u)du) \cdot s^j < \infty$.

First consider the case $b_0 \geq m_b$. Solving the ordinary differential equation (2.3.7) for $s \in [0,1)$ immediately yields

$$F_i(s)e^{\int_0^s \frac{A(x)}{B(x)}dx} = \int_0^s \frac{a_{i0} - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy, \quad s \in [0,1), \ i \geq 1. \tag{2.3.8}$$

Now we claim that if $J = \infty$, then $a_{i0} = 1$. Indeed, if $a_{i0} < 1$, then by letting $s \uparrow 1$ in (2.3.8) we see that the right hand side of (2.3.8) tends to $-\infty$, while the left hand side is always nonnegative, which is a contradiction. Hence (i) is proven. We turn to (ii). First note that

$J < \infty$ implies $\int_0^1 \frac{A(x)}{B(x)}dx = -\infty$. Since the left hand side of (2.3.8) is clearly nonnegative and thus so is the right hand side of (2.3.8). It follows that $a_{i0} \geq J^{-1} \cdot \int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Therefore, in order to prove (2.3.5) in (ii), we only need to show that $a_{i0} \leq J^{-1} \cdot \int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Take $x_j^* = J^{-1} \cdot \int_0^1 \frac{y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$, $j \geq 1$, then for $i \geq 1$,

$$\begin{aligned}
&\sum_{j=1}^{\infty} q_{ij}x_j^* + q_{i0} \\
= \; & J^{-1} \int_0^1 \frac{\sum_{j=0}^{\infty} q_{ij}y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \; & J^{-1} \int_0^1 \frac{iy^{i-1}B(y) + y^iA(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \; & J^{-1} \cdot \int_0^1 iy^{i-1} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy + J^{-1} \int_0^1 \frac{y^iA(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \; & 0.
\end{aligned}$$

Here the last equality follows from applying the method of integration by parts. Hence $(x_j^*; j \geq 1)$ is a solution of the equation

$$\sum_{j=1}^{\infty} q_{ij}x_j + q_{i0} = 0, \quad 0 \leq x_i \leq 1, \; i \geq 1.$$

By Lemma 2.3.3, we then have $a_{i0} \leq x_i^*$ $(i \geq 1)$ since $(a_{i0}; i \geq 1)$ is the minimal solution of the above equation. This completes the proof of (ii).

Finally, consider (iii). Suppose that $b_0 < m_b \leq +\infty$. By (ii) of Lemma 2.2.1, we know that $B(s) = 0$ has a root $q \in (0,1)$ and (2.3.8) holds for all $s \in [0,q)$. Similarly as in the above, we only need to show that

$$a_{i0} \leq \lim_{s \uparrow q} \left[ \int_0^s \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right]^{-1} \cdot \int_0^s \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy.$$

Since $a > 0$, we know by Lemma 2.2.1 that

$$A(x) < 0 \; (\forall x \in [0,q]), \;\; B(x) > 0 \; (\forall x \in [0,q)), \;\; B(q) = 0$$

and

$$B(x) = (q-x) \cdot \left[ -b_1 - \sum_{j=2}^{\infty} b_j (\sum_{k=1}^{j} q^{j-k}x^{k-1}) \right] \leq -b_1(q-x), \; x \in [0,1].$$

Hence, $\int_0^q \frac{A(x)}{B(x)}dx = -\infty$ and for any $y \in [0, q)$,

$$\begin{aligned}
\int_0^y \frac{A(x)}{B(x)}dx &\le -\inf\{|A(x)|; x \in [0,q]\} \cdot \int_0^y \frac{dx}{B(x)} \\
&\le \frac{\inf\{|A(x)|; x \in [0,q]\}}{b_1} \cdot \int_0^y \frac{dx}{q-x} \\
&= \rho \ln \frac{q-y}{q},
\end{aligned}$$

where $\rho = -\inf\{|A(x)|; x \in [0,q]\}/b_1 > 0$. Therefore, the integral $\int_0^q \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx}dy$, denoted by $D$, is convergent. Now let $y_j^* = D^{-1} \cdot \int_0^q \frac{y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx}dy$, $j \ge 1$, then for $i \ge 1$,

$$\begin{aligned}
& \sum_{j=1}^{\infty} q_{ij} y_j^* + q_{i0} \\
= \ & D^{-1} \int_0^q \frac{\sum_{j=0}^{\infty} q_{ij} y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \ & D^{-1} \int_0^q \frac{iy^{i-1}B(y) + y^i A(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \ & D^{-1} \cdot \int_0^q iy^{i-1} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy + D^{-1} \int_0^q \frac{y^i A(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
= \ & 0.
\end{aligned}$$

Thus, $(y_j^*; j \ge 1)$ is a solution of the equation

$$\sum_{j=1}^{\infty} q_{ij} x_j + q_{i0} = 0, \quad 0 \le x_i \le 1, \ i \ge 1.$$

Again by Lemma 2.3.3, we have $a_{i0} \le y_i^*$ $(i \ge 1)$ which proves the first equality in (2.3.6). The last two assertions in (2.3.6) are obvious. □

By Theorem 2.3.4, we see that if $a > 0$ (i.e., if immigration occurs when the process is at positive state), the condition $b_0 \ge m_b$ (i.e., the death rate is greater than or equal to the mean birth rate) is no longer sufficient, though still necessary, for the process to be finally extinct. A further condition $J = +\infty$ is necessary to guarantee the final extinction. Note that the larger the immigration is, the smaller the value of $J$ will be. Therefore, this $J$ reflects the effect of immigration.

Although Theorem 2.3.4 has answered our basic question, the probabilistic interpretation of this latter condition may not be too clear. Also, this condition may not be easy to check in some situations. The following corollaries therefore provide some further sufficient conditions for $a_{i0} = 1\ (i \geq 1)$. These conditions are much easier to be checked and also possess a clear probabilistic interpretation. To this end, we consider the cases $b_0 > m_b$ and $b_0 = m_b$ separately.

**Corollary 2.3.5.** If $b_0 > m_b$ and $\sum_{k=1}^{\infty} a_k \ln k < \infty$, then $a_{i0} = 1\ (i \geq 1)$. In particular, if $b_0 > m_b$ and $m_a < \infty$, then $a_{i0} = 1\ (i \geq 1)$.

**Proof.** Note that for $x \in [0,1]$,

$$\begin{aligned} B(x) &= \sum_{j=0}^{\infty} b_j(x^j - 1) \\ &= (1-x)(b_0 - \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1})x^k) \\ &\geq (b_0 - m_b)(1-x) \end{aligned}$$

and also

$$\begin{aligned} A(x) &= \sum_{j=0}^{\infty} a_j(x^j - 1) \\ &= (1-x)(a_0 - \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_{j+1})x^k) \\ &= (1-x)\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j)x^{k-1}. \end{aligned}$$

Hence

$$\int_0^1 \frac{A(x)}{B(x)}dx \geq -\frac{1}{b_0 - m_b} \cdot \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j) \cdot k^{-1} = -\frac{1}{b_0 - m_b} \cdot \sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} \frac{1}{k}$$

which is finite since $\sum_{k=1}^{\infty} a_k \ln k < \infty$. Thus $\int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = +\infty$ and hence $a_{i0} = 1\ (i \geq 1)$. $\square$

**Remark 2.3.1.** If $b_0 > m_b$ and $\sum_{k=1}^{\infty} a_k \ln k = +\infty$, then it is fairly easy to construct examples such that either $a_{i0} = 1\ (i \geq 1)$ or $a_{i0} < 1\ (i \geq 1)$ may occur. In this case, the condition $J = +\infty$ should be checked.

Corollary 2.3.5 tells us that if $b_0 > m_b$ (the death rate is strictly greater than the mean birth rate), then in order to "rescue" a species from extinction, a considerably large immigration is necessary. In particular, if the mean immigration rate is finite, then it can never "rescue" a species from extinction. Intuitively speaking, condition $b_0 > m_b$ (i.e., the death rate is strictly greater than the mean birth rate) implies that the species in the system will tend to be extinct. Therefore, in order to rescue this species, the immigration must be very large. The following corollary shows that, however, the situation is very different if $b_0 = m_b$.

**Corollary 2.3.6.** Suppose that $b_0 = m_b$.

(i) If $m_a < \frac{1}{2}B''(1) \le \infty$, then $a_{i0} = 1$ $(i \ge 1)$.

(ii) If $\frac{1}{2}B''(1) < m_a \le \infty$, then $a_{i0} < 1$ $(i \ge 1)$.

**Proof.** We first prove (i). Suppose that $m_a < \frac{1}{2}B''(1) \le +\infty$. If we further have $\int_0^1 \frac{A(x)}{B(x)}dx > -\infty$, which can happen only in the case $B''(1) = +\infty$ due to the reason that $m_a < +\infty$, then $\int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \ge e^{\int_0^1 \frac{A(x)}{B(x)}dx} \cdot \int_0^1 \frac{1}{B(y)}dy = +\infty$ and hence $a_{i0} = 1$ $(i \ge 1)$. Now assume that $\int_0^1 \frac{A(x)}{B(x)}dx = -\infty$. For this latter case, it follows from the expression of $B(x)$ and $b_0 = m_b$ that

$$\begin{aligned} B(x) &= (1-x)(b_0 - \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1})x^k) \\ &= (1-x)\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1})(1-x^k) \\ &= (1-x)^2 \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1}) \sum_{l=1}^{k} x^{l-1}. \end{aligned}$$

Using the expression of $A(x)$ (see, again, the proof of Corollary 2.3.5) and the above equality yield that

$$\frac{A(x)(1-x)}{B(x)} = -\frac{\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j)x^{k-1}}{\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1}) \sum_{l=1}^{k} x^{l-1}}.$$

Therefore,

$$\lim_{x \uparrow 1} \frac{A(x)(1-x)}{B(x)} = -\frac{2m_a}{B''(1)}. \tag{2.3.9}$$

Hence, by the assumed condition, there exists an $\varepsilon \in (0,1)$ such that

$$\frac{A(x)}{B(x)} \ge -\frac{1}{1-x}, \quad x \in (\varepsilon, 1) \tag{2.3.10}$$

and therefore for all $y \in (\varepsilon, 1)$ we have

$$e^{\int_0^y \frac{A(x)}{B(x)}dx} = e^{\int_0^\varepsilon \frac{A(x)}{B(x)}dx} \cdot e^{\int_\varepsilon^y \frac{A(x)}{B(x)}dx}$$

$$
\begin{aligned}
&\geq \ (1-y) \cdot e^{\int_0^{\varepsilon} \frac{A(x)}{B(x)} dx} \\
&> \ -m_a^{-1} A(y) \cdot e^{\int_0^{\varepsilon} \frac{A(x)}{B(x)} dx},
\end{aligned}
$$

where the last inequality in the above expressions follows from the fact that $-A(y) \leq m_a(1-y)$ for $y \in [0,1]$. It follows immediately from the above inequalities that

$$
\int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \geq -\frac{1}{m_a} e^{\int_0^{\varepsilon} \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{A(y)}{B(y)} dy = +\infty
$$

and hence $a_{i0} = 1 \ (i \geq 1)$.

We now prove (ii). Suppose that $\frac{1}{2}B''(1) < m_a \leq +\infty$. Then by (2.3.9), there exists an $\bar{\varepsilon} \in (0,1)$ and an $\eta \in (1, \frac{2m_a}{B''(1)})$ such that

$$
\frac{A(x)(1-x)}{B(x)} \leq -\eta < -1, \quad x \in (\bar{\varepsilon}, 1).
$$

Therefore

$$
e^{\int_0^y \frac{A(x)}{B(x)} dx} \leq (1-y)^{\eta} \cdot e^{\int_0^{\bar{\varepsilon}} \frac{A(x)}{B(x)} dx}, \quad y \in (\bar{\varepsilon}, 1).
$$

which implies that $J < +\infty$ and hence $a_{i0} < 1 \ (i \geq 1)$. □

**Remark 2.3.2.** By the above we see that even if $m_a = \frac{1}{2}B''(1) < +\infty$, so long as there exists an $\varepsilon \in (0,1)$ such that (2.3.10) holds, then we still have $a_{i0} = 1 \ (i \geq 1)$.

It is interesting to compare the conclusions obtained in Corollaries 2.3.5 and 2.3.6. Indeed, in the case $b_0 = m_b$ (the death rate is equal to the mean birth rate), only a mild immigration requirement, as stated in (ii) of Corollary 2.3.6, may rescue a species from extinction which is significantly different from the case $b_0 > m_b$.

It is also interesting to investigate the so-called balance situation of $b_0 = m_b$ and $m_a = \frac{1}{2}B''(1)$ which is not covered by Corollary 2.3.6. In contrast with Corollary 2.3.6, now both $a_{i0} = 1 \ (i \geq 1)$ and $a_{i0} < 1 \ (i \geq 1)$ may happen for this most subtle case, as the following Corollary 2.3.7 shows.

**Corollary 2.3.7.** Suppose that $b_0 = m_b$ and $m_a = \frac{1}{2}B''(1) < \infty$.

(i) If $B'''(1) < \infty$, then $a_{i0} = 1 \ (i \geq 1)$.

(ii) If $B'''(1) = \infty$ and assume that the following limit

$$\rho := \lim_{y \uparrow 1} \frac{[A(y)(1-y) + B(y)] \cdot \ln(1-y)}{B(y)}$$

exists, then if $\rho < 1$, then $a_{i0} = 1$ $(i \geq 1)$ while if $\rho > 1$, then $a_{i0} < 1$ $(i \geq 1)$.

**Proof.** We first consider the case $B'''(1) < \infty$. Since $b_0 = m_b$, we know from the proof of Corollary 2.3.6 that

$$B(x) = (1-x)^2 \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} b_{j+1}) \sum_{l=1}^{k} x^{l-1}.$$

Using the expression of $A(x)$ (see the proof of Corollary 2.3.5) and the above equality yield that

$$\frac{A(x)(1-x) + B(x)}{B(x)(1-x)} = \frac{\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j) x^{k-1} + \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1}) \sum_{l=1}^{k} x^{l-1}}{(1-x)\sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1}) \sum_{l=1}^{k} x^{l-1}}.$$

Since $A'(1) = m_a = \frac{1}{2}B''(1) < \infty$, it follows by using L'Hospital's rule that

$$\lim_{x \uparrow 1} \frac{A(x)(1-x) + B(x)}{B(x)(1-x)} = \frac{3A''(1) - B'''(1)}{3B''(1)} > -\infty.$$

Choose $c \in (-\infty, \frac{3A''(1)-B'''(1)}{3B''(1)})$. Then there exists an $\varepsilon \in (0,1)$ such that

$$\frac{A(x)}{B(x)} > -\frac{1}{1-x} + c, \quad x \in (\varepsilon, 1)$$

and hence

$$e^{\int_0^y \frac{A(x)}{B(x)} dx} \geq M(1-y)e^{cy}, \quad y \in (\varepsilon, 1)$$

where $M = e^{-c\varepsilon + \int_0^{\varepsilon} \frac{A(x)}{B(x)} dx}$ is a finite positive constant. The above inequality clearly implies that $\int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy = +\infty$ and hence $a_{i0} = 1$ $(i \geq 1)$.

Next we consider the case $B'''(1) = \infty$. Assume that the stated limit $\rho$ does exist and that $\rho < 1$. Noting that

$$\lim_{x \uparrow 1} \frac{[A(x)(1-x) + B(x)] \cdot (1 - \ln(1-x))}{B(x)} = -\rho,$$

we may find an $\varepsilon \in (0,1)$ such that

$$\frac{[A(x)(1-x)+B(x)]\cdot(1-\ln(1-x))}{B(x)} \geq -\rho_1 > -1, \quad x \in (\varepsilon, 1),$$

where $\rho_1 \in (\rho, 1)$. That is

$$\frac{A(x)}{B(x)} \geq -\frac{1}{1-x} - \frac{\rho_1}{(1-x)(1-\ln(1-x))}, \quad x \in (\varepsilon, 1).$$

Hence

$$e^{\int_0^y \frac{A(x)}{B(x)}dx} \geq (1-y)(1-\ln(1-y))^{-\rho_1}, \quad y \in (\varepsilon, 1)$$

and so

$$\begin{aligned}
& \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
\geq\ & \int_0^1 \frac{(1-y)(1-\ln(1-y))^{-\rho_1}}{B(y)} dy \\
\geq\ & C\int_0^1 \frac{dy}{(1-y)(1-\ln(1-y))^{\rho_1}} \\
=\ & C\int_1^\infty u^{-\rho_1} du = \infty.
\end{aligned}$$

Thus, $a_{i0} = 1$ $(i \geq 1)$. If $\rho > 1$, then similarly we may find an $\varepsilon \in (0,1)$ such that

$$\frac{[A(x)(1-x)+B(x)]\cdot(1-\ln(1-x))}{B(x)} \leq -\rho_2 < -1, \quad x \in (\varepsilon, 1),$$

where $\rho_2 \in (1, \rho)$. That is

$$\frac{A(x)}{B(x)} \leq -\frac{1}{1-x} - \frac{\rho_2}{(1-x)(1-\ln(1-x))}, \quad x \in (\varepsilon, 1).$$

Hence

$$e^{\int_0^y \frac{A(x)}{B(x)}dx} \leq (1-y)(1-\ln(1-y))^{-\rho_2}, \quad y \in (\varepsilon, 1)$$

and so

$$\begin{aligned}
& \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \\
\leq\ & \int_0^1 \frac{(1-y)(1-\ln(1-y))^{-\rho_2}}{B(y)} dy \\
\leq\ & C_1\int_0^1 \frac{dy}{(1-y)(1-\ln(1-y))^{\rho_2}} \\
=\ & C_1\int_1^\infty u^{-\rho_2} du < \infty.
\end{aligned}$$

Therefore, $a_{i0} < 1$ $(i \geq 1)$. $\square$

Having obtained the extinction probability, we are now in a position to consider the extinction time. We shall use $E_i[\tau_0]$ to denote the mean extinction time when the process starts at state $i \geq 1$.

**Theorem 2.3.8.** Suppose that $b_0 \geq m_b$ and $J = +\infty$ where $J$ is given in (2.3.4) and thus the extinction probability $a_{i0} = 1$ $(i \geq 1)$. Then for any $i \geq 1$, $E_i[\tau_0] < \infty$ if and only if

$$\int_0^1 \frac{1 - y - A(y)}{B(y)} dy < \infty \tag{2.3.11}$$

and in which case, $E_i[\tau_0]$ is given by

$$E_i[\tau_0] = \int_0^1 \frac{1 - y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)} dx} dy. \tag{2.3.12}$$

**Proof.** Suppose that $b_0 \geq m_b$ and $J = \infty$. By Theorem 2.3.4, $a_{i0} = 1$. Hence it follows from (2.3.8) that for $s \in [0, 1)$,

$$F_i(s) \cdot e^{\int_0^s \frac{A(x)}{B(x)} dx} = \int_0^s \frac{1 - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy,$$

where $F_i(s) = \sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(t) dt) s^j$. Thus,

$$\sum_{j=1}^{\infty} \left( \int_0^{\infty} p_{ij}(t) dt \right) \cdot s^j = \int_0^s \frac{1 - y^i}{B(y)} \cdot e^{-\int_y^s \frac{A(x)}{B(x)} dx} dy.$$

Letting $s \uparrow 1$, using the honesty condition and applying the Monotone Convergence Theorem then

$$\begin{aligned} E_i[\tau_0] &= \int_0^{\infty} (1 - p_{i0}(t)) dt \\ &= \sum_{j=1}^{\infty} \int_0^{\infty} p_{ij}(t) dt \\ &= \int_0^1 \frac{1 - y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)} dx} dy. \end{aligned}$$

Thus (2.3.12) is proved. Finally, we turn to prove that for any $i \geq 1$, $E_i[\tau_0] < \infty$ if and only if (2.3.11) holds. If (2.3.11) holds, i.e.,

$$\int_0^1 \frac{1 - y}{B(y)} dy < \infty, \quad \int_0^1 \frac{-A(y)}{B(y)} dy < \infty.$$

It follows from (2.3.12) that

$$E_i[\tau_0] = \int_0^1 \frac{1-y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)}dx} dy < e^{-\int_0^1 \frac{A(x)}{B(x)}dx} \cdot \int_0^1 \frac{1-y^i}{B(y)} dy < \infty.$$

Conversely, suppose that (2.3.11) fails, i.e., $\int_0^1 \frac{1-y}{B(y)}dy = \infty$ or $\int_0^1 \frac{-A(y)}{B(y)}dy = \infty$. Note that

$$E_i[\tau_0] = \int_0^1 \frac{1-y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)}dx} dy \geq \int_0^1 \frac{1-y^i}{B(y)} dy,$$

$$\begin{aligned} E_i[\tau_0] &= \int_0^1 \frac{1-y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)}dx} dy \\ &\geq \int_0^{\frac{1}{2}} \frac{1-y^i}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)}dx} dy \\ &\geq e^{-\int_{\frac{1}{2}}^1 \frac{A(x)}{B(x)}dx} \cdot \int_0^{\frac{1}{2}} \frac{1-y^i}{B(y)} dy. \end{aligned}$$

and that $\int_0^1 \frac{-A(y)}{B(y)}dy = \infty$ is equivalent to $\int_{\frac{1}{2}}^1 \frac{-A(y)}{B(y)}dy = \infty$. We get $E_i[\tau_0] = \infty$. The proof is complete. □

When the extinction probability is less than 1, the mean extinction time is trivially infinite and thus uninformative. For this case we are naturally more interested in finding the more informative conditional mean extinction time. The following two theorems discuss the two different cases separately.

**Theorem 2.3.9.** Suppose that $b_0 \geq m_b$ and $J < +\infty$, and thus $a_{i0} < 1$ $(i \geq 1)$. Then for any $i \geq 1$, $E_i[\tau_0|\tau_0 < \infty] < \infty$ if and only if

$$\int_0^1 \frac{e^{\int_0^y \frac{A(x)}{B(x)}dx}}{B(y)} \cdot \left(\int_0^y \frac{ds}{B(s)}\right) dy < \infty, \tag{2.3.13}$$

and in which case, $E_i[\tau_0|\tau_0 < \infty]$ is given by

$$E_i[\tau_0|\tau_0 < \infty] = \frac{\int_0^1 \frac{1}{B(y)} \cdot \left(\int_0^y \frac{a_{i0}-u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du\right) dy}{\int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy} \tag{2.3.14}$$

where $E_i[\tau_0|\tau_0 < \infty]$ is the conditional mathematical expectation under the condition $\{\tau_0 < \infty\}$.

**Proof.** Suppose that $b_0 \geq m_b$ and $J < +\infty$ where $J$ is given in (2.3.4). Noting that by (2.3.5) we have $a_{i0} = J^{-1} \cdot \int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Thus, by applying the Fubini Theorem, we obtain

$$
\begin{aligned}
&\int_0^{\infty} (a_{i0} - p_{i0}(t))dt \\
= \;& J^{-1} \cdot \int_0^{\infty} \left( \int_0^1 \frac{y^i}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy - J \cdot p_{i0}(t) \right) dt \\
= \;& J^{-1} \cdot \int_0^{\infty} \left( \int_0^1 \frac{y^i}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy - \int_0^1 \frac{p_{i0}(t)}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right) dt \\
= \;& J^{-1} \cdot \int_0^{\infty} \left( \int_0^1 \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right) dt. \qquad (2.3.15)
\end{aligned}
$$

On the other hand, it follows from (2.2.1) that

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i = A(s) \cdot G_i(t,s) + B(s) \cdot \frac{\partial G_i(t,s)}{\partial s}, \quad t \geq 0, \; s \in [0,1),$$

where $G_i(t,s) = \sum_{j=1}^{\infty} (\int_0^t p_{ij}(u)du) \cdot s^j$. Therefore for any $t \geq 0, \; s \in [0,1)$,

$$G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = \int_0^s \frac{\sum_{j=0}^{\infty} p_{ij}(t)y^j - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy. \qquad (2.3.16)$$

By (2.3.8), we know that for $s \in [0,1)$,

$$\sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(u)du) \cdot s^j \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = \int_0^s \frac{a_{i0} - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy.$$

Combining above equality and (2.3.16) yields that for $t \geq 0$ and $s \in [0,1)$,

$$
\begin{aligned}
0 \leq G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} &\leq \sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(u)du) \cdot s^j \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} \\
&= \int_0^s \frac{a_{i0} - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy.
\end{aligned}
$$

Letting $s \uparrow 1$ in the above inequality yields that

$$\lim_{s \uparrow 1} G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = 0$$

since $J < \infty$ and $a_{i0} = J^{-1} \cdot \int_0^1 \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Noting (2.3.16) yields

$$\int_0^1 \frac{\sum_{j=0}^{\infty} p_{ij}(t)y^j - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = 0, \quad t \geq 0,$$

i.e.,

$$\int_0^1 \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = \int_0^1 \frac{\sum_{j=1}^\infty p_{ij}(t) y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy, \quad t \geq 0.$$

Integrating above equality with respect to $t$ on $[0, \infty)$ yields that

$$\begin{aligned} &\int_0^\infty \left( \int_0^1 \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right) dt \\ = &\int_0^1 \frac{\sum_{j=1}^\infty (\int_0^\infty p_{ij}(t)dt) \cdot y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy. \end{aligned} \tag{2.3.17}$$

On the second hand, letting $t \uparrow \infty$ in (2.3.16) yields that for $y \in [0, 1)$,

$$\begin{aligned} G_i(\infty, y) &= \sum_{j=1}^\infty (\int_0^\infty p_{ij}(t)dt) \cdot y^j \\ &= e^{-\int_0^y \frac{A(x)}{B(x)}dx} \cdot \int_0^y \frac{a_{i0} - u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \end{aligned} \tag{2.3.18}$$

since $\lim_{t\to\infty} p_{i0}(t) = a_{i0}$ and $\lim_{t\to\infty} p_{ij}(t) = 0$ $(j \geq 1)$. Now, substituting (2.3.17) and (2.3.18) into (2.3.15) yields

$$\int_0^\infty (a_{i0} - p_{i0}(t))dt = J^{-1} \cdot \int_0^1 \frac{1}{B(y)} \cdot \left( \int_0^y \frac{a_{i0} - u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \right) dy$$

which, in turn, yields (2.3.14) by noting that $E_i[\tau_0|\tau_0 < \infty] = a_{i0}^{-1} \cdot \int_0^\infty (a_{i0} - p_{i0}(t))dt$. Finally, noting that

$$\lim_{y\uparrow 1} \frac{\int_0^y \frac{a_{i0}-u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du}{\int_y^1 \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du} = \lim_{y\uparrow 1} (y^i - a_{i0}) = 1 - a_{i0} > 0,$$

we know that (2.3.14) is finite if and only if

$$\int_0^1 \frac{1}{B(y)} \cdot \left( \int_y^1 \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \right) dy < \infty,$$

but

$$\begin{aligned} &\int_0^1 \frac{1}{B(y)} \cdot \left( \int_y^1 \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \right) dy \\ = &\int_0^1 \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} \cdot \left( \int_0^u \frac{dy}{B(y)} \right) du \end{aligned}$$

by the Fubini Theorem. Thus the proof is complete. $\square$

Next we consider the case $b_0 < m_b \le +\infty$.

**Theorem 2.3.10.** Suppose that $b_0 < m_b \le +\infty$. Then for any $i \ge 1$, $E_i[\tau_0|\tau_0 < \infty] < \infty$ if and only if

$$\int_0^q \frac{e^{\int_0^y \frac{A(x)}{B(x)}dx}}{B(y)} \cdot \left(\int_0^y \frac{ds}{B(s)}\right)dy < \infty \tag{2.3.19}$$

and in which case, $E_i[\tau_0|\tau_0 < \infty]$ is given by

$$E_i[\tau_0|\tau_0 < \infty] = \frac{\int_0^q \frac{1}{B(y)} \cdot \left(\int_0^y \frac{a_{i0}-u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du\right) dy}{\int_0^q \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy}. \tag{2.3.20}$$

**Proof.** Suppose that $b_0 < m_b \le \infty$. By (2.3.6) we have $a_{i0} = D^{-1} \cdot \int_0^q \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$, where $D = \int_0^q \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Thus, by applying the Fubini Theorem, we obtain

$$\begin{aligned}
&\int_0^\infty (a_{i0} - p_{i0}(t))dt \\
= \;& D^{-1} \cdot \int_0^\infty \left(\int_0^q \frac{y^i}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy - D \cdot p_{i0}(t)\right) dt \\
= \;& D^{-1} \cdot \int_0^\infty \left(\int_0^q \frac{y^i}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy - \int_0^q \frac{p_{i0}(t)}{B(y)} e^{\int_0^y \frac{A(x)}{B(x)}dx} dy\right) dt \\
= \;& D^{-1} \cdot \int_0^\infty \left(\int_0^q \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy\right) dt.
\end{aligned} \tag{2.3.21}$$

On the other hand, it follows from (2.2.1) that

$$\sum_{j=0}^\infty p_{ij}(t)s^j - s^i = A(s) \cdot G_i(t,s) + B(s) \cdot \frac{\partial G_i(t,s)}{\partial s}, \quad t \ge 0, \; s \in [0,q),$$

where $G_i(t,s) = \sum_{j=1}^\infty (\int_0^t p_{ij}(u)du) \cdot s^j$. Therefore for any $t \ge 0$, $s \in [0,q)$,

$$G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = \int_0^s \frac{\sum_{j=0}^\infty p_{ij}(t)y^j - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy. \tag{2.3.22}$$

By (2.3.8), we know that for $s \in [0,q)$,

$$\sum_{j=1}^\infty \left(\int_0^\infty p_{ij}(u)du\right) \cdot s^j \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = \int_0^s \frac{a_{i0} - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy.$$

Combining above equality and (2.3.22) yields that for $t \ge 0$ and $s \in [0,q)$,

$$\begin{aligned}
0 \le G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} &\le \sum_{j=1}^\infty \left(\int_0^\infty p_{ij}(u)du\right) \cdot s^j \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} \\
&= \int_0^s \frac{a_{i0} - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy.
\end{aligned}$$

Letting $s \uparrow q$ in the above inequality yields that

$$\lim_{s\uparrow q} G_i(t,s) \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} = 0$$

since $D < \infty$ and $a_{i0} = D^{-1} \cdot \int_0^q \frac{y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy$. Noting (2.3.22) yields

$$\int_0^q \frac{\sum_{j=0}^{\infty} p_{ij}(t) y^j - y^i}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = 0, \quad t \geq 0,$$

i.e.,

$$\int_0^q \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = \int_0^q \frac{\sum_{j=1}^{\infty} p_{ij}(t) y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy, \quad t \geq 0.$$

Integrating above equality with respect to $t$ on $[0, \infty)$ yields that

$$\begin{aligned} & \int_0^{\infty} \left( \int_0^q \frac{y^i - p_{i0}(t)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right) dt \\ = & \int_0^q \frac{\sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(t)dt) \cdot y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy. \end{aligned} \tag{2.3.23}$$

On the other hand, letting $t \uparrow \infty$ in (2.3.16) yields that for $y \in [0, q)$,

$$\begin{aligned} G_i(\infty, y) &= \sum_{j=1}^{\infty} (\int_0^{\infty} p_{ij}(t)dt) \cdot y^j \\ &= e^{-\int_0^y \frac{A(x)}{B(x)}dx} \cdot \int_0^y \frac{a_{i0} - u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \end{aligned} \tag{2.3.24}$$

since $\lim_{t\to\infty} p_{i0}(t) = a_{i0}$ and $\lim_{t\to\infty} p_{ij}(t) = 0$ $(j \geq 1)$. Now, substituting (2.3.23) and (2.3.24) into (2.3.21) yields

$$\int_0^{\infty} (a_{i0} - p_{i0}(t))dt = D^{-1} \cdot \int_0^q \frac{1}{B(y)} \cdot \left( \int_0^y \frac{a_{i0} - u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \right) dy$$

which, in turn, yields (2.3.20) by noting that $E_i[\tau_0 | \tau_0 < \infty] = a_{i0}^{-1} \cdot \int_0^{\infty} (a_{i0} - p_{i0}(t))dt$. Finally, noting that

$$\lim_{y\uparrow q} \frac{\int_0^y \frac{a_{i0} - u^i}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du}{\int_y^q \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du} = \lim_{y\uparrow q} (y^i - a_{i0}) = q^i - a_{i0} > 0,$$

we know that (2.3.20) is finite if and only if

$$\int_0^q \frac{1}{B(y)} \cdot \left( \int_y^q \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)}dx} du \right) dy < \infty,$$

but

$$\int_0^q \frac{1}{B(y)} \cdot \left( \int_y^q \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)} dx} du \right) dy$$
$$= \int_0^q \frac{1}{B(u)} \cdot e^{\int_0^u \frac{A(x)}{B(x)} dx} \cdot \left( \int_0^u \frac{dy}{B(y)} \right) du$$

by the Fubini Theorem. Thus the proof is complete. □

## 2.4. Ergodicity Property

Different from the previous section, in this section we shall always assume that $h > 0$ and thus 0 is no longer an absorbing state. Considering we have also assumed that $b_0 > 0$ and $a > 0$ throughout this chapter, it is easy to see that our BIR $q$-matrix $Q$ is irreducible and thus so is the corresponding MBPIR. For this latter process the most important problem is the recurrence and ergodicity on which we shall concentrate from now on. We shall also, again, assume that the BIR $q$-matrix $Q$ is regular and thus the MBPIR is honest.

The following Lemma 2.4.1 is just the Lemma 4.46 in Chen (1992), we state it here for convenience.

**Lemma 2.4.1.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be a transition function with regular $q$-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ and $\{X(t); t \geq 0\}$ be the corresponding Markov chain. Suppose that $\sigma_n$ is the $n$'th transition time of $X(t)$, $H$ is a non-empty finite subset of $\mathbf{Z}_+$ and $\tau_H = \inf\{t \geq \sigma_1; X(t) \in H\}$. Then

$$f_{iH}^{(1)} = \pi_{iH}, \quad f_{iH}^{(n+1)} = \sum_{k \in \mathbf{Z}_+ \setminus H} \pi_{ik} f_{kH}^{(n)}, \quad n \geq 1,$$

and $(f_{iH}; i \in \mathbf{Z}_+)$ is the minimal solution to the equation

$$x_i = \sum_{k \in \mathbf{Z}_+ \setminus H} \pi_{ik} x_k + \pi_{iH}, \quad i \in \mathbf{Z}_+,$$

where $(\pi_{ij};\ i, j \in \mathbf{Z}_+)$ is the transition probabilities of the embedding chain of $X(t)$, $\pi_{iH} = \sum_{k \in H} \pi_{ik}$ and

$$f_{iH}^{(n)} = P(\tau_H = \sigma_n > 0 | X(0) = i), \quad f_{iH} = \sum_{n=1}^{\infty} f_{iH}^{(n)}.$$

Furthermore, the Markov chain is recurrent if and only if $f_{iH} = 1$ for all $i \in \mathbf{Z}_+$.

**Theorem 2.4.2.** The MBPIR is recurrent if and only if $b_0 \geq m_b$ and $J = +\infty$, where $J$ is given in (2.3.4).

**Proof.** We first prove the "if" part. Suppose that $b_0 \geq m_b$ and $J = +\infty$. By Lemma 2.4.1 (i.e., Lemma 4.46 of Chen (1992)), it is sufficient to prove that the minimal solution of the equation

$$x_i = \sum_{j=1}^{\infty} \tilde{\pi}_{ij} x_j + \tilde{\pi}_{i0}, \quad i \geq 0 \tag{2.4.1}$$

equals 1 identically, where $(\tilde{\pi}_{ij}; i, j \in \mathbf{Z}_+)$ denote the transition probability of the embedding chain of the MBPIR. Denote

$$\pi_{ij} = \begin{cases} \delta_{0j}, & if\ i = 0,\ j \geq 0 \\ \tilde{\pi}_{ij}, & if\ i > 0,\ j \geq 0. \end{cases}$$

If $(x_i^*; i \geq 0)$ is the minimal solution of (2.4.1), then it can be easily checked that $(x_i^*; i \geq 1)$ is a solution of the equation

$$x_i = \sum_{j=1}^{\infty} \pi_{ij} x_j + \pi_{i0}, \quad 0 \leq x_i \leq 1,\ i \geq 1.$$

By Lemma 2.3.3 and Theorem 2.3.4 we immediately see that $x_i^* = 1,\ (i \geq 1)$ and hence $x_i^* = 1,\ (i \geq 0)$.

We now prove the "only if" part. Assume that either $b_0 \geq m_b$ together with $J < +\infty$ or $b_0 < m_b \leq +\infty$. We shall prove that the process is transient. By Theorem 4.34 and Theorem 4.25 in Chen (1992), a irreducible Markov chain is transient if and only if the equation

$$\sum_{j=0}^{\infty} \pi_{ij} x_j = x_i, \quad i \geq 1$$

has a non-constant bounded solution. So, in the present case, it is sufficient to show that the above equation has a non-constant bounded solution. By the Comparison Lemma (stated in, say, Lemma 3.14 of Chen (1992)), the above equation has a non-constant bounded solution if and only if the inequality

$$\sum_{j=0}^{\infty} \pi_{ij} x_j \geq x_i, \quad i \geq 1 \tag{2.4.2}$$

has a non-constant bounded solution. Thus we only need to show that the inequality (2.4.1) has a non-constant bounded solution. Now if $0 < b_0 < m_b \le +\infty$, then $B(s) = 0$ has a root $q \in (0,1)$. Let $x_0 = 1-q,\ x_i = 1 - q^i \quad (i \ge 1)$, then $(x_i; i \ge 0)$ is a non-constant bounded solution of (2.4.2). Indeed, for $i > 1$

$$
\begin{aligned}
&\sum_{j=0}^{\infty} \pi_{ij} x_j \\
&= \pi_{ii-1} x_{i-1} + \sum_{k=1}^{\infty} \pi_{ii+k} x_{i+k} \\
&= \frac{1}{-ib_1 - a_0} \cdot \left[ ib_0(1-q^{i-1}) + \sum_{k=1}^{\infty} (ib_{k+1} + a_k) \cdot (1 - q^{i+k}) \right] \\
&= 1 - q^i + \frac{(iB(q) + qA(q))q^{i-1}}{ib_1 + a_0} \\
&\ge 1 - q^i = x_i.
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{\infty} \pi_{1j} x_j &= \pi_{10} x_0 + \sum_{k=1}^{\infty} \pi_{11+k} x_{1+k} \\
&= \frac{1}{-b_1 - a_0} \cdot \left[ b_0(1-q) + \sum_{k=1}^{\infty} (b_{k+1} + a_k) \cdot (1 - q^{1+k}) \right] \\
&= 1 - q + \frac{B(q) + qA(q) + b_0(q-1)}{b_1 + a_0} \\
&\ge 1 - q = x_1.
\end{aligned}
$$

If $b_0 \ge m_b$ and $J < +\infty$. Let $x_j^* = J^{-1} \cdot \int_0^1 \frac{y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy,\ j \ge 0$, then for $i \ge 1$,

$$
\begin{aligned}
\sum_{j=0}^{\infty} \pi_{ij} x_j^* &= \frac{1}{Jq_i} \int_0^1 \frac{\sum_{j=0}^{\infty} q_{ij} y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \\
&= \frac{1}{Jq_i} \int_0^1 \frac{iy^{i-1}B(y) + y^i A(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \\
&= \frac{1}{Jq_i} \cdot \int_0^1 iy^{i-1} e^{\int_0^y \frac{A(x)}{B(x)} dx} dy + \frac{1}{Jq_i} \int_0^1 \frac{y^i A(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \\
&= 0.
\end{aligned}
$$

Here the last equality follows from applying the method of integration by parts. Hence $(x_j^*; j \ge 1)$ is a non-constant bounded solution of (2.4.2). The proof is thus complete. $\square$

**Corollary 2.4.3.** The MBPIR is recurrent if any one of the following conditions holds.

(i) $b_0 \geq m_b$ and $\int_0^1 \frac{-A(y)}{B(y)} dy < +\infty$.

(ii) $b_0 > m_b$ and $\sum_{k=1}^{\infty} a_k \ln k < \infty$.

(iii) $b_0 = m_b$ and $m_a < \frac{1}{2}B''(1) \leq \infty$.

(iv) $b_0 = m_b$, $m_a = \frac{1}{2}B''(1) < \infty$ and $B'''(1) < \infty$.

(v) $b_0 = m_b$, $m_a = \frac{1}{2}B''(1) < \infty$, $B'''(1) = \infty$ and

$$\lim_{y \uparrow 1} \frac{[A(y)(1-y) + B(y)] \cdot \ln(1-y)}{B(y)} < 1.$$

**Proof.** If (i) holds, then

$$J = \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \geq e^{\int_0^1 \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{dy}{B(y)} = \infty$$

since $A(x) \leq 0$ and $B(x) > 0$ for $x \in (0, 1)$. Hence by Theorem 2.4.2, the MBPIR is recurrent.

If (ii) holds, note that for $x \in [0, 1]$,

$$\begin{aligned} B(x) &= \sum_{j=0}^{\infty} b_j (x^j - 1) \\ &= (1-x)(b_0 - \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} b_{j+1}) x^k) \\ &\geq (b_0 - m_b)(1-x) \end{aligned}$$

and also

$$\begin{aligned} A(x) &= \sum_{j=0}^{\infty} a_j (x^j - 1) \\ &= (1-x)(a_0 - \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} a_{j+1}) x^k) \\ &= (1-x) \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} a_j) x^{k-1}. \end{aligned}$$

Hence

$$\int_0^1 \frac{A(x)}{B(x)} dx \geq -\frac{1}{b_0 - m_b} \cdot \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} a_j) \cdot k^{-1} = -\frac{1}{b_0 - m_b} \cdot \sum_{j=1}^{\infty} a_j \sum_{k=1}^{j} \frac{1}{k}$$

which is finite since $\sum_{k=1}^{\infty} a_k \ln k < \infty$. Thus $J = \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = +\infty$ and hence the MBPIR is recurrent.

If (iii) holds, then exactly repeating the proof of Corollary 2.3.6(i) yields $J = \infty$ and hence the MBPIR is recurrent.

If (iv) holds, then exactly repeating the proof of Corollary 2.3.7(i) yields $J = \infty$ and hence the MBPIR is recurrent.

If (v) holds, then exactly repeating the proof of Corollary 2.3.7(ii) in the case $\rho < 1$ still yields $J = \infty$ and hence the MBPIR is recurrent. Thus the proof is complete. □

**Remark 2.4.1.** Note that the recurrence criterion obtained in Theorem 2.4.2 is the same as the conditions obtained in Theorem 2.3.4 regarding the extinction probability of the corresponding absorbing MBPIR. In particular, this recurrence criterion does not depend on the resurrection sequence $\{h_i\}$. In fact this is a special case of a more general result. Indeed, let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be an irreducible conservative $q$-matrix with $q_0 > 0$. Define $\widetilde{Q} = (\tilde{q}_{ij})$ as follows.

$$\tilde{q}_{ij} = \begin{cases} 0, & if\ i = 0,\ j \geq 0, \\ q_{ij}, & if\ i > 0,\ j \geq 0. \end{cases}$$

Then the Feller minimal $Q$-process is recurrent if and only if the extinction probability of the corresponding absorbing $\widetilde{Q}$-process is 1.

Note that (i) in Corollary 2.4.3 is a sufficient but not necessary condition for recurrence. Indeed, let $B(s) = (1-s)^2$, $H(s) = A(s) = s - 1$. Then $b_0 = m_b = 1$ together with

$$\int_0^y \frac{A(x)}{B(x)} dx = \ln(1-y), \quad \text{and} \quad \int_0^1 \frac{1}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy = \int_0^1 \frac{dy}{1-y} = +\infty.$$

By Theorem 2.4.2, the process is recurrent but $\int_0^1 \frac{-A(y)}{B(y)} dy = +\infty$.

Now we consider the positive recurrence of the MBPIR. We need the following preparation.

**Lemma 2.4.4.** Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a regular irreducible $q$-matrix and $H \neq \emptyset$ be a finite subset of $\mathbf{Z}_+$, where $\emptyset$ is the empty set.

(i) The process is positive recurrent (i.e., ergodic) if and only if the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -1, & i \in \mathbf{Z}_+ \setminus H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty. \end{cases}$$

has a finite non-negative solution.

(ii) The process is exponentially ergodic if and only if for some $\lambda > 0$ with $\lambda < \inf\{q_i; i \geq 0\}$, the equation

$$\begin{cases} \sum_j q_{ij} y_j \leq -\lambda y_i - 1, & i \in \mathbf{Z}_+ \setminus H \\ \sum_{i \in H} \sum_{j \neq i} q_{ij} y_j < \infty. \end{cases}$$

has a finite non-negative solution.

(iii) The process is strongly ergodic if and only if the solution to the equation in (i) is uniformly bounded.

Lemma 2.4.4 is just the Theorem 4.45 in Chen (1992). The following Lemma 2.4.5 is just the Corollary 4.49 in Chen (1992).

**Lemma 2.4.5.** Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a regular conservative irreducible $q$-matrix. Suppose that there exist constants $C_1 > 0,\ C_2 > 0$ and a nonnegative function $(f_i; i \geq 0)$ with $\lim_{i \to \infty} f_i = +\infty$ such that

$$\sum_{j=0}^{\infty} q_{ij}(f_j - f_i) \leq C_1 - C_2 f_i, \quad i \geq 0.$$

Then the $Q$-process is exponentially ergodic.

**Proof.** By the property of $(f_i; i \geq 0)$, we can choose a finite subset of $\mathbf{Z}_+$ so that $C_1 - \frac{1}{2}C_2 f_i \leq -1$ for all $i \in \mathbf{Z}_+ \setminus H$. Now the assertion follows from Lemma 2.4.4. □

**Theorem 2.4.6.** The MBPIR is positive recurrent (i.e., ergodic) if and only if $b_0 \geq m_b$ and

$$\int_0^1 \frac{-(A(y) + H(y))}{B(y)} dy < +\infty. \tag{2.4.3}$$

Moreover, If $b_0 > m_b$ and $m_a + m_h < \infty$, then the process is exponentially ergodic.

**Proof.** Suppose that $b_0 \geq m_b$ and (2.4.3) holds. In order to prove the positive recurrence, we only need to show that the equation

$$\begin{cases} \sum_{j=0}^{\infty} q_{ij} y_j \leq -1, & i \geq 1 \\ \sum_{j=1}^{\infty} q_{0j} y_j < \infty \end{cases}$$

has a finite nonnegative solution (see Lemma 2.4.4, i.e., Theorem 4.45 in Chen (1992)). Noting that for $y \in [0, 1]$,

$$H(y) = -(1-y) \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} h_j) y^{k-1} \leq -h(1-y)$$

where $h = -h_0 = \sum_{j=1}^{\infty} h_j$. Since $h > 0$, we then have $1 - y \leq -h^{-1} H(y)$ which, together with (2.4.3), implies that $\int_0^1 \frac{1-y^j}{B(y)} dy < \infty$ for any $j \geq 0$. Thus, by (2.4.3), we have

$$\int_0^1 \frac{1-y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \leq \int_0^1 \frac{1-y^j}{B(y)} dy < \infty, \quad j \geq 0.$$

Now let $y_j = e^{-\int_0^1 \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{1-y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy$, $j \geq 0$, then $0 \leq y_j < \infty$ $(j \geq 0)$ and for any $i \geq 1$,

$$\begin{aligned} \sum_{j=0}^{\infty} q_{ij} y_j &= e^{-\int_0^1 \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{-\sum_{j=0}^{\infty} q_{ij} y^j}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \\ &= -e^{-\int_0^1 \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{i y^{i-1} B(y) + y^i A(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \\ &= -1. \end{aligned}$$

As to $i = 0$, it is easy to see that $\sum_{j=1}^{\infty} q_{0j} y_j \leq e^{-\int_0^1 \frac{A(x)}{B(x)} dx} \cdot \int_0^1 \frac{-H(y)}{B(y)} dy < \infty$. Therefore the MBPIR is positive recurrent.

Conversely, suppose that the process is positive recurrent and thus possesses an equilibrium distribution $(\pi_j; j \geq 0)$, say. That is that

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j > 0 \quad \textit{and} \quad \sum_{j=0}^{\infty} \pi_j = 1.$$

Letting $t \to \infty$ in (2.2.1) yields that for $s \in [0, 1)$,

$$H(s)\pi_0 + A(s) \cdot \lim_{t \to \infty} \sum_{j=1}^{\infty} p_{ij}(t) s^j + B(s) \cdot \lim_{t \to \infty} \sum_{j=1}^{\infty} p_{ij}(t) \cdot j s^{j-1} = 0.$$

Since both $\sum_{j=1}^{\infty} s^j$ and $\sum_{j=1}^{\infty} j s^{j-1}$ are convergent for $s \in [0, 1)$, therefore using the Dominated Convergence Theorem yields

$$H(s)\pi_0 + A(s) \cdot \sum_{j=1}^{\infty} \pi_j s^j + B(s) \cdot \sum_{j=1}^{\infty} \pi_j \cdot j s^{j-1} = 0, \quad s \in [0, 1). \quad (2.4.4)$$

Note that the above expression implies that $b_0 \geq m_b$. Indeed, since $H(s) \leq 0$ and $A(s) \leq 0$ for all $s \in [0,1)$ (see Lemma 2.2.1), we obtain that $B(s) \cdot \sum_{j=1}^{\infty} \pi_j \cdot j s^{j-1} = -(H(s)\pi_0 + A(s) \cdot \sum_{j=1}^{\infty} \pi_j s^j) \geq 0$ $(s \in [0,1))$ and hence $B(s) \geq 0$ $(s \in [0,1))$, which, by (ii) of Lemma 2.2.1, implies that $b_0 \geq m_b$. Denote $\pi(s) = \sum_{j=0}^{\infty} \pi_j s^j$, then (2.4.4) can be written as

$$B(s) \cdot \pi'(s) + A(s) \cdot \pi(s) + \pi_0 \cdot (H(s) - A(s)) = 0, \quad s \in [0,1)$$

and hence

$$\pi(s) = \pi_0 \left[ 1 + e^{-\int_0^s \frac{A(x)}{B(x)}dx} \cdot \int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \right], \quad s \in [0,1). \quad (2.4.5)$$

Letting $s \uparrow 1$ in (2.4.5) yields

$$\lim_{s \uparrow 1} \frac{\int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy}{e^{\int_0^s \frac{A(x)}{B(x)}dx}} < \infty.$$

Since $\int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy \geq \int_0^{s_0} \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy > 0$ for some $s_0 \in (0,1)$ as $s \uparrow 1$, we must have $\int_0^1 \frac{-A(x)}{B(x)} dx < \infty$. Hence

$$\lim_{s \uparrow 1} \int_0^s \frac{-H(y)}{B(y)} dy \leq \lim_{s \uparrow 1} \frac{\int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)}dx} dy}{e^{\int_0^s \frac{A(x)}{B(x)}dx}} < \infty.$$

Combining the above two expressions yields (2.4.3), which completes the proof of the first part. Now suppose that $b_0 > m_b$ and $m_a + m_h < \infty$. We prove that the MBPIR is exponentially ergodic. By Lemma 2.4.5 (i.e., Corollary 4.49 in Chen (1992)), it is sufficient to show that there exist two constants $C_1 \geq 0$, $C_2 > 0$ and a finite nonnegative function $(f_i; i \geq 0)$ with $\lim_{i \to \infty} f_i = +\infty$ such that

$$\sum_{j=0}^{\infty} q_{ij}(f_j - f_i) \leq C_1 - C_2 f_i, \quad i \geq 0.$$

Let $C_1 = m_a \vee m_h > 0$, $C_2 = b_0 - m_b > 0$ and $f_i = i$ $(i \geq 0)$. Then for any $i \geq 0$,

$$\begin{aligned} & \sum_{j=0}^{\infty} q_{ij}(f_j - f_i) \\ = \; & q_{ii-1}(f_{i-1} - f_i) \cdot I_{\mathbf{Z}_+}(i) + \sum_{k=1}^{\infty} q_{ii+k}(f_{i+k} - f_i) \\ \leq \; & C_1 - C_2 f_i, \end{aligned}$$

where $I_{\mathbf{Z}_+}(\cdot)$ is the indicator function of $\mathbf{Z}_+$. Thus the proof is complete. $\square$

**Remark 2.4.2.** Yamazato (1975) considered the special case $a = 0$ and obtained that the process is recurrent (or positive recurrent) if and only if $b_0 \geq m_b$ (or $b_0 \geq m_b$ and $\int_0^1 \frac{-H(y)}{B(y)} dy < \infty$ respectively). As mentioned in the beginning of this chapter, although we have assumed that $a > 0$, our results apply perfect well if $a = 0$. In particular, if we let $a = 0$ (and thus $A(s) \equiv 0$) in Theorems 2.4.2 and 2.4.6, we then regain the results obtained in Yamazato (1975).

**Corollary 2.4.7.** The MBPIR is positive recurrent if and only if one of the following conditions holds.

(i) $b_0 > m_b$, $\sum_{k=1}^{\infty} a_k \ln k < \infty$ and $\sum_{k=1}^{\infty} h_k \ln k < \infty$.

(ii) $b_0 = m_b$ and

$$\int_0^1 \frac{-(A(y) + H(y))}{B(y)} dy < +\infty.$$

**Proof.** If (i) holds, note that for $y \in [0, 1]$,

$$B(y) = (1 - y)(b_0 - \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1})y^k) \geq (b_0 - m_b)(1 - y),$$

$$A(y) = -(1 - y) \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j)y^{k-1}$$

and

$$H(y) = -(1 - y) \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} h_j)y^{k-1},$$

we know that

$$\begin{aligned}\int_0^1 \frac{-A(y)}{B(y)} dy &\leq \frac{1}{b_0 - m_b} \cdot \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j) \cdot k^{-1} \\ &= \frac{1}{b_0 - m_b} \cdot \sum_{j=1}^{\infty} a_j \sum_{j=k}^{\infty} k^{-1} < \infty\end{aligned}$$

since $\sum_{k=1}^{\infty} a_k \ln k < \infty$. A exactly same argument yields that

$$\begin{aligned}\int_0^1 \frac{-H(y)}{B(y)} dy &\leq \frac{1}{b_0 - m_b} \cdot \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} h_j) \cdot k^{-1} \\ &= \frac{1}{b_0 - m_b} \cdot \sum_{j=1}^{\infty} h_j \sum_{j=k}^{\infty} k^{-1} < \infty\end{aligned}$$

since $\Sigma_{k=1}^{\infty} h_k \ln k < \infty$. Hence by Theorem 2.4.6, the MBPIR is positive recurrent.

If (ii) holds, then the condition is just same as that in Theorem 2.4.6 for the case $b_0 = m_b$ and hence the MBPIR is positive recurrent.

Conversely, suppose that the MBPIR is positive recurrent. By Theorem 2.4.6, we have $b_0 \geq m_b$ and

$$\int_0^1 \frac{-(A(y)+H(y))}{B(y)} dy < +\infty.$$

If $b_0 = m_b$, then this condition is just (ii). If $b_0 > m_b$, then for $y \in [0,1]$,

$$B(y) = (1-y)(b_0 - \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} b_{j+1})y^k) \leq b_0(1-y)$$

and hence

$$\begin{aligned}\int_0^1 \frac{-A(y)}{B(y)} dy &\geq \frac{1}{b_0} \cdot \sum_{k=1}^{\infty}(\sum_{j=k}^{\infty} a_j) \cdot k^{-1} \\ &= \frac{1}{b_0} \cdot \sum_{j=1}^{\infty} a_j \sum_{j=k}^{\infty} k^{-1}.\end{aligned}$$

Therefore,

$$\sum_{j=1}^{\infty} a_j \sum_{j=k}^{\infty} k^{-1} < \infty$$

since $\int_0^1 \frac{-A(y)}{B(y)} dy < \infty$. Hence $\Sigma_{k=1}^{\infty} a_k \ln k < \infty$. A exactly same argument yields that

$$\sum_{j=1}^{\infty} h_j \sum_{j=k}^{\infty} k^{-1} < \infty$$

since $\int_0^1 \frac{-H(y)}{B(y)} dy < \infty$. Hence $\Sigma_{k=1}^{\infty} h_k \ln k < \infty$. The proof is complete. □

As a direct consequence of Theorem 2.4.6, we have the following Corollary 2.4.8, note that if $a = 0$, i.e., $m_a = 0$, then the following Corollary 2.4.8(i) is just the results obtained in Yamazato (1975) (see Remark 2.4.2).

**Corollary 2.4.8.** Suppose that $m_a < \infty$.

(i) The MBPIR is positive recurrent if and only if $b_0 \geq m_b$ and

$$\int_0^1 \frac{-H(y)}{B(y)} dy < +\infty.$$

(ii) If $b_0 > m_b$ and $m_h < \infty$, then the process is exponentially ergodic.

**Proof.** (ii) is just the last assertion in Theorem 2.4.6. By the first assertion in Theorem 2.4.6, we see that in order to prove (i), we only need to show that $\int_0^1 \frac{-A(y)}{B(y)} dy < \infty$ under the conditions $b_0 \geq m_b$, $m_a < \infty$ and $\int_0^1 \frac{-H(y)}{B(y)} dy < \infty$. Indeed, since $-H(y) = (1-y) \cdot \sum_{k=1}^{\infty} (\sum_{j=k}^{\infty} h_j) y^{k-1} \geq h(1-y)$ we have $\int_0^1 \frac{1-y}{B(y)} dy < \infty$. On the other hand, $-A(y) = (1-y) \cdot \sum_{k=0}^{\infty} (\sum_{j=k+1}^{\infty} a_j) y^k \leq m_a(1-y)$ which implies that $\int_0^1 \frac{-A(y)}{B(y)} dy \leq m_a \int_0^1 \frac{1-y}{B(y)} dy < \infty$. The proof is complete. □

**Theorem 2.4.9.** Suppose that the MBPIR is positive recurrent. Then its equilibrium distribution $(\pi_j; j \in \mathbf{Z}_+)$ is given by

$$\pi(s) = \pi_0 \left[ 1 + e^{-\int_0^s \frac{A(x)}{B(x)} dx} \cdot \int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \right], \quad s \in [0,1). \quad (2.4.6)$$

where $\pi(s) = \sum_{j=0}^{\infty} \pi_j s^j$.

**Proof.** Suppose that the MBPIR is positive recurrent. In the second part of the proof of Theorem 2.4.6, we proved that if the MBPIR is positive recurrent then (2.4.5) holds, i.e.,

$$\pi(s) = \pi_0 \left[ 1 + e^{-\int_0^s \frac{A(x)}{B(x)} dx} \cdot \int_0^s \frac{-H(y)}{B(y)} \cdot e^{\int_0^y \frac{A(x)}{B(x)} dx} dy \right], \quad s \in [0,1).$$

The above equality is just (2.4.6). The proof is complete. □

Finally, we have the following conclusion.

**Theorem 2.4.10.** The MBPIR is never strongly ergodic.

**Proof.** Assume that the MBPIR is strongly ergodic, then by Theorem 2.4.2 or Theorem 2.4.6, $b_0 \geq m_b$. By Proposition 6.3.3 in Anderson (1991), the process is strongly ergodic if and only if $\sup_{i \geq 1} E_i[\sigma_0] < \infty$, where $\sigma_0$ is the first hitting time of state 0. Suppose that $Q = (q_{ij})$ is the $q$-matrix of the MBPIR, define $\widetilde{Q} = (\tilde{q}_{ij})$ as follows:

$$\tilde{q}_{ij} = \begin{cases} 0, & if\ i = 0,\ j \geq 0 \\ q_{ij}, & if\ i \geq 1,\ j \geq 0. \end{cases}$$

By Lemma 2.4 in Chen (2002b), $E_i[\sigma_0] = E_i[\tau_0]$, $(i \geq 1)$, where $\tau_0$ is the extinction time for the $\widetilde{Q}$-process. However, by Theorem 2.3.8, we have

$$\sup_{i\geq 1} E_i[\tau_0] = \int_0^1 \frac{1}{B(y)} \cdot e^{-\int_y^1 \frac{A(x)}{B(x)}dx} dy \geq \int_0^1 \frac{1}{B(y)} = \infty.$$

Therefore, the MBPIR must not be strongly ergodic. $\square$

## 2.5. Examples

In this section, we use two examples to illustrate our conclusions obtained in the previous sections. Our first example is that both state-independent immigration and resurrection are presented with the special property that $a_j \equiv h_j$ $(j \geq 0)$ and thus $A(s) \equiv H(s)$. Then the basic equation (2.2.1) takes the special form of

$$\frac{\partial F_i(t,s)}{\partial t} = A(s) \cdot F_i(t,s) + B(s) \cdot \frac{\partial F_i(t,s)}{\partial s} \tag{2.5.1}$$

where $F_i(t,s) = \Sigma_{j=0}^{\infty} p_{ij}(t)s^j$ $(i \geq 0,\ -1 < s < 1)$.

Equation (2.5.1) is a first order partial differential equation. The method of solving such first order partial differential equation can be found in Anderson (1991) (page 104–109). However, we shall not use such routine techniques here, but rather, just point out some special properties which can be obtained by using our method adopted in this chapter.

We first point out that the transition functions of the process possess the following interesting property.

**Theorem 2.5.1.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function, where $Q$ is a BIR $q$-matrix. Suppose further that $H(s) = A(s)$. Then

$$F_i(t,s) = [F_1(t,s)]^i/[F_0(t,s)]^{i-1}, \quad i \geq 1 \tag{2.5.2}$$

and in particular,

$$p_{i0}(t) = p_{10}^i(t)/p_{00}^{i-1}(t), \quad i \geq 1, \tag{2.5.3}$$

where $F_i(t,s) = \Sigma_{j=0}^{\infty} p_{ij}(t)s^j$ $(i \geq 0,\ -1 < s < 1)$.

**Proof.** To prove (2.5.2), we only need to show that

$$F_i(t,s) \cdot F_0(t,s) = F_{i-1}(t,s) \cdot F_1(t,s), \quad i \geq 1,$$

or equivalently,

$$\sum_{k=0}^{j} p_{0k}(t)p_{ij-k}(t) = \sum_{k=0}^{j} p_{1k}(t)p_{i-1j-k}(t), \quad i \geq 1,\ j \geq 0.$$

Let $i \geq 1$ be fixed and denote

$$f_j(t) = \sum_{k=0}^{j} p_{0k}(t)p_{ij-k}(t), \quad g_j(t) = \sum_{k=0}^{j} p_{1k}(t)p_{i-1j-k}(t), \quad j \geq 0.$$

It is clear that $f_j(0) = g_j(0) = \delta_{ij}$. By the Kolmogorov forward equations, we have

$$\begin{aligned}
& f_j'(t) \\
= \ & \sum_{k=0}^{j} p_{0k}(t)\left(\sum_{r=0}^{j-k+1} p_{ir}(t)q_{rj-k}\right) + \sum_{k=0}^{j}\left(\sum_{r=0}^{j-k+1} p_{0r}(t)q_{rj-k}\right)p_{ik}(t) \\
= \ & \sum_{r=0}^{j+1}\sum_{k=0}^{j-r+1} p_{0k}(t)p_{ir}(t)q_{rj-k} + \sum_{r=0}^{j+1}\sum_{k=0}^{j-r+1} p_{0r}(t)p_{ik}(t)q_{rj-k} \\
= \ & \sum_{n=0}^{j+1}\sum_{m=0}^{j+1} p_{0n}(t)p_{im}(t)[q_{mj-n} + q_{nj-m}],
\end{aligned}$$

where $q_{kl} = 0$ $(k \geq 0,\ l < 0)$. It follows from the identity $q_{nj-m} + q_{mj-n} = q_{n+mj} + a_{j-(n+m)} := \bar{q}_{n+mj}$ (here $a_k = 0$ if $k < 0$) that

$$f_j'(t) = \sum_{k=0}^{j+1} f_k(t)\bar{q}_{kj}.$$

Similarly, by the Kolmogorov forward equations, we have

$$\begin{aligned}
& g_j'(t) \\
= \ & \sum_{k=0}^{j} p_{1k}(t)\left(\sum_{r=0}^{j-k+1} p_{i-1r}(t)q_{rj-k}\right) + \sum_{k=0}^{j}\left(\sum_{r=0}^{j-k+1} p_{1r}(t)q_{rj-k}\right)p_{i-1k}(t) \\
= \ & \sum_{r=0}^{j+1}\sum_{k=0}^{j-r+1} p_{1k}(t)p_{i-1r}(t)q_{rj-k} + \sum_{r=0}^{j+1}\sum_{k=0}^{j-r+1} p_{1r}(t)p_{i-1k}(t)q_{rj-k} \\
= \ & \sum_{n=0}^{j+1}\sum_{m=0}^{j+1} p_{1n}(t)p_{i-1m}(t)[q_{mj-n} + q_{nj-m}],
\end{aligned}$$

i.e.,

$$g_j'(t) = \sum_{k=0}^{j+1} g_k(t)\bar{q}_{kj}.$$

On the other hand, note that

$$\bar{q}_{ij} = q_{ij} + a_{j-i} = \begin{cases} \bar{h}_j, & if\ i = 0, j \geq 0 \\ ib_{j-i+1} + \bar{a}_{j-i}, & if\ \ i \geq 1, j \geq i \\ ib_0, & if\ \ i \geq 1, j = i-1 \\ 0, & otherwise \end{cases}$$

where $\bar{h}_j = h_j + a_j$ and $\bar{a}_j = 2a_j$. i.e., $\bar{Q}$ is of the same form as the $Q$ defined in (2.1.1)–(2.1.2). Thus, a same argument as in the proof of Theorem 2.2.4 yields that the solution of the Kolmogorov forward equations for the $q$-matrix $\bar{Q} = (\bar{q}_{ij})$ is unique, the only difference is that $h_j$ and $a_j$ should be replaced with $\bar{h}_j$ and $\bar{a}_j$ respectively. Thus $f_j(t) = g_j(t)$ $(j \geq 1)$ which implies (2.5.2). Letting $s = 0$ yields (2.5.3). □

Secondly, we point out that for this special case (i.e., $A(s) \equiv H(s)$), the moments and the equilibrium distribution have also some special properties. Let $\{X(t); t \geq 0\}$ be the corresponding process and denote its first and second moments as

$$M_1(i,t) = E_i X(t), \quad M_2(i,t) = E_i X^2(t), \qquad i \geq 0.$$

**Theorem 2.5.2.** Suppose that $A(s) \equiv H(s)$.

(i) For any $i \geq 0$, $M_1(i,t) < \infty$ if and only if $A'(1) + B'(1) < \infty$.

Moreover, under this condition, $M_1(i,t)$ is given by

$$M_1(i,t) = \begin{cases} i + A'(1)t, & if\ B'(1) = 0, \\ ie^{B'(1)t} + \frac{A'(1)}{B'(1)} \cdot (e^{B'(1)t} - 1), & if\ B'(1) \neq 0. \end{cases} \tag{2.5.4}$$

(ii) For any $i \geq 0$, $M_2(i,t) < \infty$ if and only if $A''(1) + B''(1) < \infty$.

Moreover, under the latter condition, if $B'(1) = 0$ then

$$\begin{aligned} M_2(i,t) &= i^2 + [A'(1) + A''(1) + i(2A'(1) + B''(1))]t \\ &\quad + A'(1)(A'(1) + \frac{1}{2}B''(1))t^2 \end{aligned} \tag{2.5.5}$$

while if $B'(1) \neq 0$ then

$$\begin{aligned} M_2(i,t) &= i^2 e^{2B'(1)t} + \frac{A'(1) + A''(1)}{2B'(1)} \cdot (e^{2B'(1)t} - 1) \\ &\quad + (2A'(1) + B''(1) - B'(1)) \cdot (e^{B'(1)t} - 1) \\ &\quad \cdot \left[ \frac{ie^{B'(1)t}}{B'(1)} + \frac{A'(1)}{2B'(1)^2} \cdot (e^{B'(1)t} - 1) \right]. \end{aligned} \tag{2.5.6}$$

(iii) Suppose further that the process is positive recurrent (as stated in Theorem 2.4.6) and that $b_0 > m_b$, $A'(1) = H'(1) < \infty$, then the equilibrium distribution $(\pi_j; \in \mathbf{Z}_+)$ satisfies

$$\sum_{k=1}^{\infty} k\pi_k = \lim_{t\to\infty} \sum_{j=1}^{\infty} p_{ij}(t) \cdot j = \frac{m_a}{b_0 - m_b}.$$

**Proof.** It follows from (2.2.1) that for $|s| < 1$,

$$\begin{aligned}\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i &= H(s) \cdot \int_0^t p_{i0}(u)du + A(s) \cdot \sum_{j=1}^{\infty}\left(\int_0^t p_{ij}(u)du\right)s^j \\ &\quad + B(s) \cdot \sum_{k=1}^{\infty}\left(\int_0^t p_{ik}(u)du\right) \cdot ks^{k-1}. \qquad (2.5.7)\end{aligned}$$

By dividing $s - 1$ on both sides of (2.5.7), we obtain that for $|s| < 1$,

$$\begin{aligned}&\sum_{j=1}^{\infty} p_{ij}(t) \cdot \frac{s^j - 1}{s - 1} + \frac{1 - s^i}{s - 1} \\ = &\frac{H(s)}{s-1} \cdot \int_0^t p_{i0}(u)du + \frac{A(s)}{s-1} \cdot \sum_{j=1}^{\infty}\left(\int_0^t p_{ij}(u)du\right)s^j \\ &+\frac{B(s)}{s-1} \cdot \sum_{k=1}^{\infty}\left(\int_0^t p_{ik}(u)du\right) \cdot ks^{k-1}.\end{aligned}$$

Letting $s \uparrow 1$ in the above equality and using the Monotone Convergence Theorem yield that

$$\begin{aligned}&M_1(i,t) - i \\ = &H'(1) \cdot \int_0^t p_{i0}(u)du + A'(1) \cdot \sum_{j=1}^{\infty} \int_0^t p_{ij}(u)du + B'(1) \cdot \int_0^t M_1(i,u)du\end{aligned}$$

Suppose that $H'(1) = A'(1)$, then

$$M_1(i,t) - i = A'(1)t + B'(1) \cdot \int_0^t M_1(i,u)du \qquad (2.5.8)$$

If $A'(1) + B'(1) < \infty$ then solving (2.5.8) immediately yields (2.5.4) and $M_1(i,t) < \infty$. Conversely, if $A'(1) + B'(1) = \infty$ then (2.5.8) implies that $M_1(i,t) = \infty$.

We now prove (ii). Differentiating (2.5.7) with respect to $s$ yields

$$\begin{aligned}&\sum_{j=1}^{\infty} p_{ij}(t) \cdot js^{j-1} - is^{i-1} \\ = &H'(s) \cdot \int_0^t p_{i0}(u)du + A'(s) \cdot \sum_{j=1}^{\infty}\left(\int_0^t p_{ij}(u)du\right) \cdot s^j\end{aligned}$$

$$+[A(s)+B'(s)]\cdot\sum_{j=1}^{\infty}(\int_0^t p_{ij}(u)du)\cdot js^{j-1}$$
$$+B(s)\cdot\sum_{j=2}^{\infty}(\int_0^t p_{ij}(u)du)\cdot j(j-1)s^{j-2}.$$

Noting (2.5.8) we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} p_{ij}(t)\cdot j\cdot\frac{s^{j-1}-1}{s-1}-i\frac{s^{i-1}-1}{s-1}\\
=\ &\frac{H'(s)-H'(1)}{s-1}\cdot\int_0^t p_{i0}(u)du\\
&+\frac{A'(s)\cdot\sum_{j=1}^{\infty}(\int_0^t p_{ij}(u)du)\cdot s^j-A'(1)\sum_{j=1}^{\infty}\int_0^t p_{ij}(u)du}{s-1}\\
&+\frac{[A(s)+B'(s)]\cdot\sum_{j=1}^{\infty}(\int_0^t p_{ij}(u)du)\cdot js^{j-1}-B'(1)\cdot\sum_{j=1}^{\infty}(\int_0^t p_{ij}(u)du)\cdot j}{s-1}\\
&+\frac{B(s)}{s-1}\cdot\sum_{j=2}^{\infty}(\int_0^t p_{ij}(u)du)\cdot j(j-1)s^{j-2}.
\end{aligned}$$

Suppose that $H''(1)=A''(1)$. If $A''(1)+B''(1)=\infty$ then letting $s\uparrow 1$ yields that $M_2(i,t)=\infty$. Now assume that $A''(1)+B''(1)<\infty$. Letting $s\uparrow 1$ in the above equality and using (2.5.8) yields that

$$\begin{aligned}
&M_2(i,t)-i^2\\
=\ &[A'(1)+A''(1)]t+[2A'(1)+B''(1)-B'(1)]\cdot\int_0^t M_1(i,u)du\\
&+2B'(1)\cdot\int_0^t M_2(i,u)du.
\end{aligned}$$

If $B'(1)=0$ then above equality becomes

$$M_2(i,t)-i^2=[A'(1)+A''(1)]t+[2A'(1)+B''(1)]\cdot\int_0^t M_1(i,u)du.$$

Noting that $\int_0^t M_1(i,u)du=it+\frac{1}{2}A'(1)t^2$. Substituting this into the above equality immediately yields (2.5.5).

If $B'(1)\neq 0$, then noting that

$$\int_0^t M_1(i,u)du=\frac{i}{B'(1)}(e^{B'(1)t}-1)+\frac{A'(1)}{B'(1)}[\frac{1}{B'(1)}(e^{B'(1)t}-1)-t].$$

Substituting this into

$$\begin{aligned}
&M_2(i,t)-i^2\\
=\ &[A'(1)+A''(1)]t+[2A'(1)+B''(1)-B'(1)]\cdot\int_0^t M_1(i,u)du\\
&+2B'(1)\cdot\int_0^t M_2(i,u)du
\end{aligned}$$

and then solve it yields (2.5.6). Finally, suppose further that the process is positive recurrent and that $b_0 > m_b$, $A'(1) = H'(1) < \infty$, then by (i),

$$\sum_{k=1}^{\infty} k\pi_k = \lim_{t\to\infty} \sum_{k=1}^{\infty} jp_{ij}(t) = \lim_{t\to\infty} M_1(i,t) = -\frac{A'(1)}{B'(1)} = \frac{m_a}{b_0 - m_b}.$$

The proof is complete. $\square$

**Remark 2.5.1.** If one checks the proof in Theorem 2.5.2 carefully, one will find that the strong condition $A(s) \equiv H(s)$ is not really necessary to get the conclusions in Theorem 2.5.2. Indeed, by checking the proof of Theorem 2.5.2, one can find that only the condition $A'(1) = H'(1)$ is needed in obtaining (2.5.8), the much more strict condition $A(s) \equiv H(s)$ is not necessary. Therefore, the weaker condition $H'(1) = A'(1)$ is sufficient to obtain (i) while a further assumption $H''(1) = A''(1)$ is enough to get (ii).

Our second example is that the underlying branching takes a simple birth-death structure and the immigration is Poisson. More specifically

Let $b_0 = \mu > 0$, $b_2 = \lambda > 0$, $b_j = 0$ $(j > 2)$, $a_1 = a > 0$, $a_j = 0$ $(j > 1)$ and $h = 0$. Then $B(s) = (1-s)(\mu - \lambda s)$, $A(s) = a(1-s)$ and $H(s) = 0$. Let us agree to call this model as a Birth-Death-Immigration process without resurrection. By our results obtained in Section 2.3, we can get the following conclusion.

**Theorem 2.5.3.** A Birth-Death-Immigration process without resurrection is always honest. Furthermore, we have the following conclusions.

(i) If $\mu > \lambda$ or if $a \le \mu = \lambda$, then $a_{i0} = 1$ $(i \ge 1)$ and

$$E_i[\tau_0] = \begin{cases} \frac{1}{(\mu-\lambda)^{a/\lambda}} \cdot \int_0^1 \frac{1-y^i}{(1-y)(\mu-\lambda y)^{1-a/\lambda}} dy, & if\ \mu > \lambda \\ \infty, & if\ \mu = \lambda. \end{cases} \tag{2.5.9}$$

(ii) If $a > \mu = \lambda$, then

$$a_{i0} = \frac{i!}{(a/\lambda + i - 1)(a/\lambda + i - 2)\cdots a/\lambda} < 1, \quad i \ge 1, \tag{2.5.10}$$

and $E_i[\tau_0 | \tau_0 < \infty] < \infty$ $(i \ge 1)$ if and only if $a > 2\lambda$.

(iii) If $\mu < \lambda$, then

$$a_{i0} = \frac{\sum_{k=i}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda}}{\sum_{k=0}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda}} < 1, \quad i \geq 1, \tag{2.5.11}$$

where $q = \mu/\lambda < 1$, and $E_i[\tau_0|\tau_0 < \infty] < \infty$ for all $i \geq 1$.

**Proof.** Note first that in our current situation we have $m_b = \lambda < \infty$ and $b_0 = \mu$. Thus by Theorem 2.2.3, the process is honest. Now, in either of the two cases in (i), the quantity $J$ defined in (2.3.4) is

$$J = \int_0^1 \frac{1}{B(s)} \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} ds = \lambda^{a/\lambda - 1}\mu^{-a/\lambda} \cdot \int_0^1 \frac{ds}{(1-s)(\frac{\mu}{\lambda} - s)^{1-a/\lambda}} = \infty.$$

Therefore, by Theorem 2.3.4, $a_{i0} = 1$ $(i \geq 1)$ and then by Theorem 2.3.8, (2.5.9) holds.

If $a > \mu = \lambda$, then

$$J = \lambda^{-1} \cdot \int_0^1 \frac{ds}{(1-s)^{2-a/\lambda}} = \frac{1}{a-\lambda} < \infty,$$

and that for any $i \geq 1$,

$$\begin{aligned} J_i : &= \int_0^1 \frac{s^i}{B(s)} \cdot e^{\int_0^s \frac{A(x)}{B(x)}dx} ds \\ &= \lambda^{-1} \int_0^1 \frac{s^i}{(1-s)^{2-a/\lambda}} ds \\ &= \frac{i}{a/\lambda - 1} \cdot \lambda^{-1} \int_0^1 \frac{s^{i-1}}{(1-s)^{1-a/\lambda}} ds, \end{aligned}$$

and thus

$$J_i = \frac{i}{a/\lambda + i - 1} \cdot J_{i-1} = \frac{i!}{(a/\lambda + i - 1)(a/\lambda + i - 2)\cdots a/\lambda} \cdot \frac{1}{a - \lambda}.$$

Therefore, by (2.3.5) in Theorem 2.3.4 we obtain

$$a_{i0} = \frac{J_i}{J} = \frac{i!}{(a/\lambda + i - 1)(a/\lambda + i - 2)\cdots a/\lambda}, \quad i \geq 1,$$

which proves (2.5.10). Also it follows from Theorem 2.3.9 that $E_i[\tau_0|\tau_0 < \infty] < \infty$ $(i \geq 1)$ if and only if $a > 2\lambda$.

Finally, if $\mu < \lambda$, then $q = \mu/\lambda < 1$. Note that for any $k \geq 0$ and $\alpha > 0$, we have

$$\begin{aligned}
\tilde{J}_k &:= \int_0^q \frac{s^k}{(q-s)^{1-\alpha}} ds \\
&= q^{k-\alpha} \int_0^1 \frac{y^k}{(1-y)^{2-(1+\alpha)}} dy \\
&= q^{k-\alpha} \cdot \frac{k!}{(k+\alpha)\cdots(1+\alpha)\alpha}.
\end{aligned}$$

Thus we get that

$$\begin{aligned}
D: &= \int_0^q \frac{1}{B(s)} \cdot e^{\int_0^s \frac{A(x)}{B(x)} dx} ds \\
&= \lambda^{-1} q^{-a/\lambda} \cdot \int_0^q \frac{1}{(1-s)(q-s)^{1-a/\lambda}} ds \\
&= \lambda^{-1} q^{-a/\lambda} \cdot \sum_{k=0}^{\infty} \int_0^q \frac{s^k}{(q-s)^{1-a/\lambda}} ds \\
&= \left( \sum_{k=0}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda} \right) \cdot \lambda^{-1} q^{-2a/\lambda}
\end{aligned}$$

and that for any $i \geq 1$,

$$\begin{aligned}
D_i: &= \int_0^q \frac{s^i}{B(s)} \cdot e^{\int_0^s \frac{A(x)}{B(x)} dx} ds \\
&= \lambda^{-1} q^{-2a/\lambda} \cdot \sum_{k=i}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda}.
\end{aligned}$$

Therefore, by (2.3.6) in Theorem 2.3.4, we know that for any $i \geq 1$,

$$a_{i0} = \frac{D_i}{D} = \frac{\sum_{k=i}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda}}{\sum_{k=0}^{\infty} \frac{q^k k!}{(k+a/\lambda)\cdots(1+a/\lambda)a/\lambda}},$$

which proves (2.5.11). Finally, by Theorem 2.3.10, it is easily seen that $E_i[\tau_0 | \tau_0 < \infty] < \infty$ for all $i \geq 1$. The proof is complete. $\square$

## 2.6. Notes

The three sequences $\{b_n; n \geq 0\}$, $\{a_n; n \geq 0\}$ and $\{h_n; n \geq 0\}$ are the basic known conditions of our model.

The special case of birth-death and immigration process was considered by many authors. For more details, refer to Anderson (1991).

Foster (1971) and Pakes (1971) considered a discrete branching process with immigration occurring only when the process hits state 0, while Yamazato (1975) considered the continuous version of the latter, i.e., the special case $a = 0$.

Here we study the much more general case by combining immigration and resurrection together.

Lemma 2.2.1(i)–(ii) are well-known, while Lemma 2.2.1(iii)–(v) are new and important to our study despite their simplicity. Lemma 2.3.3 comes from Chen (1992). All the results (except Lemma 2.2.1(i)–(ii) and Lemma 2.3.3) in this chapter are new. This chapter has been submitted for publication in Li and Chen (2004).

The model considered in this chapter is a natural generalisation of ordinary Markov branching model and hence it is significant in applications. In this model, the underlying structure is an ordinary Markov branching process and the particles act independently. We mainly concentrated on the study of the influence of immigration and resurrection. From the next chapter on, we shall consider generalising branching models from another point of view. In the models we will consider in the following chapters, the independence property is no longer true (i.e., particles act dependently) and the corresponding $q$-matrix takes quite different form from the model considered in this chapter. Hence, the models considered in this chapter and in the following chapter possess quite different properties.

# Chapter 3. Weighted Markov Branching Processes

## 3.1. Basic Concepts

Markov branching processes form one of the most important classes of Markov chains and have a vast range of applications. With the development of the modern science, it is necessary to consider more general branching models. In this chapter, we consider the following model.

**Definition 3.1.1** A $q$-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called a weighted branching $q$-matrix (henceforth referred to as a WB-$q$-matrix), if

$$q_{ij} = \begin{cases} w_i b_{j-i+1}, & if\ \ i \geq 1\ \ j \geq i-1 \\ 0, & otherwise \end{cases} \tag{3.1.1}$$

where

$$b_j \geq 0\ (j \geq 1),\ b_0 > 0,\ \ 0 < -b_1 = \sum_{j \neq 1} b_j < \infty,\ \ w_j > 0\ \ (j \geq 1).$$

**Definition 3.1.2** A weighted Markov branching process (henceforth referred to as a WMBP) is a $\mathbf{Z}_+$-valued continuous time Markov chain whose transition function $P(t) = (p_{ij}(t);\ i, j \in \mathbf{Z}_+)$ satisfies the Kolmogorov forward equations

$$P'(t) = P(t)Q \tag{3.1.2}$$

where $Q$ is a WB-$q$-matrix as in (3.1.1).

Note that MBP is a very special case of WMBP, i.e., $w_n = n$ $(n \geq 1)$. It is well-known that MBP has the branching property. Conversely, Chen (2001) proved that if a $Q$-process satisfies the branching property, then its $q$-matrix must take (1.4.1)–(1.4.2). Therefore, the branching property is no longer held for the general WMBP.

## 3.2. Preliminary

As same as in Chapter 2, let

$$B(s) = \sum_{j=0}^{\infty} b_j s^j$$

and

$$m_b = \sum_{j=1}^{\infty} j b_{j+1}.$$

The property of the generating function $B(s)$ can be seen in Lemma 2.2.1. And also let $q(< 1)$ denote the smallest positive root of $B(s) = 0$ as in the previous chapter.

Now, further define a sequence of functions $\{G_i(s);\ i \geq 1\}$ as

$$G_i(s) = \begin{cases} (1 - s^i)/B(s)\ , & if\ \ b_0 \geq m_b \\ (q^i - s^i)/B(s)\ , & if\ \ b_0 < m_b \leq +\infty \end{cases}$$

and denote $G(s) = G_1(s)$. Each $G_i(s)$ is well-defined at least in $(-1, 1)$. For convenience, sometimes we shall view $G_i(s)$ $(i \geq 1)$ as a complex function.

**Lemma 3.2.1.** The complex function $G(z)$ is analytic on the disk $\{z;\ |z| < 1\}$ and thus $G(s)$, as a real function, can be expanded as a Taylor series

$$G(s) = \sum_{k=0}^{\infty} g_k s^k\ , \qquad |s| < 1 \tag{3.2.1}$$

where $g_k = G^{(k)}(0)/k!$ $(k \geq 0)$ satisfies the following properties:

(i) $0 < g_k \leq g_0$ $(k \geq 0)$.

(ii) If $b_0 < m_b \leq +\infty$, (and thus $B(s) = 0$ has a root $q \in (0, 1)$), then the limit $\lim_{n\to\infty} g_n$ exists, denoted by $g_\infty$, and that

$$g_\infty = \frac{1-q}{m_b - b_0}. \tag{3.2.2}$$

In particular, $g_\infty > 0$ if and only if $m_b < +\infty$.

**Proof.** Denote

$$\rho_0 = \begin{cases} b_0/q, & if\ \ b_0 < m_b \leq +\infty \\ b_0, & if\ \ b_0 \geq m_b \end{cases}$$

and

$$\rho_k = \begin{cases} \sum_{m=1}^{\infty} b_{k+m} q^{m-1}, & if\ \ b_0 < m_b \leq +\infty \\ \sum_{m=1}^{\infty} b_{k+m}, & if\ \ b_0 \geq m_b, \end{cases} \qquad k \geq 1.$$

It is clear that $\{\rho_k; k \geq 0\}$ is a nonnegative sequence with $\rho_0 > 0$ and

$$\sum_{k=1}^{\infty} \rho_k \begin{cases} = \rho_0, & if\ \ b_0 \leq m_b \leq +\infty \\ < \rho_0, & if\ \ b_0 > m_b. \end{cases}$$

A little algebra then immediately yields

$$G(z) = (\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k)^{-1}. \tag{3.2.3}$$

Since $\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k$ is analytic on the disk $\{z; |z| < 1\}$ and $|\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k| \geq \rho_0(1 - |z|) > 0$ for all $|z| < 1$, we obtain that $G(z)$ is analytic on the disk $\{z; |z| < 1\}$ and thus can be expanded as a Taylor series (3.2.1).

Since (3.2.1) and (3.2.3) hold for all $|s| < 1$ we obtain

$$\rho_0 g_0 = 1, \quad \rho_0 g_n = \sum_{k=1}^{n} \rho_k g_{n-k}, \quad n \geq 1 \tag{3.2.4}$$

and then (i) immediately follows.

Now suppose $b_0 < m_b \leq +\infty$. After rewriting (3.2.4) as

$$g_n - \sum_{k=0}^{n} g_k a_{n-k} = c_n \quad (n \geq 0)$$

where $a_0 = 0$, $a_k = \rho_0^{-1} \rho_k \quad (k \geq 1)$ and $c_n = \rho_0^{-1} \delta_{0n}$ and noting that $\{g_n;\ n \geq 0\}$ is bounded and $\sum_{k=1}^{\infty} a_k = 1$, we recognize that (3.2.4) is just a renewal equation. It follows (see Theorem 3.1.1 in Karlin(1966)) that $\lim_{n\to\infty} g_n = g_\infty$ exists and

$$\lim_{n\to\infty} g_n = \frac{c_0}{\sum_{k=1}^{\infty} k a_k} = \frac{1}{\sum_{k=1}^{\infty} k \rho_k}.$$

Let $\rho(s) = \sum_{k=1}^{\infty} \rho_k s^k$, $s \in [0, 1)$. By the definition of $\rho_k$,

$$\begin{aligned}
\rho(s) &= \sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} b_j q^{j-k-1} s^k \\
&= \sum_{j=2}^{\infty} b_j q^{j-1} \sum_{k=1}^{j-1} \frac{s}{q}^k \\
&= \sum_{j=2}^{\infty} b_j q^{j-1} \cdot \frac{s - q^{1-j} s^j}{q - s} \\
&= \frac{s(-b_1 q - b_0)}{q(q-s)} - \frac{B(s) - b_1 s - b_0)}{q - s} \\
&= \frac{qB(s) + b_0(s - q)}{q(s - q)}, \quad s \in (q, 1).
\end{aligned}$$

So,

$$q(s - q)\rho(s) = qB(s) + b_0(s - q).$$

Differentiating the above equality yields that

$$q\rho(1) + q(s-q)\rho'(s) = qB'(s) + b_0(s-q).$$

Letting $s \uparrow 1$ and noting that $\rho(1) = b_0/q$ yields that

$$(1-q)\rho'(1) = B'(1).$$

Thus,

$$\sum_{k=1}^{\infty} k\rho_k = \rho'(1) = \frac{B'(1)}{1-q} = \frac{m_b - b_0}{1-q}.$$

The proof is complete. □

**Remark 3.2.1.** It follows from Lemma 3.2.1 that for any $i \geq 1$, $G_i(z)$ is also analytic on the disk $\{z; |z| < 1\}$ and thus can be expanded as a Taylor series

$$G_i(z) = \sum_{k=0}^{\infty} \frac{G_i^{(k)}(0)}{k!} z^k, \qquad (|z| < 1, i \geq 1)$$

where $G_i^{(k)}(0)$ is the $k$'th degree derivative of $G_i(z)$ evaluated at 0. By using (3.2.1), it is easily seen that for any $i \geq 1$,

$$\frac{G_i^{(n)}(0)}{n!} = \begin{cases} \sum_{k=0}^{n\wedge(i-1)} g_{n-k} q^{i-1-k}, & if \;\; b_0 < m_b \leq +\infty \\ \sum_{k=0}^{n\wedge(i-1)} g_{n-k}, & if \;\; b_0 \geq m_b \end{cases} \qquad n \geq 0$$

which implies that $\{\frac{G_i^{(n)}(0)}{n!}; n \geq 0\}$ is also nonnegative and bounded. In particular, by (3.2.4) we have that $g_{n-k} \leq (\rho_0/\rho_1)^k \cdot g_n$ for $0 \leq k \leq n$, and hence for each $i \geq 1$,

$$C_1 g_n \leq \frac{G_i^{(n)}(0)}{n!} \leq C_2 g_n, \qquad n \geq 0$$

where the positive constants $C_1$ and $C_2$, which may depend on $i \geq 1$, are independent of $n$.

The following key lemma plays an important role in our future analysis.

**Lemma 3.2.2.** Let $(p_{ij}(t); i,j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j \in \mathbf{Z}_+)$ be, respectively, the Feller minimal $Q$-function and $Q$-resolvent where $Q$ is a WB-$q$-matrix given in (3.1.1). Then for any $i \geq 1$ and $0 \leq s < 1$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \cdot \sum_{k=1}^{\infty} p_{ik}(t)w_k s^{k-1} \tag{3.2.5}$$

and

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j = s^i + B(s) \cdot \sum_{k=1}^{\infty} \phi_{ik}(\lambda)w_k s^{k-1}. \tag{3.2.6}$$

**Proof.** It follows immediately from the integral recursion scheme (1.3.2) that

$$(\lambda + q_j)\phi_{ij}^{(n+1)}(\lambda) \le \delta_{ij} + \sum_{k=1}^{j+1} \phi_{ik}^{(n)}(\lambda)w_k \cdot |b_{j-k+1}|, \quad j \ge 0, \quad n \ge 0.$$

Noting $\Sigma_{k=0}^{\infty} |b_k| \cdot |s|^k \le -2b_1$ for all $|s| \le 1$ and using the mathematical induction principle yields that for any $n \ge 0$, $i \ge 0$ and $0 \le s \le 1$,

$$\sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) \cdot w_k \cdot s^{k-1} < +\infty.$$

Also by the integral recursion scheme (1.3.2) and (3.1.1), we have

$$\sum_{j=0}^{\infty}(\lambda+q_j)\phi_{ij}^{(n+1)}(\lambda)s^j = s^i + \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda)w_k s^{k-1} \cdot (b_0 + \sum_{m=1}^{\infty} b_m s^{m+1}). \tag{3.2.7}$$

It follows from (3.2.7) and the above inequaility that

$$-b_1 \sum_{j=1}^{\infty} \phi_{ij}^{(n+1)}(\lambda) \cdot w_j s^j \le s^i - b_1 \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda)w_k s^{k-1} < +\infty \tag{3.2.8}$$

since $-b_1 w_j < \lambda + q_j$ and $b_0 + \Sigma_{m=1}^{\infty} b_m s^{m+1} \le -b_1$ for all $s \in [0,1]$.

Now if we define $A_{ij}^{(n+1)}(\lambda) = \phi_{ij}^{(n+1)}(\lambda) - \phi_{ij}^{(n)}(\lambda)$ $(n \ge 0)$. Then $A_{ij}^{(n)}(\lambda) \ge 0$ and

$$\lim_{n\to\infty} A_{ij}^{(n)}(\lambda) = 0 \qquad for\ \ all\ \ i,j \in \mathbf{Z}_+. \tag{3.2.9}$$

Using this notation, (3.2.7) can be rewritten as

$$\begin{aligned} \lambda \sum_{j=0}^{\infty} \phi_{ij}^{(n+1)}(\lambda)s^j \;&=\; s^i + B(s) \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda)w_k s^{k-1} \\ &\quad + b_1 s \sum_{j=1}^{\infty} A_{ij}^{(n+1)}(\lambda)w_j s^{j-1}. \end{aligned} \tag{3.2.10}$$

Also, by (3.2.8),

$$-b_1 \sum_{j=1}^{\infty} \left(\phi_{ij}^{(n+1)}(\lambda)s - \phi_{ij}^{(n)}(\lambda)\right) \cdot w_j s^{j-1} \le s^i, \quad s \in [0,1].$$

Letting $s = 1$ in above inequality and noting $A_{ij}^{(n+1)}(\lambda) = \phi_{ij}^{(n+1)}(\lambda) - \phi_{ij}^{(n)}(\lambda)$ $(n \ge 0)$ yields that

$$\sum_{k=1}^{\infty} A_{ik}^{(n)}(\lambda) w_k \le -1/b_1, \quad n \ge 1.$$

Hence applying Dominated Convergence Theorem and using (3.2.9) we obtain that for $0 \le s < 1$,

$$\lim_{n\to\infty} \sum_{j=1}^{\infty} A_{ij}^{(n+1)}(\lambda) w_j s^{j-1} = 0.$$

Letting $n \uparrow \infty$ in (3.2.10) and using the above limit leads to the fact that for $0 \le s < 1$,

$$\lambda \lim_{n\to\infty} \sum_{j=0}^{\infty} \phi_{ij}^{(n+1)}(\lambda) s^j = s^i + B(s) \cdot \lim_{n\to\infty} \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-1}.$$

Noting $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ (see (1.3.2)) yields that for $0 \le s < 1$,

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j = s^i + B(s) \cdot \lim_{n\to\infty} \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-1} \tag{3.2.11}$$

provided that $B(s) \neq 0$. However by Lemma 2.2.1(ii), we may find an $\varepsilon > 0$ such that $B(s) \neq 0$ for all $1 - \varepsilon \le s < 1$. Hence by (3.2.11) we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-1} < +\infty, \quad \textit{for} \quad s \in [1-\varepsilon, 1).$$

The above inequality trivially implies that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-1} < +\infty, \quad \textit{for} \quad s \in [0, 1).$$

Using Monotone Convergence Theorem and noting the above inequality then yields that for $s \in [0, 1)$,

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda) w_k s^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-1} < +\infty. \tag{3.2.12}$$

Substituting (3.2.12) into (3.2.11) yields (3.2.6).

Finally, since $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i$ and $\sum_{j=1}^{\infty} \phi_{ij}(\lambda) w_j s^{j-1}$ are just the Laplace transforms of $\sum_{j=0}^{\infty} p'_{ij}(t) s^j$ and $\sum_{j=1}^{\infty} p_{ij}(t) w_j s^{j-1}$ respectively, i.e., (3.2.6) is just the Laplace transform of (3.2.5). However, both sides of (3.2.5) are continuous functions of $t$, and thus (3.2.5) holds for all $t \geq 0$. $\square$

**Lemma 3.2.3.** Let $(p_{ij}(t);\ i, j \in \mathbf{Z}_+)$ be the transition function of the Feller minimal WMBP.

(i) For any $i,\ k \geq 1$, we have

$$\int_0^{\infty} p_{ik}(t)dt = \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} < +\infty \tag{3.2.13}$$

where $G_i^{(k-1)}(0)$ denotes the $(k-1)$'th degree derivative of $G_i(s)$ at 0, and hence $\lim_{t\to\infty} p_{ik}(t) = 0$. Moreover, for any $i \geq 1$,

$$\lim_{t\to\infty} p_{i0}(t) = \begin{cases} q^i < 1, & if\ \ b_0 < m_b \leq +\infty \\ 1, & if\ \ b_0 \geq m_b \end{cases} \tag{3.2.14}$$

where $0 < q < 1$ is the smallest positive root of $B(s) = 0$ on $[0, 1]$.

(ii) If $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$, in particular, if $\sum_{k=1}^{\infty}(1/w_k) < +\infty$, then for any $i \geq 1$

$$\int_0^{\infty} (\sum_{k=1}^{\infty} p_{ik}(t))dt < +\infty \tag{3.2.15}$$

and hence

$$\lim_{t\to\infty} \sum_{k=1}^{\infty} p_{ik}(t) = 0. \tag{3.2.16}$$

**Proof.** For any fixed $i \geq 0$, it follows from the Kolomogorv forward equations that

$$p_{i0}(t) = \delta_{i0} + b_0 \int_0^t p_{i1}(u)du$$

which clearly implies that $\int_0^{\infty} p_{i1}(t)dt < \infty$. Suppose that $\int_0^{\infty} p_{ik}(t)dt < \infty$ for $k \leq j$. ¿From Kolmogorov forward equations we can see that

$$p_{ij}(t) - \delta_{ij} = \sum_{k=1}^{j} w_k b_{j-k+1} \cdot \int_0^{\infty} p_{ik}(t)dt + w_{j+1} b_0 \int_0^{\infty} p_{ij+1}(t)dt$$

and hence $\int_0^\infty p_{ij+1}(t)dt < \infty$. Therefore, the left-most quantity in (3.2.13) is finite by the mathematical induction principle. Hence all states $j \geq 1$ are transient. It then follows that

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad (\forall i \geq 1, j \geq 1). \tag{3.2.17}$$

We now prove (3.2.14). If $b_0 \geq m_b$, then $B(s) \geq 0$ for all $s \in [0,1]$. Therefore, from (3.2.5) we can see that for $s \in [0,1)$,

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0.$$

Letting $t \uparrow \infty$ and noting $\lim_{t\to\infty} p_{ij}(t) = 0$ for all $j \geq 1$ yields that

$$\lim_{t\to\infty} p_{i0}(t) \geq s^i, \qquad s \in [0,1)$$

which implies (3.2.14). Next consider the case $b_0 < m_b \leq +\infty$. Recall that for this case, the equation $B(s) = 0$ possesses a smallest positive root $q$ such that $0 < q < 1$. Letting $s = q$ in (3.2.5), we obtain that for all $t \geq 0$,

$$\sum_{j=0}^{\infty} p_{ij}(t)q^j = q^i, \quad i \geq 1. \tag{3.2.18}$$

Now letting $t \to \infty$ yields

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} \sum_{j=1}^{\infty} p_{ij}(t)q^j = q^i. \tag{3.2.19}$$

Note that here both limits in (3.2.19) do exist. Now since $0 < q < 1$, we may apply the Dominated Convergence Theorem in (3.2.19). This together with (3.2.17) immediately yields (3.2.14).

We now proceed to prove the equality in (3.2.13). Integrating with $t$ in (3.2.5) yields

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i = B(s) \sum_{k=1}^{\infty} \left(\int_0^t p_{ik}(u)du\right) \cdot w_k \cdot s^{k-1}. \tag{3.2.20}$$

For $s \in [0,1)$, letting $t \uparrow +\infty$ in (3.2.20) , using the Dominated Convergence Theorem and (3.2.17) in the left-hand side of (3.2.20) and applying Monotone Convergence Theorem in the right-hand side of (3.2.20) yields that

$$\lim_{t\to\infty} p_{i0}(t) - s^i = B(s) \sum_{k=1}^{\infty} \left(\int_0^\infty p_{ik}(u)du\right) \cdot w_k \cdot s^{k-1}.$$

Dividing $B(s)$ on the both sides of the above equality yields

$$\frac{\lim_{t\to\infty} p_{i0}(t) - s^i}{B(s)} = \sum_{k=1}^{\infty}\left(\int_0^\infty p_{ik}(u)du\right) \cdot w_k \cdot s^{k-1}.$$

Noting (3.2.14) and the definition of $G_i(s)$, we rewrite the above equality as

$$G_i(s) = \sum_{k=1}^{\infty}\left(\int_0^\infty p_{ik}(t)dt\right) \cdot w_k \cdot s^{k-1}. \tag{3.2.21}$$

Since (3.2.21) holds at least for all $0 \le s < 1$, we obtain, by using the uniqueness of Taylor expansion, the equality in (3.2.13).

Finally, we prove (ii). Suppose that $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$. Then by Remark 3.2.1 we see that for any $i \ge 1$, $\sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} < +\infty$. Now using (3.2.13) immediately yields (3.2.15). In particular, if $\sum_{k=1}^{\infty}(1/w_k) < +\infty$, then since $\{g_k\}$ is bounded we obtain

$$\sum_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$$

and thus (ii) holds. □

## 3.3. Regularity and Uniqueness

We now discuss the regularity and uniqueness of the process. First consider the case $b_0 \ge m_b$.

**Theorem 3.3.1.** If $b_0 \ge m_b$, then the WB-$q$-matrix $Q$ is regular. That is, the Feller minimal $Q$-process is honest and thus there exists only one WMBP.

**Proof.** If $b_0 \ge m_b$, then by Lemma 2.2.1(ii), $B(s) > 0$ for all $0 \le s < 1$ . It then follows from (3.2.6) that

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j \ge s^i, \qquad 0 \le s < 1.$$

Letting $s \uparrow 1$ yields that $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) \ge 1$. However, the converse inequality always holds and thus $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) = 1$. Hence the Feller minimal $Q$-process is honest. □

**Theorem 3.3.2.** Suppose $\sum_{n=1}^{\infty}(1/w_n) = +\infty$.

(i) If $m_b < +\infty$, then $Q$ is regular, i.e., the Feller minimal WMBP is honest and thus there exists only one WMBP.

(ii) If $m_b = +\infty$ and $\Sigma_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$, then $Q$ is not regular, i.e., the Feller minimal WMBP is dishonest.

**Proof.** We prove (ii) first. Suppose the contrary is true, then

$$1 - p_{i0}(t) = \sum_{k=1}^{\infty} p_{ik}(t), \quad \forall i \geq 1$$

which, together with (3.2.15) in Lemma 3.2.3, yields

$$\int_0^{\infty}(1 - p_{i0}(t))dt < \infty.$$

Hence we obtain $\lim_{t\to\infty} p_{i0}(t) = 1$ which contradicts with (3.2.14) in Lemma 3.2.3 since we have assumed that $m_b = +\infty$.

We now prove (i). If $b_0 \geq m_b$, the assertion is just Theorem 3.3.1. So we only need to consider the case $b_0 < m_b < +\infty$. Using the definition of $G(s)$ we may rewrite (3.2.6) for $i = 1$ as

$$(q - s)\sum_{k=1}^{\infty} \phi_{1k}(\lambda) w_k s^{k-1} = G(s)[\lambda \sum_{k=0}^{\infty} \phi_{1k}(\lambda) s^k - s].$$

Since the above equality holds for all $s \in [0, 1)$, we derive that, by comparing the coefficients of both sides and noting (3.2.1), for $n \geq 1$

$$q\phi_{1n+1}(\lambda)w_{n+1} - \phi_{1n}(\lambda)w_n = \sum_{k=0}^{n} \lambda\phi_{1k}(\lambda)g_{n-k} - g_{n-1}. \tag{3.3.1}$$

Noting Lemma 3.2.3 and the fact that $\Sigma_{k=0}^{\infty} \lambda\phi_{1k}(\lambda) \leq 1$, we obtain (see for example Theorem 2.5.5 in Hunter (1983)) that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda\phi_{1k}(\lambda)g_{n-k} = g_{\infty} \cdot \sum_{k=0}^{\infty} \lambda\phi_{1k}(\lambda), \tag{3.3.2}$$

where $g_{\infty} > 0$, guaranteed by the condition $b_0 < m_b < +\infty$, is given in (3.2.2). We now claim that for any $\lambda > 0$ we have

$$\sum_{k=0}^{\infty} \lambda\phi_{1k}(\lambda) = 1. \tag{3.3.3}$$

Indeed, if (3.3.3) does not hold then there exists a $\lambda > 0$ such that $1 - \lambda\Sigma_{k=0}^{\infty} \phi_{1k}(\lambda) > 0$. Letting $n \to \infty$ in (3.3.1) and using (3.3.2) and

(ii) of Lemma 3.2.1 we obtain

$$\lim_{n\to\infty}(\phi_{1n}(\lambda)w_n - q\phi_{1n+1}(\lambda)w_{n+1}) = g_\infty \cdot (1 - \sum_{k=0}^{\infty} \lambda\phi_{1k}(\lambda)) > 0.$$

Hence there exists a constant $\delta > 0$ and an integer $N > 1$ such that for all $n \geq N$, we have

$$\phi_{1n}(\lambda) \geq \delta \cdot w_n^{-1}.$$

This is a contradiction since $\sum_{n=1}^{\infty} w_n^{-1} = +\infty$. Thus (3.3.3) holds for all $\lambda > 0$. It follows from (3.3.3) that

$$\sum_{k=0}^{\infty} \lambda\phi_{ik}(\lambda) = 1, \qquad (\forall\lambda > 0)$$

for all $i \geq 1$ since the set of all positive states forms a communicating class. As for $i = 0$, it is trivially true. This completes the proof. □

**Theorem 3.3.3.** Suppose $\sum_{n=1}^{\infty}(1/w_n) < \infty$. Then $Q$ is regular if and only if $b_0 \geq m_b$.

**Proof.** By Theorem 3.3.1, we only need to prove that if $b_0 < m_b \leq +\infty$, then $Q$ is not regular. However, this is easy. Indeed, since $\{g_k;\ k \geq 0\}$ is bounded we know that $\sum_{k=1}^{\infty}(1/w_k) < \infty$ implies $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < \infty$ and thus the conclusion follows from the first part of the proof of Theorem 3.3.2. □

The previous three theorems established regularity criteria. If a WB-$q$-matrix $Q$ is regular then there exists only one WMBP. However, the converse may not be always true. Indeed, if a WB-$q$-matrix $Q$ is not regular, then although there exist infinitely many (even honest) $Q$-functions, there may still exist only one WMBP since our WMBP must satisfy the Kolmogorov forward equation (3.1.2). Therefore, in addition to the regularity criteria, we also need to establish uniqueness criteria. We first consider the case $\sum_{n=1}^{\infty}(1/w_n) = +\infty$. Interestingly, although there exists infinitely many honest $Q$-functions, there always exists only one WMBP, as the following result shows. Although by Theorem 3.3.2 we only need to consider the case $m_b = +\infty$, we shall not confine ourself to this case since the proof is the same.

**Theorem 3.3.4**. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a WB-$q$-matrix. If $\sum_{n=1}^{\infty}(1/w_n) = +\infty$, then there always exists only one WMBP.

**Proof.** By Anderson (1991) (see Theorem 2.2.8 there) or Yang (1990), we know that for any $q$-matrix $Q$, if the equation

$$\begin{cases} \mathbf{Y}(\lambda I - Q) = 0, \\ \mathbf{0} \le \mathbf{Y}, \mathbf{Y}\mathbf{1} < +\infty \end{cases} \tag{3.3.4}$$

has only a trivial solution for some (and therefore for all) $\lambda > 0$ where $\mathbf{1}$ denotes the column vector on $\mathbf{Z}_+$ whose components are all equal to 1, then there exists only one $Q$-function satisfying the Kolmogorov forward equation. For our case, WMBP satisfies the Kolmogorov forward equation, therefore, we only need to prove that (3.3.4) has only a trivial solution for $\lambda = 1$.

Suppose $\mathbf{Y} = (y_i; i \ge 0)$ is a non-trivial solution of the equation (3.3.4) for $\lambda = 1$. Then (3.3.4) can be rewritten as

$$y_n = \sum_{j=1}^{n+1} y_j w_j b_{n-j+1}, \quad n \ge 0$$

or, equivalently,

$$\begin{cases} y_1 w_1 b_0 = y_0, \\ y_{n+1} w_{n+1} b_0 = \Sigma_{k=0}^{n} y_k + \Sigma_{k=1}^{n} y_k w_k \cdot (\Sigma_{j=n-k+2}^{\infty} b_j), \quad n \ge 1. \end{cases} \tag{3.3.5}$$

It is easily seen that if $y_0 = 0$, then $y_n = 0, \forall n \ge 0$. So we may assume that $y_0 > 0$. It follows from (3.3.5) that

$$y_{n+1} \ge \frac{y_0}{b_0 w_{n+1}}.$$

Since $\Sigma_{n=1}^{\infty}(1/w_n) = +\infty$, we obtain that $\Sigma_{n=0}^{\infty} y_n = +\infty$ which shows that $\mathbf{Y} = \{y_i;\ i \ge 0\}$ is not a non-trivial solution of Equation (3.3.4) for $\lambda = 1$. This is a contradiction. The proof is thus complete. □

In contract to Theorem 3.3.4, the conclusion regarding uniqueness is much more subtle for the case $\Sigma_{n=1}^{\infty}(1/w_n) < +\infty$.

**Theorem 3.3.5.** Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a WB-$q$-matrix. If $\Sigma_{n=1}^{\infty}(1/w_n) < +\infty$ and $b_0 < m_b \le +\infty$, then there exists only one WMBP if and only if

$$\sum_{n=1}^{\infty} R_n = +\infty \tag{3.3.6}$$

where

$$R_0 = 1, \quad R_n = \frac{1}{b_0 w_{n+1}} + \sum_{k=1}^{n} \frac{w_k \tau_{n-k+1}}{b_0 w_{n+1}} R_{k-1}, \quad (n \geq 1) \tag{3.3.7}$$

with

$$\tau_n = \sum_{j=n+1}^{\infty} b_j, \quad (n \geq 1). \tag{3.3.8}$$

**Proof.** By Theorem 3.3.3 we see that if $b_0 < m_b \leq +\infty$ and $\Sigma_{n=1}^{\infty}(1/w_n) < +\infty$, then the WMBP $q$-matrix $Q$ is not regular. Hence there exists only one $Q$-function which satisfies the Kolmogorov forward equation (3.1.2) if and only if the equation (3.3.4) has only a trivial solution for some (and therefore for all) $\lambda > 0$ (see, for example, Anderson (1991)). However by the proof of Theorem 3.3.4 we have seen that any non-trivial solution $\mathbf{Y} = (y_i; \quad i \geq 0)$ of the equation (3.3.4) for $\lambda = 1$ can be obtained by (3.3.5). Now denote $\sigma_n = \Sigma_{k=0}^{n} y_k, \quad (n \geq 0)$. It is clear that $(\sigma_n; n \geq 0)$ is an increasing sequence and, by (3.3.5),

$$(\sigma_{n+1} - \sigma_n) w_{n+1} b_0 = \sigma_n + \sum_{k=1}^{n} (\sigma_k - \sigma_{k-1}) w_k \tau_{n-k+1}, \quad n \geq 1, \tag{3.3.9}$$

if we define the sequence $\{\tau_n; \quad n \geq 1\}$ as in (3.3.8). It follows that there exist more than one WMBP if and only if $\{\sigma_n; \quad n \geq 0\}$ is bounded. Now we claim that

$$F_n(\sigma_1 - \sigma_0) \leq \sigma_{n+1} - \sigma_n \leq F_n \sigma_n, \quad n \geq 1 \tag{3.3.10}$$

where

$$F_1 = \frac{1 + w_1 \tau_1}{b_0 w_2} \tag{3.3.11}$$

and

$$F_n = \frac{1 + w_1 \tau_n}{b_0 w_{n+1}} + \sum_{k=2}^{n} \frac{w_k \tau_{n-k+1}}{b_0 w_{n+1}} F_{k-1}, \quad n \geq 2. \tag{3.3.12}$$

Indeed, by (3.3.9),

$$\begin{aligned} \sigma_2 - \sigma_1 &= \frac{\sigma_1}{b_0 w_2} + \frac{(\sigma_1 - \sigma_0) w_1 \tau_1}{b_0 w_2} \\ &\geq \frac{1 + w_1 \tau_1}{b_0 w_2} \cdot (\sigma_1 - \sigma_0) \\ &= F_1 (\sigma_1 - \sigma_0) \end{aligned}$$

and

$$\sigma_2 - \sigma_1 \le \frac{\sigma_1}{b_0 w_2} + \frac{\sigma_1 w_1 \tau_1}{b_0 w_2} = F_1 \sigma_1.$$

Suppose that (3.3.10) holds for $n \le m$. Then by (3.3.9),

$$\begin{aligned}
&\sigma_{m+2} - \sigma_{m+1} \\
= \ & \frac{1}{b_0 w_{m+2}} \cdot \left[ \sigma_{m+1} + \sum_{k=1}^{m+1} (\sigma_k - \sigma_{k-1}) w_k \tau_{m+1-k+1} \right] \\
\ge \ & \frac{\sigma_1 - \sigma_0}{b_0 w_{m+2}} \cdot \left[ 1 + w_1 \tau_{m+1} + \sum_{k=2}^{m+1} F_{k-1} \cdot w_k \tau_{m+2-k} \right] \\
= \ & (\sigma_1 - \sigma_0) \cdot \left[ \frac{1 + w_1 \tau_{m+1}}{b_0 w_{m+2}} + \sum_{k=2}^{m+1} \frac{w_k \tau_{m+2-k}}{b_0 w_{m+2}} \cdot F_{k-1} \right] \\
= \ & F_{m+1} \cdot (\sigma_1 - \sigma_0)
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_{m+2} - \sigma_{m+1} \\
\le \ & \frac{1}{b_0 w_{m+2}} \cdot \left[ \sigma_{m+1} + (\sigma_1 - \sigma_0) w_1 \tau_{m+1} + \sum_{k=2}^{m+1} \sigma_{k-1} w_k \tau_{m+2-k} \right] \\
\le \ & \left[ \frac{1 + w_1 \tau_{m+1}}{b_0 w_{m+2}} + \sum_{k=2}^{m+1} \frac{w_k \tau_{m+2-k}}{b_0 w_{m+2}} \cdot F_{k-1} \right] \cdot \sigma_{m+1} \\
= \ & F_{m+1} \cdot \sigma_{m+1}.
\end{aligned}$$

By the mathematical induction principle, (3.3.10) holds. Note that $\sigma_1 - \sigma_0 = y_1 = y_0 (w_1 b_0)^{-1} > 0$ and thus the boundedness of $\{\sigma_n; \quad n \ge 0\}$ implies

$$\sum_{n=1}^{\infty} F_n < +\infty \tag{3.3.13}$$

by using the left-hand side inequality in (3.3.10). Conversely, if (3.3.13) is true then using the right hand side inequality in (3.3.10) we obtain

$$\sum_{n=1}^{\infty} \left(\frac{\sigma_{n+1}}{\sigma_n} - 1\right) < +\infty$$

which implies $\{\sigma_n; \quad n \ge 0\}$ is bounded. Therefore there exist more than one WMBP if and only if (3.3.13) holds. However, it is trivial to see, by comparing (3.3.7) with (3.3.11)–(3.3.12), that if we let $R_0 = 1$ then $F_n \equiv R_n, \quad (n \ge 1)$. This completes the proof. □

The advantage of criterion (3.3.6) is that the sequence $\{R_n\}$ in (3.3.6) can be obtained by using (3.3.7) and (3.3.8) since $\{b_k\}$ and $\{w_k\}$ are known sequences. In some cases, even the closed form of the sequence $\{R_n; \quad n \geq 1\}$ can be given and thus our criterion is satisfactory. For example, although we shall not provide details here, we point out that in the case when there exists a positive integer $N$ such that $b_n = 0 \quad (\forall n \geq N)$, then (3.3.7) can be easily transformed into a finite difference equation with constant coefficients and hence a general closed form for $\{R_n; \quad n \geq 1\}$ is completely available. That is, all the values of $R_n$ are known and therefore (3.3.6) can be checked directly.

For general cases, however, checking criterion (3.3.6) may not be always simple since the sequence $\{R_n; \quad n \geq 1\}$ is given recursively in (3.3.7). Fortunately, in many cases particularly in the models which have important applications, we do not need to check (3.3.6). Indeed, a much better sufficient condition can be given as the following result shows. Because of its importance, we shall write it as a theorem rather than a corollary of Theorem 3.3.5.

**Theorem 3.3.6**. Let $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ be a WB-$q$-matrix satisfying $\Sigma_{n=1}^{\infty}(1/w_n) < +\infty$ and $b_0 < m_b \leq +\infty$ and thus the equation $B(s) = 0$ possesses a positive root $q$ such that $0 < q < 1$.

(i) If $\limsup_{n\to\infty} \sqrt[n]{w_{n+1}} < 1/q$, then there exists only one WMBP and this unique WMBP is just the Feller minimal $Q$-process which is dishonest.

(ii) If $\liminf_{n\to\infty} \sqrt[n]{w_{n+1}} > 1/q$, then there exist infinitely many WMBPs. Exactly one of them is honest which is not the Feller minimal $Q$-process.

(iii) Suppose that $\lim_{n\to\infty} \sqrt[n]{w_{n+1}} = w$ exists. Then if $w < 1/q$ there exists only one WMBP which is the (dishonest) Feller minimal $Q$-process while if $w > 1/q$, then there exist infinitely many WMBPs with exactly one of them being honest.

**Proof**. We only need to prove (i) and (ii) since (iii) is just the special case of (i)–(ii) (i.e., the case $\lim_{n\to\infty} \sqrt[n]{w_{n+1}} = w$ exists). Suppose

(i) is not true then by Theorem 3.3.5, $\sum_{n=1}^{\infty} R_n < +\infty$, where $\{R_n\}$ is given in (3.3.7). It follows that $\limsup_{n\to\infty} \sqrt[n]{R_n} \le 1$ (otherwise, $\sum_{n=1}^{\infty}$ is convergent), so if we define $h_n = R_n w_{n+1}$ $(n \ge 0)$, then

$$\limsup_{n\to\infty} \sqrt[n]{h_n} \le \limsup_{n\to\infty} \sqrt[n]{R_n} \cdot \limsup_{n\to\infty} \sqrt[n]{w_{n+1}} < 1/q. \tag{3.3.14}$$

Thus there exists an $\varepsilon \in (0, 1-q)$ such that for any $s \in [0, q+\varepsilon)$ the generating function of $\{h_n\}$, denoted by $H(s) = \sum_{n=0}^{\infty} h_n s^n$, is well-defined and finite, in particular, $H(q) < \infty$. In consideration of the fact that the sequence $\{\tau_n\}$ defined in (3.3.8) is decreasing and thus bounded, the generating function $\sum_{n=1}^{\infty} \tau_n s^n$ is also well-defined and finite at least on $[-1, 1)$. Now, we rewrite (3.3.7) as

$$b_0 w_{n+1} R_n = 1 + \sum_{k=1}^{n} w_k \tau_{n-k+1} R_{k-1}, \quad (n \ge 1),$$

i.e.,

$$b_0 h_n = 1 + \sum_{k=1}^{n} \tau_{n-k+1} h_{k-1}, \quad (n \ge 1).$$

Therefore for any $s \in [0, q+\varepsilon) \subset [0, 1)$,

$$\begin{aligned}
b_0 \sum_{n=1}^{\infty} h_n s^n &= \frac{s}{1-s} + \sum_{n=1}^{\infty} \Big(\sum_{k=1}^{n} \tau_{n-k+1} h_{k-1}\Big) s^n \\
&= \frac{s}{1-s} + \sum_{k=1}^{\infty} h_{k-1} s^{k-1} \cdot \sum_{n=k}^{\infty} \tau_{n-k+1} s^{n-k+1} \\
&= \frac{s}{1-s} + H(s) \cdot \sum_{m=1}^{\infty} \tau_m s^m \\
&= \frac{s}{1-s} + H(s) \cdot \sum_{m=1}^{\infty} \sum_{j=m}^{\infty} b_{j+1} s^m \\
&= \frac{s}{1-s} + H(s) \cdot \sum_{j=1}^{\infty} b_{j+1} \sum_{m=1}^{j} s^m \\
&= \frac{1}{1-s} [s + sH(s) \sum_{j=1}^{\infty} b_{j+1} (1 - s^j)]
\end{aligned}$$

since $\tau_m = \sum_{j=m}^{\infty} b_{j+1}$. By reviewing the definition of $B(s)$ and noting that $h_0 = R_0 w_1 = w_1$, we may rewrite the above equation as

$$B(s) H(s) = s + b_0 w_1 (1-s), \quad 0 \le s \le q + \varepsilon < 1. \tag{3.3.15}$$

Letting $s = q$ in (3.3.15) and noting $B(q) = 0$ we obtain that $H(q) = +\infty$ which contradicts with (3.3.14). This completes the proof of (i).

In order to prove the first part of (ii), we only need to prove that if $\liminf_{n\to\infty} \sqrt[n]{w_{n+1}} > q^{-1}$, then $\sum_{n=1}^{\infty} R_n < +\infty$ (see Theorem 3.3.5). Similarly as in the proof of (i), by using (3.3.7) and again denoting $h_n = R_n w_{n+1}$ $(n \geq 0)$, we may get for $s \in [0,1)$ and any positive integer $N > 1$,

$$\begin{aligned}
b_0 \sum_{n=1}^{N} h_n s^n &= \frac{s - s^{N+1}}{1-s} + \sum_{n=1}^{N} (\sum_{k=1}^{n} \tau_{n-k+1} h_{k-1}) s^n \\
&= \frac{s - s^{N+1}}{1-s} + \sum_{k=1}^{N} h_{k-1} s^{k-1} \cdot \sum_{n=k}^{N} \tau_{n-k+1} s^{n-k+1} \\
&\leq \frac{s}{1-s} + \sum_{k=1}^{N} h_{k-1} s^{k-1} \cdot \sum_{m=1}^{\infty} \tau_m s^m \\
&\leq \frac{s}{1-s} + \sum_{k=0}^{N} h_k s^k \cdot \sum_{m=1}^{\infty} \sum_{j=m}^{\infty} b_{j+1} s^m \\
&= \frac{s}{1-s} + \sum_{k=0}^{N} h_k s^k \cdot \sum_{j=1}^{\infty} b_{j+1} \sum_{m=1}^{j} s^m \\
&= \frac{s}{1-s} [1 + \sum_{k=0}^{N} h_k s^k \cdot \sum_{j=1}^{\infty} b_{j+1} (1 - s^j)].
\end{aligned}$$

i.e.,

$$b_0 (1-s) \sum_{n=1}^{N} h_n s^n \leq s + s \sum_{k=0}^{N} h_k s^k \cdot \sum_{j=1}^{\infty} b_{j+1} (1 - s^j)$$

and so

$$B(s) \cdot \sum_{n=0}^{N} h_n s^n \leq s + b_0 w_1 (1-s).$$

Since $B(s) > 0$ for $s \in [0, q)$, (see Lemma 2.2.1(ii)), we obtain that for $s \in [0, q)$ and any positive integer $N$

$$\sum_{n=0}^{N} h_n s^n \leq \frac{s + b_0 w_1 (1-s)}{B(s)}.$$

It follows that $\sum_{n=0}^{\infty} h_n s^n < +\infty$ for $s \in [0, q)$ and thus

$$\liminf_{n\to\infty} \sqrt[n]{w_{n+1}} \cdot \limsup_{n\to\infty} \sqrt[n]{R_n} \leq \limsup_{n\to\infty} \sqrt[n]{R_n w_{n+1}} \leq q^{-1}.$$

Since $\liminf_{n\to\infty} \sqrt[n]{w_{n+1}} > q^{-1}$, we then obtain from the above that

$$\limsup_{n\to\infty} \sqrt[n]{R_n} < 1$$

which implies that $\sum_{n=1}^{\infty} R_n < +\infty$. The first part of (ii) is thus proved. Finally, by Theorem 14.2.7 in Hou and Guo (1988), for a conservative

$q$-matrix $Q$, if the equation (3.3.4) has just one linearly independent solution, then there exists exactly one honest $Q$-function satisfying the Kolmogorov forward equation. In the present case, since we have known that there exist infinitely many WMBPs, i.e., the equation (3.3.4) has a nontrivial solution. By (3.3.5), we see that this solution is linearly independent. Therefore, the other part in (ii) holds. The proof is complete. $\square$

Criterion (iii) in Theorem 3.3.6 is very useful and simple in applications. Indeed, in many applicable models, we actually have $\lim_{n\to\infty} \sqrt[n]{w_{n+1}} = 1$ which is less than $1/q$ when $b_0 < m_b \le +\infty$. Hence Theorem 3.3.6 is immediately applicable. Therefore Theorem 3.3.6, together with Theorem 3.3.4, can answer nearly all the uniqueness questions. For example, for the models discussed in Chen (2002a, b), we have $w_n = n^\theta$ where $\theta > 0$ and thus $\lim_{n\to\infty} \sqrt[n]{w_{n+1}} = 1$. Now suppose $b_0 < m_b \le +\infty$ and $\theta > 1$ we know that there exists only one WMBP by applying Theorem 3.3.6. On the other hand if $0 < \theta \le 1$, then using Theorem 3.3.4 we get the same conclusion.

## 3.4. Hitting Times

We now turn to consider the hitting times, particularly the extinction time and explosion time. From now on, we shall only consider the Feller minimal WMBP. Let $\{X(t); t \ge 0\}$ denote the Feller minimal WMBP with a given WB-$q$-matrix $Q$ and $\tau_0$ be the extinction time defined as

$$\tau_0 = \begin{cases} \inf\{t > 0; X(t) = 0\}, & if \;\; X(t) = 0 \;\; for \;\; some \;\; t > 0 \\ +\infty, & if \;\; X(t) \ne 0 \;\; for \;\; all \;\; t > 0. \end{cases}$$

For every $i \ge 1$, let

$$a_{i0} \equiv \lim_{t\to\infty} p_{i0}(t) = P_i(\tau_0 < \infty)$$

denote the extinction probability when the process starts at state $i \ge 1$. Here and henceforth $P_i(\cdot)$ denote the conditional probability $P(\cdot|X(0) = i)$.

**Theorem 3.4.1.** For the Feller-minimal WMBP starting at state $i \ge 1$, the extinction probability is given by

$$a_{i0} = \begin{cases} q^i < 1, & if \;\; b_0 < m_b \leq +\infty \\ 1, & if \;\; b_0 \geq m_b \end{cases} \tag{3.4.1}$$

where $0 < q < 1$ is the smallest positive root of $B(s) = 0$ on $[0,1]$. The mean extinction time $E_i[\tau_0]$ is finite if and only if both $b_0 \geq m_b$ and $\Sigma_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$ hold and, under these conditions, is given by

$$E_i[\tau_0] = \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!}, \tag{3.4.2}$$

where $E_i$ is the mathematical expectation under $P_i$. More specifically, if $b_0 \geq m_b$, then $E_i[\tau_0] < +\infty$ if and only if $\Sigma_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$ and when this holds, $E_i[\tau_0]$ is given in (3.4.2). In particular, if $\Sigma_{k=1}^{\infty}(1/w_k) < +\infty$, then for all $i \geq 1$, $E_i[\tau_0] < +\infty$. On the other hand, if $b_0 < m_b \leq +\infty$, then $E_i[\tau_0] = +\infty$.

**Proof.** The proof of (3.4.1) has been already given in Lemma 3.2.3, see (3.2.14). We now prove the latter part of the theorem. Suppose $b_0 \geq m_b$, then by Theorem 3.3.1 the Feller minimal WMBP is honest. Hence by Lemma 3.2.3, we have

$$E_i[\tau_0] = \int_0^{\infty}(1 - p_{i0}(t))dt = \int_0^{\infty}(\sum_{k=1}^{\infty} p_{ik}(t))dt = \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!}$$

which is finite if and only if $\Sigma_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$ because $\{\frac{G_i^{(k-1)}(0)}{(k-1)!};\ k \geq 1\}$ is bounded for each $i \geq 1$. See Remark 3.2.1. The last statement in the current case also follows from Lemma 3.2.3. On the other hand, if $b_0 < m_b \leq +\infty$, then $a_{i0} = 1 - q^i < 1$, $(i \geq 1)$, and therefore $E_i[\tau_0] = +\infty$. □

By Theorem 3.4.1 we see that if $b_0 < m_b \leq +\infty$, then $E_i[\tau_0] = +\infty$. The reason for this uninformative fact is that in this case we have $P_i(\tau_0 < +\infty) = q^i < 1$ for $i \geq 1$. Hence instead of considering the uninformative $E_i[\tau_0]$ itself, we are now interested in finding the conditional mean extinction time $E_i[\tau_0|\tau_0 < \infty]$ when the process starts at state $i \geq 1$, where $E_i[\cdot|\tau_0 < \infty]$ is the conditional mathematical expectation under the condition $\{\tau_0 < \infty\}$ with respect to $P_i$. By conditional mathematical expectation, $E_i[\tau_0|\tau_0 < \infty] = E_i[\tau_0 I_{\{\tau_0<\infty\}}]/P_i(\tau_0 < \infty)$. Note that

$\tau_0 I_{\{\tau_0<\infty\}}$ is random variable and thus, although it is finite almost surely, the mathematical expectation may not be finite.

**Theorem 3.4.2.** Suppose $b_0 < m_b \le +\infty$.

(i) If $\sum_{k=1}^{\infty}(g_{k-1}q^k/w_k) < +\infty$, then for $i \ge 1$, $E_i[\tau_0|\tau_0 < \infty] < +\infty$ and in which case

$$E_i[\tau_0|\tau_0 < \infty] = \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} q^{k-i}. \tag{3.4.3}$$

In particular, if $\sum_{k=1}^{\infty}(q^k/w_k) < +\infty$ (especially, if $\sum_{k=1}^{\infty}(1/w_k)$ $< +\infty$), then for $i \ge 1$, $E_i[\tau_0|\tau_0 < \infty] < +\infty$.

(ii) If $\sum_{k=1}^{\infty}(g_{k-1}q^k/w_k) = +\infty$, then for $i \ge 1$, $E_i[\tau_0|\tau_0 < \infty] = +\infty$.

**Proof.** Noting that $P_i(\tau_0 < t|\tau_0 < \infty) = p_{i0}(t)/q^i$ and thus by (3.2.18) and Lemma 3.2.3 we have

$$\begin{aligned} E_i[\tau_0|\tau_0 < \infty] &= \int_0^{\infty}(1 - P_i(\tau_0 < t|\tau_0 < \infty))dt \\ &= q^{-i}\int_0^{\infty}(q^i - p_{i0}(t))dt = q^{-i}\int_0^{\infty}(\sum_{k=1}^{\infty} p_{ik}(t)q^k)dt \\ &= \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} q^{k-i}. \end{aligned}$$

By Remark 3.2.1 we see that $\sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} q^k < +\infty \quad (i \ge 1)$ is equivalent to $\sum_{k=1}^{\infty}(g_{k-1}q^k/w_k) < +\infty$ and this together with Lemma 3.2.1 completes the proof. □

We are now in a position to consider the mean explosion time. By Theorem 3.3.1, we only need to consider the case $b_0 < m_b \le +\infty$. Let $\tau_\infty$ denote the explosion time, i.e., the time epoch that the WMBP tends to infinity and let $a_{i\infty} = P_i(\tau_\infty < \infty)$ denote the explosion probability. In addition, let $p_{i\infty}(t)$ denote the explosion probability by time $t$.

**Theorem 3.4.3.** Suppose that $b_0 < m_b \le +\infty$ and $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < \infty$. Then for any $i \ge 1$,

$$a_{i\infty} = 1 - q^i > 0 \tag{3.4.4}$$

and

$$E_i[\tau_\infty|\tau_\infty < \infty] = \frac{1}{1-q^i} \cdot \sum_{k=1}^{\infty} \frac{1}{w_k} \frac{G_i^{(k-1)}(0)}{(k-1)!}(1-q^k) < +\infty. \tag{3.4.5}$$

In particular, if $\sum_{k=1}^{\infty}(1/w_k) < +\infty$, then for all $i \geq 1$, $E_i[\tau_\infty|\tau_\infty < \infty] < +\infty$.

**Proof.** Since $b_0 < m_b \leq +\infty$ and $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < \infty$, we know by Theorems 3.3.2 and 3.3.3 that $p_{i\infty}(t) > 0$ for all $t > 0$ and

$$p_{i\infty}(t) = 1 - \sum_{j=0}^{\infty} p_{ij}(t) \tag{3.4.6}$$

which yields (3.4.4) by letting $t \to \infty$ and using (3.2.14) and (3.2.16). Also noting that

$$P_i(\tau_\infty \leq t|\tau_\infty < \infty) = p_{i\infty}(t)/(1-q^i) \tag{3.4.7}$$

by (3.4.4) and (3.4.7),

$$\begin{aligned} E_i[\tau_\infty|\tau_\infty < \infty] &= \int_0^\infty (1 - P_i(\tau_\infty \leq t|\tau_\infty < \infty))dt \\ &= \frac{1}{1-q^i}\int_0^\infty (1 - q^i - p_{i\infty}(t))dt. \end{aligned}$$

Using (3.4.6) and (3.3.20) in above equality yields that

$$\begin{aligned} & E_i[\tau_\infty|\tau_\infty < \infty] \\ = {} & \frac{1}{1-q^i}\int_0^\infty [\sum_{k=1}^{\infty} p_{ik}(t) - q^i + p_{i0}(t)]dt \\ = {} & \frac{1}{1-q^i}\left[\sum_{k=1}^{\infty}\int_0^\infty p_{ik}(t)dt - \int_0^\infty (q^i - p_{i0}(t))dt\right] \\ = {} & \frac{1}{1-q^i}\sum_{k=1}^{\infty}(1-q^k)\int_0^\infty p_{ik}(t)dt. \end{aligned}$$

Noting (3.2.13) yields that

$$E_i[\tau_\infty|\tau_\infty < \infty] = \frac{1}{1-q^i} \cdot \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!}(1-q^k)$$

which is finite by the assumed condition. In particular, if $\sum_{k=1}^{\infty}(1/w_k) < +\infty$, then $\sum_{k=1}^{\infty}(g_{k-1}/w_k) < +\infty$ since $\{g_k; k \geq 0\}$ is bounded, thus the last statement holds. The proof is complete. □

Finally, we consider the following question. Suppose a WMBP starts at state $i \geq 1$, then before the final extinction or explosion, the process, with probability 1, will "enjoy" its life by wandering over the positive states. We are now interested in obtaining the overall mean holding time at each positive state $j \geq 1$, since it provides important and very useful information regarding the evolution behaviour of the WMBP. Let us agree to call them the global holding times. More specifically, for any $i \geq 1$, $j \geq 1$, let

$$\mu_{ij} =: E_i[I_{\{X(t)=j\}}] = \int_0^\infty p_{ij}(t)dt$$

denote the mean global holding time at state $j$ before extinction or explosion. Clearly $\mu_i = \sum_{j=1}^\infty \mu_{ij}$ is the mean total survival time of the WMBP when the process starts at state $i \geq 1$. However, the solutions to this question have been implied by our previous work and thus we only need to summarize them here.

**Theorem 3.4.4.** Suppose the Feller minimal WMBP starts at state $i \geq 1$.

(i) For any $j \geq 1$, the mean global holding time at state $j$ is always finite and given by

$$\mu_{ij} = \frac{1}{w_j} \cdot \frac{G_i^{(j-1)}(0)}{(j-1)!}. \tag{3.4.8}$$

(ii) The mean total survival time of the WMBP is finite if and only if $\sum_{j=1}^\infty (g_{j-1}/w_j)$ is convergent and, in which case, is given by

$$\mu_i = \sum_{j=1}^\infty \frac{1}{w_j} \cdot \frac{G_i^{(j-1)}(0)}{(j-1)!}. \tag{3.4.9}$$

**Proof.** By the definition of $\mu_{ij}, (i, j \geq 1)$, $\mu_{ij} = \int_0^\infty p_{ij}(t)dt$. On the other hand, by (3.2.13) in Lemma 3.2.3, $\int_0^\infty p_{ij}(t)dt = \frac{1}{w_j} \cdot \frac{G_i^{(j-1)}(0)}{(j-1)!}$. Hence (i) holds. Summing (3.4.8) over $j \geq 1$ yields that (3.4.9). Using Remark 3.2.1, we see that (3.4.9) is finite if and only if $\sum_{j=1}^\infty (g_{j-1}/w_j) < \infty$. The proof is complete. $\square$

## 3.5. Probabilistic Approach

As in nearly every branch of random processes, there are two methods, i.e., analytic method and probabilistic method. In analytic method, we can use the related results in different mathematical branches and thus more explicit results can be obtained in general. However, probabilistic method can bring us deeper probabilistic and intuitive insight.

The methods used in the previous sections are mainly analytic. In this section we shall use probabilistic method to regain most of the results obtained before. From the definition of the $q$-matrix $Q$ of WMBP, we can see that WMBP is closely related with compound Poisson process. The latter is well-discussed and has many deep properties. Therefore, we will use compound Poisson process to study the WMBP.

For this approach, it is more convenient to denote the weight function $w_i$ as $w(i)$ and define $w(0) = 0$. Let $\{Y(t);\ t \geq 0\}$ be a compound Poisson process on the set $\mathbf{Z}$ of all integers with the generator, i.e. the conservative $q$-matrix $Q^* = (q^*_{ij})$, where the elements of $Q^*$ are given by (here only the non-zero off-diagonal elements are specified)

$$q^*_{ij} = \begin{cases} b_{j-i+1}, & if\ \ j > i \\ b_0, & if\ \ j = i - 1. \end{cases} \tag{3.5.1}$$

The properties of this process are, of course, well-known.

Now let $T_0$ be the first hitting time of state 0 of the process $\{Y(t)\}$ and define

$$\eta(t) = \int_0^{t \wedge T_0} \frac{ds}{w(Y(s))}, \qquad t \geq 0.$$

It is clear that $\eta(t)$ is an increasing function of $t \geq 0$ with probability 1 and thus possesses a limit, denoted by $\eta(T_0)$, as $t \to \infty$ as well as a right-inverse function, denoted by $\theta(t)$. It is easily seen that

$$\eta(T_0) = \int_0^{T_0} \frac{ds}{w(Y(s))}$$

is finite on the set $\{T_0 < \infty\}$, for if $T_0 < \infty$ then $Y(t)$ has only finitely many jump points until $T_0$. On the other hand, $\eta(T_0)$ can be either finite or infinite on the set $\{T_0 = \infty\}$. Note that for any $t > 0$, $\theta(t)$ is only well-defined on the set $\{\eta(T_0) > t\}$. If $\eta(T_0) < \infty$, then the process $X(t)$ we are studying stops at time $\eta(T_0)$. For this reason, we shall extend its definition to the set $\{T_0 < \infty, \eta(T_0) \leq t\}$ by defining it as $T_0$. It follows that for any $t \geq 0$ we have

$$\theta(t) = \begin{cases} \inf\{u;\ \int_0^{u\wedge T_0} \frac{ds}{w(Y(s))} = t\}, & if\ \ \eta(T_0) > t \\ T_0, & if\ \ \eta(T_0) \leq t\ and\ T_0 < \infty. \end{cases} \tag{3.5.2}$$

We shall leave $\theta(t)$ undefined on the set $\{\eta(T_0) \leq t,\ T_0 = \infty\}$. Furthermore, for any stopping time $\tau$, particularly the jump times $\sigma_n$, of the compound Poisson process $Y(t)$, we define $\eta(\tau) = \int_0^{\tau\wedge T_0} \frac{ds}{w(Y(s))}$. Now suppose $Y(t)$ starts from some positive state and define

$$X(t) = Y(\theta(t)), \qquad t \geq 0, \tag{3.5.3}$$

then by (3.5.2) and (3.5.3) we have

$$\theta(t) = \int_0^t w(X(s))ds. \tag{3.5.4}$$

Indeed, by (3.5.2) we see that $\theta(t) \leq T_0$ on the set $\{\eta(T_0) > t\}$ and so

$$\int_0^{\theta(t)} \frac{ds}{w(Y(s))} = \int_0^{\theta(t)\wedge T_0} \frac{ds}{w(Y(s))} = \eta(\theta(t)) \equiv t.$$

Hence

$$1 = \frac{1}{w(Y(\theta(t)))} \cdot \frac{d\theta(t)}{dt} = \frac{1}{w(X(t))} \cdot \frac{d\theta(t)}{dt}.$$

This yields (3.5.4) by noting $\theta(0) = 0$. Note that (3.5.4) also holds on the set $\{\eta(T_0) \leq t,\ T_0 < \infty\}$ since we have defined $w(0) = 0$.

By (3.5.2) and (3.5.3) it is clear that the process $X(t)$ possesses the same jump behaviour as $Y(t)$ until $Y(t)$ first hits the state 0 at $T_0$. Furthermore, with probability 1, the process $X(t)$ shares the same sample path behaviour with $Y(t)$ until, again, $Y(t)$ first hits the state 0 at $T_0$ except that the length of sojourn times may not be the same. It is easily seen, however, that the sojourn time at any state $k \geq 1$ of $X(t)$ is the scaled, by $1/w(k)$, sojourn time of the compound Poisson process $Y(t)$ at the same state before the latter first hits the state 0 at $T_0$. The difference is that the process $X(t)$ will stay at 0 forever after it hits the state 0 and thus 0 is an absorbing state for $X(t)$. Therefore $X(t)$ is just the Feller minimal WMBP whose $q$-matrix $Q$ is given in (3.1.1) (see (3.5.1)). Hence we have proved the following obvious conclusion.

**Theorem 3.5.1.** The process $\{X(t);\ t \geq 0\}$ defined in (3.5.3) is the Feller minimal WMBP whose $q$-matrix $Q$ is given in (3.1.1).

In other words, any WMBP, including the well-discussed ordinary Markov branching process, can be viewed as a random time change of some compound Poisson process. This explains the reason why WMBPs share so many common properties with the ordinary Markov branching process. These properties will be shown more clearly below. From now on, we shall always assume that both $X(t)$ (the WMBP process) and $Y(t)$ (the compound Poisson process) start at the same state $i \geq 1$. As defined in Section 4, let $\tau_0 = \inf\{t > 0;\ X(t) = 0\}$ and $\tau_\infty$ denote the hitting time at 0 and the explosion time of $X(t)$, respectively. Also recall that $T_0$ denotes the first hitting time of state 0 of $Y(t)$. It is then easily seen that

$$\eta(T_0) = \begin{cases} \tau_0, & if\ \ T_0 < \infty \\ \tau_\infty, & if\ \ T_0 = \infty. \end{cases} \tag{3.5.5}$$

Furthermore, let $T_k(Y)$ be the time spent at state $k(\geq 1)$ of $Y(t)$ until $Y(t)$ first hits the state 0. Similarly, let

$$T_k(X) = \int_0^\infty I_{\{X(t)=k\}} dt, \quad k \geq 1$$

denote the total time spent at state $k \geq 1$ of the WMBP $X(t)$ where $I_{\{\cdot\}}$ is the indicator function. From the definition of $X(t)$, we know that if $w(n) \equiv 1$ then $T_k(Y) = T_k(X)$. Suppose that $w(n) \equiv 1$ and let $(p_{ij}(t))$ be the $Q$-function of $X(t)$. By Kolmogorov forward equation,

$$\sum_{j=0}^{\infty} p_{ij}(t) s^j - s^i = B(s) \cdot \sum_{k=1}^{\infty} \left(\int_0^t p_{ik}(u) du\right) \cdot s^{k-1}, \quad |s| < 1.$$

Letting $t \uparrow \infty$ in the above equality and noting the definition of $G_i(s)$ yields that

$$G_i(s) = \sum_{k=1}^{\infty} \left(\int_0^\infty p_{ik}(t) dt\right) \cdot s^{k-1}, \quad |s| < 1.$$

Hence,

$$E_i[T_k(Y)] = E_i[T_k(X)] = \int_0^\infty p_{ik}(t) dt = \frac{G_i^{(k-1)}(0)}{(k-1)!}, \qquad i, k \geq 1$$

where $E_i$ denote the conditional expectation when the process starts at state $i \geq 1$ and $G_i(s)$ is defined in Section 3.2. Note that the above result is not new, see, for instance, Chen (1986).

The following result establishes a close link between the properties of the Feller minimal WMBP $X(t)$ and the compound Poisson process $Y(t)$.

**Theorem 3.5.2.** The Feller minimal WMBP $X(t)$ is honest, i.e., the corresponding $q$-matrix $Q$ is regular, if and only if $P_i(\eta(T_0) < \infty,\ T_0 = \infty) = 0$ $(i \geq 1)$. The extinction probability of $X(t)$ is just $P_i(T_0 < \infty)$ under the condition that $X(t)$ starts at $i \geq 1$. In particular,

(i) if $b_0 \geq m_b$, then $Q$ is regular, i.e., the Feller minimal WMBP is honest and, furthermore, the extinction probability is 1.

(ii) If $\Sigma_{n=1}^{\infty} 1/w(n) < \infty$ then $Q$ is regular (i.e., the Feller minimal WMBP is honest) if and only if $b_0 \geq m_b$.

(iii) If $b_0 < m_b \leq +\infty$ then the extinction probability is $q^i$ when the process starts at $i \geq 1$ where $q < 1$ is the smallest positive root of $B(s) = 0$ on $[0, 1]$.

Furthermore, for any $k \geq 1$, the following relation holds

$$T_k(X) = T_k(Y)/w(k). \tag{3.5.6}$$

**Proof.** By (3.5.3) it is easily seen that the WMBP is honest if and only if for all $t \geq 0$, $\theta(t)$ is well-defined almost surely. However, the latter happens if and only if for all $t \geq 0$, $P_i(\eta(T_0) \leq t,\ T_0 = \infty) = 0$, see (3.5.2) and the sentence immediately follows it. It is easily seen that this last condition is equivalent to $P_i(\eta(T_0) < \infty,\ T_0 = \infty) = 0$. The extinction probability of $X(t)$ is $P_i(\tau_0 < \infty)$, which is just $P_i(T_0 < \infty)$ by (3.5.5). This proves the first part of the theorem.

Now, if $b_0 = m_b$ then the compound Poisson process $Y(t)$ is recurrent and thus will visit state 0 infinitely many times with probability 1 by irreducibility. Hence for all $i \geq 1$, $P_i(T_0 < \infty) = 1$ which yields all the conclusions in (i). If $b_0 > m_b$, although transient, the compound Poisson process will eventually drift to $-\infty$ and thus will hit 0 with probability 1 (assuming the process starts at $i \geq 1$). Therefore again $P_i(T_0 < \infty) = 1$ which proves (i).

In order to prove (ii), we first prove (3.5.6). Denote the successive jump times of $Y(t)$ until $T_0$ as $\{\sigma_n;\ n \geq 1\}$ and let $\sigma_0 = 0$. Now recall that $\eta(\sigma_n)$ is defined as $\int_0^{\sigma_n \wedge T_0}(w(Y(s)))^{-1}ds$, we see that $\eta(\sigma_0) = 0$ and $\{\eta(\sigma_n);\ n \geq 1\}$ are the jump times of $X(t)$. Hence

$$T_k(X) = \sum_{n=0}^{\infty}(\eta(\sigma_{n+1}) - \eta(\sigma_n)) \cdot I_{\{X(\eta(\sigma_n))=k\}}$$
$$= \sum_{n=0}^{\infty} \frac{\sigma_{n+1} - \sigma_n}{w(k)} \cdot I_{\{Y(\sigma_n)=k\}} \cdot I_{\{\sigma_n < T_0\}} = \frac{T_k(Y)}{w(k)}.$$

We are now ready to prove (ii). Note first that we actually have

$$\sum_{k=1}^{\infty} T_k(X) = \eta(T_0). \tag{3.5.7}$$

Now, if $\sum_{k=1}^{\infty}(1/w(k)) < \infty$, then by (3.5.6) we have that for any $i \geq 1$

$$E_i[\eta(T_0)] = \sum_{k=1}^{\infty} \frac{1}{w(k)} \cdot E_i[T_k(Y)] < \infty$$

since $\{E_i[T_k(Y)];\ k \geq 1\}$ is bounded. Therefore we have $P_i(\eta(T_0) < \infty) = 1$ and hence $P_i(\eta(T_0) < \infty,\ T_0 = \infty) = P_i(T_0 = \infty)$ which is zero if and only if $b_0 \geq m_b$. This proves (ii).

The proof of (iii) is immediate. Indeed, by the just proven result, the extinction probability of $X(t)$ is the hitting probability of state 0 of the corresponding compound Poisson process $Y(t)$. For the latter, the conclusion (iii) is well-known. It can also be easily obtained by using the corresponding result of the ordinary MBP. $\square$

The following result can now be easily proven.

**Theorem 3.5.3**. The conclusions in Theorems 3.4.1–3.4.4 hold.

**Proof.** It follows from (3.5.6) that for any $i, k \geq 1$

$$E_i[T_k(X)] = \frac{1}{w(k)} \cdot E_i[T_k(Y)]$$

which is just (3.4.8) in Theorem 3.4.4. Expression (3.4.9) then easily follows by summation over $k \geq 1$. Hence Theorem 3.4.4 is proven.

If $b_0 \geq m_b$ then $\tau_0 = \sum_{k=1}^{\infty} T_k(X)$ and hence

$$E_i[\tau_0] = \sum_{k=1}^{\infty} \frac{1}{w(k)} \cdot E_i[T_k(Y)].$$

If $b_0 < m_b \leq +\infty$ then

$$\tau_0 = \sum_{k=1}^{\infty} T_k(X) \qquad \text{on} \;\; \{\tau_0 < \infty\}$$

since $\{\tau_0 < \infty\}$ happens if and only if $\{T_0 < \infty\}$. So

$$E_i[\tau_0 | \tau_0 < \infty] = \sum_{k=1}^{\infty} \frac{1}{w(k)} \cdot E_i[T_k(Y) | T_0 < \infty].$$

On the other hand, it is easy to see that

$$E_i[T_k(Y) | T_0 < \infty] = q^{k-i} E_i[T_k(Y)]. \tag{3.5.8}$$

Indeed,

$$\begin{aligned} E_i[T_k(Y) \cdot I_{\{T_0<\infty\}}] &= \int_0^{\infty} P_i(Y(t) = k, \; t < T_0 < \infty) dt \\ = & \int_0^{\infty} E_i[I_{\{Y(t)=k, \; t<T_0\}} \cdot P_k(T_0 < \infty)] dt \\ = & \; q^k \int_0^{\infty} P_i(Y(t) = k, \; t < T_0) dt = q^k E_i[T_k(Y)]. \end{aligned}$$

This yields (3.5.8). Theorems 3.4.1 and 3.4.2 then immediately follow. We now consider Theorem 3.4.3. Assume that $b_0 < m_b \leq +\infty$. It is easily seen from (3.5.5) and (3.5.7) that for any $i \geq 1$

$$P_i(\sum_{k=1}^{\infty} T_k(X) < \infty) = P_i(\tau_0 < \infty) + P_i(\tau_\infty < \infty).$$

If $\sum_{k=1}^{\infty}(g_k/w(k)) < \infty$ then $E_i[\sum_{k=1}^{\infty} T_k(X)] < \infty$ which implies that $P_i(\sum_{k=1}^{\infty} T_k(X) < \infty) = 1$. By Theorem 3.5.2(iii),

$$a_{i\infty} = P_i(\tau_\infty < \infty) = P_i(\sum_{k=1}^{\infty} T_k(X) < \infty) - P_i(\tau_0 < \infty) = 1 - q^i$$

which is (3.4.4). Also clearly, we have

$$E_i[\tau_\infty | \tau_\infty < \infty] = \sum_{k=1}^{\infty} E_i[T_k(X) | \tau_\infty < \infty].$$

However,

$$\begin{aligned} E_i[T_k(X) \cdot I_{\{\tau_\infty<\infty\}}] &= \int_0^{\infty} P_i[X(t) = k, \; t < \tau_\infty < \infty] dt \\ = & \int_0^{\infty} E_i[I_{\{X(t)=k\}} \cdot P_k[\tau_\infty < \infty]] dt = (1 - q^k) E_i[T_k(X)], \end{aligned}$$

and thus

$$E_i[\tau_\infty | \tau_\infty < \infty] = \frac{1}{1-q^i} \cdot \sum_{k=1}^{\infty} \frac{1-q^k}{w(k)} \cdot E_i[T_k(Y)].$$

This proves (3.4.5) and the proof of Theorem 3.4.3 is thus complete. □

## 3.6. Non-linear Markov Branching Processes

An important class of weighted branching model, which was considered in Chen (2002a,b), is non-linear Markov branching process, i.e., $w_i = i^\theta$ $(i \geq 1)$, where $\theta$ is a positive constant. Note that the ordinary MBP is a special case of the non-linear Markov branching processes, i.e., $\theta = 1$. The key property of ordinary MBP is the branching property, i.e., different particles act independently when they give offspring. However, different from the ordinary MBP, the non-linear branching process no longer has this independence property, i.e., the particles may act dependently when they give offspring. One of the basic questions in studying the non-linear Markov branching process is the regularity problem. For the ordinary MBP, this question was first answered by Harris (1963) who obtained a very satisfactory criterion, known as the Harris condition.

Unfortunately this regularity problem still remains open for the general non-linear Markov branching processes, in spite of the fact that extensive progress has already been made for such processes. This seriously affects the analysis of the non-linear branching process. Indeed without knowing whether the corresponding process is honest or not, it is hard to deepen the further investigation.

The aim of this section is to answer this important question. We shall provide a criterion which is very easy to check. It is a natural generalisation of the Harris condition and so we call it the general Harris criterion.

A $q$-matrix $Q = (q_{ij}; i, j \in \mathbf{Z}_+)$ defined in (3.1.1) is called non-linear branching $q$-matrix if

$$w_i = i^\theta, \quad i \geq 1. \tag{3.6.1}$$

where $\theta > 0$ is a constant.

A non-linear branching $q$-matrix is called super-linear if $\theta > 1$, and sub-linear if $0 < \theta \leq 1$. Note that this classification is based on whether $\sum_{n=1}^{\infty} n^{-\theta}$ is finite or not.

A WMBP is called a non-linear Markov branching process if its $q$-matrix is a non-linear branching $q$-matrix. The non-linear Markov branching process is called super-linear or sub-linear based on the corresponding $q$-matrix $Q$ is super-linear or sub-linear.

Before giving the main results, we need the following lemma which can be seen in Chen (2002a).

**Lemma 3.6.1.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a non-linear branching $q$-matrix given in (3.6.1). Then for any $i \geq 1$ and $0 \leq s \leq 1$,

$$\sum_{k=1}^{\infty} p_{ik}(t) s^k = \frac{1}{\Gamma(\theta)} \int_0^s \frac{\sum_{j=0}^{\infty} p'_{ij}(t) y^j}{B(y)} \cdot (\ln \frac{s}{y})^{\theta-1} dy \tag{3.6.2}$$

and

$$\sum_{k=1}^{\infty} \phi_{ik}(\lambda) s^k = \frac{1}{\Gamma(\theta)} \int_0^s \frac{\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) y^j - y^i}{B(y)} \cdot (\ln \frac{s}{y})^{\theta-1} dy. \tag{3.6.3}$$

where $\Gamma(\theta)$ is the gamma function.

Now we can present our main results in the following.

**Theorem 3.6.2.** Let $Q$ be the non-linear branching $q$-matrix. If $b_0 \geq m_b$ then $Q$ is regular. While if $b_0 < m_b \leq +\infty$ then the following statements are equivalent.

(i) The non-linear branching $q$-matrix $Q$ is regular, i.e., the corresponding non-linear Markov branching process is honest.

(ii) For some (or equivalently for all ) $\varepsilon \in (q, 1)$, we have

$$\int_\varepsilon^1 \frac{(1-s)^{\theta-1}}{-B(s)} ds = +\infty. \tag{3.6.4}$$

(iii) For some (or equivalently for all ) $\varepsilon \in (q, 1)$, we have

$$\int_\varepsilon^1 \frac{(-\ln s)^{\theta-1}}{-B(s)} ds = +\infty. \tag{3.6.5}$$

(iv) The following integral is infinite, i.e.,

$$\int_0^1 \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds = +\infty. \tag{3.6.6}$$

(v) For any $i \geq 1$,

$$\lim_{t\to\infty} \sum_{j=1}^{\infty} p_{ij}(t) > 0 \tag{3.6.7}$$

where $(p_{ij}(t);\ i,j \in \mathbf{Z}_+)$ is the $Q$-function of the corresponding non-linear Markov branching process.

**Proof.** If $b_0 \geq m_b$, the statement is just Theorem 3.3.1. We now assume that $b_0 < m_b \leq +\infty$. Note that $\lim_{s\uparrow 1} \frac{-\ln s}{1-s} = 1$, we see that (ii)$\Leftrightarrow$(iii). By the definition of $G(s)$ and Lemma 3.2.1, we see that for $\varepsilon \in (q,1)$,

$$\frac{q-s}{B(s)} = G(s) \leq G(\varepsilon) < \infty, \quad s \in [0,\varepsilon].$$

Thus, for $\varepsilon \in (q,1)$,

$$\begin{aligned} & \int_0^{\varepsilon} \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds \\ \leq\ & G(\varepsilon) \cdot \int_0^1 (-\ln s)^{\theta-1} ds \\ =\ & G(\varepsilon) \cdot \int_1^{\infty} x^{\theta-1} e^{-x} dx s)^{\theta-1} ds \\ =\ & G(\varepsilon) \cdot \Gamma(\theta) < +\infty. \end{aligned}$$

i.e., for any $\varepsilon \in (q,1)$, we have

$$\int_0^{\varepsilon} \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds < +\infty. \tag{3.6.8}$$

Now we rewrite the integral in (3.6.6) as

$$\begin{aligned} & \int_0^1 \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds \\ =\ & \int_0^{\varepsilon} \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds + \int_{\varepsilon}^1 \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds, \end{aligned}$$

where $\varepsilon \in (q,1)$. By (3.6.8), (iv) is equivalent to $\int_{\varepsilon}^1 \frac{(q-s)(-\ln s)^{\theta-1}}{B(s)} ds = \infty$. But $\lim_{s\uparrow 1} \frac{(q-s)(\ln s)^{\theta-1}}{(1-s)^{\theta-1}} = q-1$, thus (iv)$\Leftrightarrow$(ii). Hence to complete the proof, we only need to show that (i)$\Rightarrow$(v)$\Rightarrow$(iii)$\Rightarrow$(i).

We first prove that (iii) implies (i). Now suppose (iii) holds but the non-linear Markov branching process is dishonest. Then $1-\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda) > 0$ for some (therefore for all) $\lambda > 0$ and some $i \geq 1$. Denote this quantity by $\rho(\lambda)$. So for a fixed $\lambda > 0$ and $i \geq 1$, there exists an $\varepsilon$ sufficiently close to 1 such that $s^i - \lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j \geq \frac{\rho(\lambda)}{2} > 0$ for all $s \in [\varepsilon, 1]$. Therefore for such $i$ and $\lambda$, we have by (iii) that

$$\int_0^1 \frac{\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j - s^i}{B(s)}(-\ln s)^{\theta-1}ds \geq \frac{\rho(\lambda)}{2}\int_\varepsilon^1 \frac{(-\ln s)^{\theta-1}}{-B(s)}ds = +\infty.$$

Using (3.6.3)for $s = 1$ we obtain

$$\sum_{j=1}^{\infty}\phi_{ij}(\lambda) = +\infty$$

which is a contradiction since we always have $\sum_{j=0}^{\infty}\phi_{ij}(\lambda) \leq 1/\lambda$. Thus $P(t)$ is honest and (i) is proved.

Secondly we prove that (v) implies (iii). Suppose that (iii) does not hold, i.e., $\int_\varepsilon^1 \frac{(-\ln y)^{\theta-1}}{-B(y)}dy < +\infty$ for some $\varepsilon \in (q, 1)$. Note that, by Theorem 1.2.2, $|\sum_{j=0}^{\infty}p'_{ij}(t)y^j| \leq \sum_{j=0}^{\infty}|p'_{ij}(t)| \leq 2q_i \equiv -2w_ib_1$ for any given $i \geq 1$, then we have that for $y \in [\varepsilon, 1)$,

$$\begin{aligned}
&\frac{|\sum_{j=0}^{\infty}p'_{ij}(t)y^j|}{|B(y)|}(-\ln y)^{\theta-1} \\
\leq\ & 2q_i\frac{(-\ln y)^{\theta-1}}{|B(y)|} \\
=\ & 2q_i\frac{(-\ln y)^{\theta-1}}{-B(y)}
\end{aligned}$$

since $B(s) < 0$ for $s \in (\varepsilon, 1)$. By the assumption, $\int_\varepsilon^1 \frac{(-\ln y)^{\theta-1}}{-B(y)}dy < \infty$, thus we are justified to apply the Dominated Convergence Theorem to obtain

$$\begin{aligned}
&\lim_{t\to\infty}\int_\varepsilon^1 \frac{\sum_{j=0}^{\infty}p'_{ij}(t)y^j}{B(y)}(-\ln y)^{\theta-1}dy \\
=\ & \int_\varepsilon^1 (\lim_{t\to\infty}\sum_{j=0}^{\infty}p'_{ij}(t)y^j)\cdot\frac{(-\ln y)^{\theta-1}}{B(y)}dy \\
=\ & \int_\varepsilon^1 (\sum_{j=0}^{\infty}\lim_{t\to\infty}p'_{ij}(t)y^j)\cdot\frac{(-\ln y)^{\theta-1}}{B(y)}dy = 0 \qquad (3.6.9)
\end{aligned}$$

since $\lim_{t\to\infty} p'_{ij}(t) = 0$ for all $i$ and $j$ (see, again, Chung (1967)).

In addition to (3.6.9), we also have from (3.2.5) (with $w_k = k^\theta$) that

$$\begin{aligned} & \left|\frac{\sum_{j=0}^{\infty} p'_{ij}(t)y^j}{B(y)}(-\ln y)^{\theta-1}\right| \\ = & \sum_{k=1}^{\infty} p_{ik}(t)k^\theta y^{k-1}(-\ln y)^{\theta-1} \\ \leq & \left(\sum_{k=1}^{\infty} k^\theta \varepsilon^{k-1}\right)(-\ln y)^{\theta-1} \\ = & M\cdot(-\ln y)^{\theta-1}, \quad y\in[0,\varepsilon) \end{aligned}$$

and

$$\int_0^\varepsilon M\cdot(-\ln y)^{\theta-1}dy = M\cdot\int_0^1(-\ln y)^{\theta-1}dy = M\Gamma(\theta) < +\infty,$$

where the positive constant $M = \sum_{k=1}^{\infty} k^\theta\varepsilon^{k-1}$ (independent of $t$) is finite. Hence we may, again, use the Dominated Convergence Theorem to get

$$\lim_{t\to\infty}\int_0^\varepsilon \frac{\sum_{j=0}^{\infty} p'_{ij}(t)y^j}{B(y)}(-\ln y)^{\theta-1}dy = 0 \tag{3.6.10}$$

since $\lim_{t\to\infty} p'_{ij}(t) = 0$ for all $i, j \geq 0$. Combining (3.6.9) with (3.6.10) yields that

$$\lim_{t\to\infty}\int_0^1 \frac{\sum_{j=0}^{\infty} p'_{ij}(t)y^j}{B(y)}(-\ln y)^{\theta-1}dy = 0.$$

Letting $s = 1$ in (3.6.2) yields that

$$\sum_{k=1}^{\infty} p_{ik}(t) = \frac{1}{\Gamma(\theta)}\int_0^1 \frac{\sum_{j=0}^{\infty} p'_{ij}(t)y^j}{B(y)}\cdot(-\ln y)^{\theta-1}dy.$$

Therefore, letting $t\to\infty$ in the above equality yields that

$$\lim_{t\to\infty}\sum_{j=1}^{\infty} p_{ij}(t) = \frac{1}{\Gamma(\theta)}\lim_{t\to\infty}\int_0^1 \frac{\sum_{j=0}^{\infty} p'_{ij}(t)y^j}{B(y)}(-\ln y)^{\theta-1}dy = 0$$

which contradicts with (v). Hence (v) does implies (iii).

Finally, we prove that (i) implies (v). Note that for any $i \geq 1$, it is easy to see that $\sum_{j=1}^{\infty} p_{ij}(t)$ is a non-increasing function of $t$ and thus when $t\to\infty$, the limit does exist. Indeed, write $\sum_{j=1}^{\infty} p_{ij}(t) = \sum_{j=0}^{\infty} p_{ij}(t) - p_{i0}(t)$. It is well known that the first term is a non-increasing function and $p_{i0}(t)$ is an increasing function of $t > 0$ and thus the difference is non-increasing. Hence if (v) is not true, then this limit must be zero. By honesty, we then obtain

$$\lim_{t\to\infty} p_{i0}(t) = 1 - \lim_{t\to\infty} \sum_{j=1}^{\infty} p_{ij}(t) = 1$$

which contradicts to $b_0 < m_b \leq +\infty$ since for the case $b_0 < m_b \leq +\infty$ we have $\lim_{t\to\infty} p_{i0}(t) = q^i < 1$ (since that by Theorem 3.3 of Chen (2002a), if $b_0 < m_b$ then the extinction probability for non-linear Markov branching process starting at state $i \geq 1$ is $q^i < 1$). The proof is thus complete. $\square$

Note that although criterion (3.6.4) is the most simple one, criteria (3.6.5)– (3.6.7) are more essential since they possess a clear probabilistic interpretation.

The following two corollaries are direct consequence of Theorem 3.6.2 but are much more informative.

**Corollary 3.6.3.** A super-linear Markov branching process (i.e., $\theta > 1$) is honest (i.e., the corresponding $q$-matrix $Q$ is regular) if and only if $b_0 \geq m_b$.

**Proof.** Just note that if $\theta > 1$ and $b_0 < m_b \leq +\infty$ then the integral in (3.6.4) is always finite since by Lemma 2.2.1(ii), 1 is a simple root of $B(s) = 0$. $\square$

**Corollary 3.6.4.** A sub-linear Markov branching process (i.e., $\theta \leq 1$) is honest (i.e., the corresponding sub-linear branching $q$-matrix $Q$ is regular) if and only if either

(i) $m_b < +\infty$ or

(ii) $m_b = +\infty$ together with the requirement that for some (or, equivalently for all) $\varepsilon \in (q, 1)$ we have

$$\int_{\varepsilon}^{1} \frac{(1-s)^{\theta-1}}{-B(s)} ds = \infty \tag{3.6.11}$$

where $q \in (0, 1)$ is the smallest root of $B(s) = 0$ on $[0, 1]$.

**Proof.** Let $0 < \theta \leq 1$. If $m_b = \infty$, then (3.6.11) is just the same integral as (3.6.4). Therefore, we only need to prove that if $b_0 < m_b < +\infty$ then the process is honest. If $b_0 \geq m_b$ then by Theorem 3.6.2, the process is honest. If $b_0 < m_b < +\infty$, then

$$\lim_{s\uparrow 1} \frac{1-s}{-B(s)} = \frac{1}{B'(1)} = \frac{1}{m_b - m_d}.$$

It then follows that

$$\int_{\varepsilon}^{1} \frac{(1-s)^{\theta-1}}{B(s)} ds = \infty$$

since $\int_{\varepsilon}^{1}(1-s)^{\theta-2}ds = \infty$. Hence by Theorem 3.6.2 (ii), the process is honest. The proof is complete. $\square$

Note that by Corollary 3.6.4, in particular, putting $\theta = 1$ in (3.6.11) we immediately recover the original Harris condition. Also by Corollaries 3.6.3 and 3.6.4 we see that in most cases, we do not need to check whether the integrals (3.6.4)–(3.6.6) are convergent or not. Indeed, only in the case $m_b = +\infty$ for the sub-linear process do we need to check the integral (3.6.11) directly. In addition, by Corollaries 3.6.3 and 3.6.4 we see that there is a substantial difference in the honesty condition between the super-linear and sub-linear Markov branching processes.

As another direct consequence of Theorem 3.6.2, we may obtain the following interesting result which provides further information regarding the limit in (3.6.7) and will be very useful in further investigation.

**Corollary 3.6.5.** Let $P(t) = (p_{ij}(t);\ i, j \geq 0)$ be the transition function of a non-linear Markov branching process and denote $\sigma_i(t) = \sum_{j=0}^{\infty} p_{ij}(t)$. If the process is super-linear, then for all $i \geq 1$,

(i) If $b_0 \geq m_b$ then

$$\sigma_i(t) \equiv 1,\ p_{i0}(t) \uparrow 1 \ \ \textit{and} \ \sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0 \ \ (\textit{as } t \to \infty).$$

(ii) If $b_0 < m_b \leq +\infty$ then

$$1 > \sigma_i(t) \downarrow q^i,\ \ p_{i0}(t) \uparrow q^i \ \ \textit{and} \ \sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0 \ \ (\textit{as } t \to \infty).$$

While if the process is sub-linear, then for all $i \geq 1$,

(iii) If $b_0 \geq m_b$ then

$$\sigma_i(t) \equiv 1, \quad p_{i0}(t) \uparrow 1 \quad and \quad \sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0 \qquad (as\ t \to \infty).$$

(iv) If $b_0 < m_b < +\infty$ or if $m_b = +\infty$ but the integral in (3.6.4) is divergent, then

$$\sigma_i(t) \equiv 1, \quad p_{i0}(t) \uparrow q^i \quad and \quad \sum_{j=1}^{\infty} p_{ij}(t) \downarrow 1 - q^i \qquad (as\ t \to \infty).$$

(v) If $m_b = +\infty$ and the integral in (3.6.4) is convergent, then

$$1 > \sigma_i(t) \downarrow q^i, \quad p_{i0}(t) \uparrow q^i \quad and \quad \sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0 \qquad (as\ t \to \infty).$$

Where $q \in (0, 1)$ appeared in (ii), (iv) and (v) is the smallest root of $B(s) = 0$ on $[0, 1]$.

**Proof.** We first note that, by the basic property of transition function, $\sigma_i(t)$ is a non-increasing function of $t$. Since 0 is the absorbing state, we know that $p_{i0}(t)$ is an increasing function of $t$ and hence $\sum_{j=1}^{\infty} p_{ij}(t) = \sigma_i(t) - p_{i0}(t)$ is a decreasing function of $t$.

Suppose that $b_0 \geq m_b$. By Theorem 3.6.2, $Q$ is regular, i.e., $\sigma_i(t) \equiv 1$. By Theorem 3.4.1, $p_{i0}(t) \uparrow 1$ and thus $\sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0$. (i) and (iii) have been proved.

Suppose that the process is super-linear and $b_0 < m_b \leq \infty$. By Theorem 3.4.1, $p_{i0}(t) \uparrow q^i$. By Corollary 3.6.3, $Q$ is not regular, i.e., $\sigma_i(t) < 1$. Furthermore, by Theorem 3.6.2, we must have $\sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0$ and thus $\sigma_i(t) \downarrow q^i$. (ii) is proved.

Suppose that the process is sub-linear. If $b_0 < m_b \leq \infty$ or if $m_b = \infty$ but the integral in (3.6.4) is divergent, then by Theorem 3.6.2, $Q$ is regular, i.e., $\sigma_i(t) \equiv 1$. Again by Theorem 3.4.1, $p_{i0}(t) \uparrow q^i$ and thus $\sum_{j=1}^{\infty} p_{ij}(t) \downarrow 1 - q^i$. (iv) is proved.

Suppose that the process is sub-linear. If $m_b = \infty$ and the integral in (3.6.4) is convergent, then by Theorem 3.6.2, $Q$ is not regular, i.e.,

$\sigma_i(t) < 1$ and $\sum_{j=1}^{\infty} p_{ij}(t) \downarrow 0$. Again by Theorem 3.4.1, $p_{i0}(t) \uparrow q^i$ and thus $\sigma_i(t) \downarrow q^i$. (v) is proved. □

Note that for the ordinary Markov branching process, by Lemma 2.2 in Chen and Renshaw( 1993b), if the process is honest then $p_{10}(t) \uparrow q^*$, $\sigma_1(t) \downarrow$ and so $\lim_{t\to\infty} \sigma_1(t) = \sigma$ exists. Furthermore,

(a) $\sum_{j=0}^{\infty} p_{ij}(t) \downarrow \sigma^i \quad (\forall k \geq 0,\ t \to \infty)$;

(b) $0 \leq q \leq \sigma \leq 1$, with $\sigma < 1$ if and only if $Q$ is not regular;

(c) if $\sigma < 1$ then $\sigma = q$;

(d) if $Q$ is regular then $\sigma_1(t) \equiv \sigma = 1$;

(e) if $Q$ is not regular then $\sigma_1(t) < 1$ $(\forall t > 0)$ and $\sigma_1(t) \downarrow \sigma < 1$.

These results agree with Corollary 3.6.5 in the case $\theta = 1$.

**Theorem 3.6.6.** For the non-linear Markov branching process. The extinction probability is given in (3.4.1) and the mean extinction time is finite if and only if both $b_0 \geq m_b$ and $\int_0^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\theta-1} dy < \infty$ and in which case is given by

$$E_i[\tau_0] = \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y^i}{B(y)} \cdot (-\ln y)^{\theta-1} dy.$$

Moreover, if $b_0 < m_b \leq +\infty$, then the conditional mean extinction time is given by

$$E_i[\tau_0 | \tau_0 < \infty] = \frac{1}{q^i \Gamma(\theta)} \cdot \int_0^q \frac{q^i - y^i}{B(y)} \cdot (\ln \frac{q}{y})^{\theta-1} dy,$$

while if $b_0 < m_b \leq +\infty$ and $\int_0^1 \frac{q-y}{B(y)} \cdot (-\ln y)^{\theta-1} dy < \infty$, then the explosion probability and mean explosion time are given by

$$a_{i\infty} = 1 - q^i$$

and

$$\begin{aligned} &E_i[\tau_\infty | \tau_\infty < \infty] \\ = \ &\frac{1}{(1-q^i)\Gamma(\theta)} \cdot \int_0^1 [\frac{q^i - y^i}{B(y)} - \frac{q^{i+1}(1-y^i)}{B(qy)}] \cdot (-\ln y)^{\theta-1} dy \end{aligned}$$

respectively. Finally, the mean global holding time and mean total survival time are given by

$$\mu_{ij} = \frac{1}{j^\theta} \cdot \frac{G_i^{(j-1)}(0)}{(j-1)!}, \quad i, j \geq 1$$

and

$$\mu_i = \begin{cases} \frac{1}{\Gamma(\theta)} \int_0^1 \frac{1-y^i}{B(y)} \cdot (-\ln y)^{\theta-1} dy, & if \;\; b_0 \geq m_b \\ \frac{1}{\Gamma(\theta)} \int_0^1 \frac{q^i-y^i}{B(y)} \cdot (-\ln y)^{\theta-1} dy, , & if \;\; b_0 < m_b \leq +\infty. \end{cases}$$

**Proof.** Recall that the extinction probability given in (3.4.1) does not depend on the sequence $\{w_k; k \geq 1\}$, so the extinction probability for the non-linear case is still given in (3.4.1).

Note that for $s \in [0, 1)$,

$$\begin{aligned} & \int_0^s G_i(y) \cdot (\ln \frac{s}{y})^{\theta-1} dy \\ = & \sum_{k=1}^{\infty} \frac{G_i^{(k-1)}(0)}{(k-1)!} \cdot \int_0^s y^{k-1} (\ln \frac{s}{y})^{\theta-1} dy \\ = & \Gamma(\theta) \cdot \sum_{k=1}^{\infty} \frac{1}{k^\theta} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} s^k. \end{aligned} \tag{3.6.12}$$

If $b_0 \geq m_b$, then letting $s \uparrow 1$ in the above equality and noting $G_i(y) = \frac{1-y^i}{B(y)}$ yields that

$$\int_0^1 \frac{1-y^i}{B(y)} \cdot (-\ln y)^{\theta-1} dy = \Gamma(\theta) \cdot \sum_{k=1}^{\infty} \frac{1}{k^\theta} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!}.$$

In particular,

$$\int_0^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\theta-1} dy = \Gamma(\theta) \cdot \sum_{k=1}^{\infty} \frac{g_{k-1}}{k^\theta}$$

since $g_k = \frac{G_1^{(k)}(0)}{k!} = \frac{G^{(k)}(0)}{k!}$. Using Theorem 3.4.1 and Substituting above two equations into the expressions in Theorem 3.4.1 yields that the assertion regarding $E_i[\tau_0]$ holds.

If $b_0 < m_b \leq \infty$, then letting $s \uparrow q$ in (3.6.12) and noting $G_i(y) = \frac{q^i-y^i}{B(y)}$ yields that

$$\int_0^q \frac{1-y^i}{B(y)} \cdot (\ln \frac{q}{y})^{\theta-1} dy = \Gamma(\theta) \cdot \sum_{k=1}^{\infty} \frac{1}{k^\theta} \cdot \frac{G_i^{(k-1)}(0)}{(k-1)!} q^k.$$

In particular,

$$\int_0^q \frac{q-y}{B(y)} \cdot (\ln \frac{q}{y})^{\theta-1} dy = \Gamma(\theta) \cdot \sum_{k=1}^{\infty} \frac{g_{k-1} q^k}{k^\theta}.$$

Using Theorem 3.4.2 and Substituting above two equations into the expressions in Theorem 3.4.2 yields that the assertion regarding $E_i[\tau_0|\tau_0 < \infty]$ holds.

If $b_0 < m_b \leq \infty$, letting $s \uparrow 1$ in (3.6.12) yields that

$$\sum_{k=1}^{\infty} \frac{g_{k-1}}{k^\theta} = \frac{1}{\Gamma(\theta)} \cdot \int_0^1 \frac{q^i - y^i}{B(y)} \cdot (-\ln y)^{\theta-1} dy.$$

By Theorem 3.4.3, the assertions regarding explosion probability and explosion time hold. Finally, by Theorem 3.4.4 and above expressions, the last two expressions are true. The proof is complete. □

### 3.7. Examples

In this section we give some examples, which cover some applicable models, to illustrate our results. It will be seen that the calculation of the quantities involved, in particular, the quantities (3.4.2), (3.4.3), (3.4.5) and (3.4.8), are usually quite simple.

We first give two examples in which the sequence $\{b_j;\ \ j \geq 1\}$ takes some special form while the sequence $\{w_j;\ \ j \geq 1\}$ is arbitrary.

**Example 3.7.1.** Let $b_0 = d > 0$, $b_j = 0$ $(\forall j \geq 3)$ and $b_2 = b > 0$. This is just the case of birth-death process with an absorbing state zero. For this example, it is easily seen that $B(s) = (1-s)(d-bs)$ and that

$$\frac{G_i^{(k-1)}(0)}{(k-1)!} = \begin{cases} \frac{1}{b}\sum_{j=1}^{k\wedge i}(\frac{d}{b})^{i-j}, & if\ \ d < b \\ \frac{k\wedge i}{b}, & if\ \ d = b \\ \frac{1}{d}\sum_{j=1}^{k\wedge i}(\frac{b}{d})^{k-j}, & if\ \ d > b. \end{cases}$$

Applying our results obtained in the previous sections, we can get the following conclusion.

**Theorem 3.7.1.** (i) There exists only one birth-death WMBP if and only if one of the following conditions is satisfied:

(a) $d \geq b$,

(b) $d < b$ and $\sum_{k=1}^{\infty} \frac{1}{w_k} = +\infty$,

(c) $d < b$, $\sum_{k=1}^{\infty} \frac{1}{w_k} < +\infty$ and $\sum_{k=1}^{\infty} \frac{1}{w_k}(\frac{b}{d})^k = +\infty$.

Moreover, $Q$ is regular if and only if either (a) or (b) occurs.

(ii) The extinction probability and the mean extinction time are given by

$$a_{i0} = \begin{cases} (\frac{d}{b})^i < 1, & if \;\; d < b \\ 1, & if \;\; d \geq b \end{cases}$$

and

$$E_i[\tau_0] = \begin{cases} \frac{1}{d}\sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \sum_{j=1}^{k\wedge i}(\frac{b}{d})^{k-j}, & if \;\; d > b, \;\; \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot (\frac{b}{d})^k < \infty \\ \frac{1}{b}\sum_{k=1}^{\infty} \frac{k\wedge i}{w_k}, & if \;\; d = b, \;\; \sum_{k=1}^{\infty} \frac{1}{w_k} < \infty \\ +\infty, & otherwise. \end{cases}$$

(iii) If $d < b$, then the conditional mean extinction time is given by

$$E_i[\tau_0 | \tau_0 < \infty] = \begin{cases} \frac{1}{b}\sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \sum_{j=1}^{k\wedge i}(\frac{d}{b})^{k-j}, & if \;\; \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot (\frac{d}{b})^k < \infty \\ +\infty, & if \;\; \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot (\frac{d}{b})^k = \infty. \end{cases}$$

(iv) If $d < b$ and $\sum_{k=1}^{\infty} \frac{1}{w_k} < \infty$, then the explosion probability and the mean explosion time are given by

$$a_{i\infty} = 1 - (\frac{d}{b})^i$$

and

$$E_i[\tau_\infty | \tau_\infty < \infty] = (b(1 - (\frac{d}{b})^i))^{-1} \cdot \sum_{k=1}^{\infty} \frac{1 - (\frac{d}{b})^k}{w_k} \cdot \sum_{j=1}^{k\wedge i} (\frac{d}{b})^{i-j}.$$

(v) The mean global holding times are given by

$$\mu_{ik} = \frac{1}{w_k} \cdot \begin{cases} \frac{1}{b}\sum_{j=1}^{k\wedge i}(\frac{d}{b})^{i-j}, & if \;\; d < b \\ \frac{k\wedge i}{b}, & if \;\; d = b \\ \frac{1}{d}\sum_{j=1}^{k\wedge i}(\frac{b}{d})^{k-j}, & if \;\; d > b. \end{cases}$$

**Proof.** Since $b_j = 0$ ($\forall j \geq 3$), we have $m_b = b$. By Theorem 3.3.2 and Theorem 3.3.3, $Q$ is regular if and only if either (a) or (b) holds.

Hence for the uniqueness criterion, we only need to prove that under the conditions $d < b$ and $\sum_{k=1}^{\infty} \frac{1}{w_k} < +\infty$, there exists only one WMBP if and only if $\sum_{k=1}^{\infty} \frac{1}{w_k}(\frac{b}{d})^k = +\infty$. To this end, note $\tau_1 = b,\ \tau_n = 0\ (n \geq 2)$, by (3.3.7) we know that $R_0 = 1$ and

$$R_n = \frac{1}{dw_{n+1}} + \frac{bw_n}{dw_{n+1}} R_{n-1}, \quad n \geq 1.$$

Denote $h_n = w_n R_{n-1}$ for $n \geq 1$, we rewrite the above equality as

$$h_{n+1} = \frac{1}{d} + \frac{b}{d} h_n, \qquad n \geq 1.$$

Therefore, by mathematical induction principle and note that $h_1 = w_1$,

$$h_{n+1} = \frac{(b/d)^n - 1}{b - d} + w_1 (\frac{b}{d})^n, \quad n \geq 0.$$

Thus

$$R_n = \frac{1}{w_{n+1}} \cdot [\frac{(b/d)^n - 1}{b - d} + w_1 (\frac{b}{d})^n], \quad n \geq 0.$$

By Theorem 3.3.5, the assertion regarding the uniqueness holds. By Theorem 3.4.1 and the expression of $\frac{G_i^{(k-1)}(0)}{(k-1)!}$ in the present case, (ii) holds. By Theorem 3.4.2, Theorem 3.4.3, Theorem 3.4.4 and the expression of $\frac{G_i^{(k-1)}(0)}{(k-1)!}$ in the present case, we see that (iii), (iv) and (v) hold respectively. □

**Example 3.7.2** Let $b_0 > 0,\ b_j = 0 \quad (j \geq 4), \quad b_2 \geq 0$ and $b_3 > 0$. We are interested in this example since it is beyond the birth-death process and little is known, even for the uniqueness question, until now. Now applying our results, much information can be obtained. Indeed, for this example, we have

$$B(s) = (1 - s)(b_0 - (b_2 + b_3)s - b_3 s^2)$$

and thus $B(s) = 0$ has exactly three roots, 1, $q_1$ and $q_2$, where

$$q_1 = \frac{-(b_2 + b_3) + \sqrt{(b_2 + b_3)^2 + 4b_0 b_3}}{2b_3} > 0 \tag{3.7.1}$$

and

$$q_2 = \frac{-(b_2 + b_3) - \sqrt{(b_2 + b_3)^2 + 4b_0 b_3}}{2b_3} < -1. \tag{3.7.2}$$

Here the positive root $q_1$ is less than, equal to or great than 1 depends upon $b_0 < b_2 + 2b_3$, $b_0 = b_2 + 2b_3$ or $b_0 > b_2 + 2b_3$, respectively. An easy calculation then yields the form of $G_i(s)$ and hence for $k \geq 1$,

$$\frac{G_i^{(k-1)}(0)}{(k-1)!} = \begin{cases} \frac{1}{(1-q_2)b_3} \cdot \sum_{j=1}^{k\wedge i}(1 - \frac{1}{q_2^{k+1-j}})q_1^{i-j}, & if \;\; b_0 < b_2 + 2b_3 \\ \frac{1}{(1-q_2)b_3} \cdot \sum_{j=1}^{k\wedge i}(1 - \frac{1}{q_2^{k+1-j}}), & if \;\; b_0 = b_2 + 2b_3 \\ \frac{1}{(q_1-q_2)b_3} \cdot \sum_{j=1}^{k\wedge i}(\frac{1}{q_1^{k+1-j}} - \frac{1}{q_2^{k+1-j}}), & if \;\; b_0 > b_2 + 2b_3. \end{cases}$$

**Theorem 3.7.2.** Suppose that the WMBP $q$-matrix $Q$ satisfies $b_j = 0 \;\; (j \geq 4)$ and $b_3 > 0$.

(i) There exists only one WMBP if and only if one of the following conditions holds.

(a) $b_0 \geq b_2 + 2b_3$;

(b) $b_0 < b_2 + 2b_3$ and $\sum_{k=1}^{\infty} \frac{1}{w_k} = +\infty$;

(c) $b_0 < b_2 + 2b_3$, $\sum_{k=1}^{\infty} \frac{1}{w_k} < +\infty$ and

$$\sum_{k=1}^{\infty} \frac{1}{w_k}(\frac{1}{q_1})^k = +\infty \tag{3.7.3}$$

where $q_1$ is given in (3.7.1).

(ii) The extinction probability and the mean extinction time are given by

$$a_{i0} = \begin{cases} q_1^i < 1, & if \;\; b_0 < b_2 + 2b_3 \\ 1, & if \;\; b_0 \geq b_2 + 2b_3 \end{cases}$$

$$E_i[\tau_0] = \begin{cases} \frac{1}{(q_1-q_2)b_3} \sum_{k=1}^{\infty} \frac{1}{w_k} \sum_{j=1}^{k\wedge i}(\frac{1}{q_1^{k+1-j}} - \frac{1}{q_2^{k+1-j}}), if \;\; b_0 > b_2 + 2b_3, \; \sum_{k=1}^{\infty} \frac{1}{w_k q_1^k} < \infty \\ \frac{1}{(1-q_2)b_3} \cdot \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \sum_{j=1}^{k\wedge i}(1 - \frac{1}{q_2^{k+1-j}}), \;\; if \;\; b_0 = b_2 + 2b_3, \; \sum_{k=1}^{\infty} \frac{1}{w_k} < \infty \\ +\infty, & otherwise. \end{cases}$$

(iii) If $b_0 < b_2 + 2b_3$, then the conditional mean extinction time is given by

$$E_i[\tau_0 | \tau_0 < \infty] = \begin{cases} \frac{1}{(1-q_2)b_3} \cdot \sum_{k=1}^{\infty} \frac{1}{w_k} \cdot \sum_{j=1}^{k\wedge i}(1 - \frac{1}{q_2^{k+1-j}})q_1^{k-j}, & if \;\; \sum_{k=1}^{\infty} \frac{q_1^k}{w_k} < \infty \\ +\infty, & if \;\; \sum_{k=1}^{\infty} \frac{q_1^k}{w_k} = \infty. \end{cases}$$

(iv) If $b_0 < b_2+2b_3$ and $\Sigma_{k=1}^{\infty}\frac{1}{w_k} < \infty$, then the explosion probability and the mean explosion time are given by

$$a_{i\infty} = 1 - q_1^i$$

and

$$E_i[\tau_\infty|\tau_\infty < \infty] = \frac{1}{b_3(1-q_2)(1-q_1^i)}\sum_{k=1}^{\infty}\frac{1-q_1^k}{w_k}\cdot\sum_{j=1}^{k\wedge i}(1-\frac{1}{q_2^{k+1-j}})q_1^{i-j}.$$

(v) The mean global holding times are given by

$$\mu_{ik} = \frac{1}{w_k}\cdot\begin{cases}\frac{1}{(1-q_2)b_3}\cdot\Sigma_{j=1}^{k\wedge i}(1-\frac{1}{q_2^{k+1-j}})q_1^{i-j}, & if\ \ b_0 < b_2+2b_3\\ \frac{1}{(1-q_2)b_3}\cdot\Sigma_{j=1}^{k\wedge i}(1-\frac{1}{q_2^{k+1-j}}), & if\ \ b_0 = b_2+2b_3\\ \frac{1}{(q_1-q_2)b_3}\cdot\Sigma_{j=1}^{k\wedge i}(\frac{1}{q_1^{k+1-j}}-\frac{1}{q_2^{k+1-j}}), & if\ \ b_0 > b_2+2b_3.\end{cases}$$

**Proof.** As in the proof of Theorem 3.5.1, we only need to prove that under the conditions $b_0 < b_2+2b_3$ and $\Sigma_{k=1}^{\infty}\frac{1}{w_k} < +\infty$, there exists only one WMBP if and only if $\Sigma_{k=1}^{\infty}\frac{1}{w_k}(\frac{1}{q_1})^k = +\infty$. However, this conclusion is an easy corollary of Theorem 3.3.5. Indeed, by this theorem, there exists only one WMBP if and only if

$$\sum_{n=1}^{\infty} R_n = +\infty$$

where $\{R_n\}$ is given in (3.3.7). However, since $b_j = 0$ $(j \geq 4)$ we have $\tau_j = 0$ $(\forall j \geq 3)$ and $\tau_1 = b_2+b_3$, $\tau_2 = b_3$. Hence if we let $f_n = b_0 w_{n+1} R_n$ $(n \geq 0)$, $c_1 = \frac{b_2+b_3}{b_0}$ and $c_2 = \frac{b_3}{b_0}$, then the recursive formula (3.3.7) can be rewritten as

$$f_{n+2} = 1 + c_1 f_{n+1} + c_2 f_n, \quad n \geq 0$$

which is a second order linear difference equation with constant coefficients. By solving this difference equation we get that

$$b_0 R_n w_{n+1} = f_n = A_1(\frac{1}{q_1})^n + A_2(\frac{1}{q_2})^n + A_3$$

where $q_1$ and $q_2$ are given in (3.7.1) and (3.7.2) and $A_1, A_2, A_3$ are three constants, determined by the known initial conditions. Now since $\Sigma_{n=1}^{\infty}\frac{1}{w_n} < +\infty$ and $|\frac{1}{q_2}| < 1$, it is easily seen that $\Sigma_{n=1}^{\infty} R_n = +\infty$ if and only if

$\sum_{n=1}^{\infty} \frac{1}{w_{n+1}}(\frac{1}{q_1})^n = +\infty$, or if and only if (3.7.3) holds. This completes the proof. □

**Remark 3.7.1.** It can be seen that the conclusion (i) and the arguments in Theorem 3.7.2 can be generalised to the case that there exists an $N \geq 1$ such that for all $n > N+1$, $b_n = 0$. In such more general case, we will obtain that

$$f_{n+N} = 1 + c_1 f_{n+N-1} + c_2 f_{n+N-2} + \cdots + c_N f_n, \quad n \geq 0,$$

where $c_k = \tau_k / b_0 > 0$ $(k = 1, 2, \cdots, N)$. The corresponding auxiliary equation is

$$x_N = c_1 x_{N-1} + c_2 x_{N-2} + \cdots + c_N.$$

Replacing $x$ with $1/s$ and noting $c_k = \tau_k / b_0$ yields that

$$b_0 - \sum_{k=1}^{N} \tau_k s^k = 0,$$

i.e.,

$$\frac{B(s)}{1-s} = 0.$$

Therefore, the roots of the above auxiliary equation are $\frac{1}{q_1}, \frac{1}{q_2}, \cdots, \frac{1}{q_N}$, where $q_1, q_2, \cdots, q_N$ are the roots of $B(s) = 0$ excepting 1. It can be proved that $q_1$, the smallest positive root of $B(s) = 0$ in $[0, 1]$ satisfies $q_1 \leq 1 < \inf\{|q_k|; 2 \leq k \leq N\}$. By the theory of difference equation, see, for example, Section 2.9 in Hunter (1983), we have

$$b_0 R_n w_{n+1} = f_n = D + \sum_{k=1}^{N} A_k (\frac{1}{q_k})^n, \quad n \geq 0$$

where $D, A_1, \cdots, A_N$ are constants. Note that although the explicit expressions of the roots $q_k$ are not available, we only need the smallest root $q_1$ to check that $\sum_{n=1}^{\infty} R_n < \infty$ or not. Examples 3.7.1 and 3.7.2 are actually special cases of $N = 2$ and $N = 3$, respectively. See (3.7.3) in Theorem 3.7.2 and (c) in Theorem 3.7.1.

Next, we give three examples in which the sequence $\{w_j; \ j \geq 1\}$ takes some special form while the sequence $\{b_j; \ j \geq 1\}$ is arbitrary.

**Example 3.7.3.** Let $w_k \equiv 1$ $(\forall k \geq 1)$. This example may be viewed as an $M^X/M/1$ queue stopped at state zero which plays an important role in the Markovian queueing models. For more details, see Chen and Renshaw (1997, 2003).

For this example, the $q$-matrix $Q$ is bounded and thus the uniqueness question is trivial. That is, the $q$-matrix $Q$ is regular and the Feller minimal WMBP is honest and thus explosion is impossible. It is also clear that if $b_0 \geq m_b$, then

$$\sum_{k=1}^{\infty} \frac{G^{(k-1)}(0)}{(k-1)!} = \lim_{s\uparrow 1} \frac{1-s}{B(s)} = \begin{cases} \frac{1}{b_0 - m_b}, & if \;\; b_0 > m_b \\ +\infty, & if \;\; b_0 = m_b \end{cases}$$

and if $b_0 < m_b \leq +\infty$, then

$$\sum_{k=1}^{\infty} \frac{G_i^{(k-1)}(0)}{(k-1)!} q^k = q \cdot \lim_{s\uparrow q} \frac{q^i - s^i}{B(s)} = \frac{i \cdot q^i}{-B'(q)} < +\infty.$$

**Theorem 3.7.3.** Suppose that $w_k = 1 \;\; (\forall k \geq 1)$. Then there always exists only one WMBP which is the honest Feller minimal process. The extinction Probability is given in (3.4.1) and the mean extinction time is given by

$$E_i[\tau_0] = \begin{cases} \frac{i}{b_0 - m_b}, & if \;\; b_0 > m_b \\ +\infty, & if \;\; b_0 \leq m_b \leq +\infty. \end{cases}$$

Moreover, if $b_0 < m_b \leq +\infty$, then the conditional extinction time is given by

$$E_i[\tau_0 | \tau_0 < \infty] = \frac{i}{-B'(q)}.$$

and the mean global holding time and mean total survival time are given by

$$\mu_{ij} = \frac{G_i^{(j-1)}(0)}{(j-1)!}, \quad j \geq 1$$

and

$$\mu_i = \begin{cases} \frac{i}{b_0 - m_b}, & if \;\; b_0 > m_b \\ \infty, & if \;\; b_0 \leq m_b \leq +\infty. \end{cases}$$

**Example 3.7.4.** Let $w_k = k \;\; (\forall k \geq 1)$. This is just the case of ordinary Markov branching process and also a special case of non-linear Markov branching process. Let

$$F_i(s) = \int_0^s \frac{a_{i0} - y^i}{B(y)} dy, \quad for \;\; i \geq 1 \tag{3.7.4}$$

and

$$F(s) = F_1(s) \tag{3.7.5}$$

then $F_i(s)$ is well-defined for $s \in [0, 1)$ and $F_i(0) = 0$. So

$$\sum_{k=1}^{\infty} \int_0^{\infty} p_{ik}(t)dt = \sum_{k=1}^{\infty} \frac{F_i^{(k)}(0)}{k!} = \int_0^1 \frac{a_{i0} - y^i}{B(y)} dy, \quad i \geq 1.$$

**Theorem 3.7.4.** There always exists only one ordinary Markov branching process which is the Feller minimal process. This Feller minimal process is honest, i.e., $Q$ is regular, if and only if either $b_0 \geq m_b$ or $b_0 < m_b \leq +\infty$ and

$$\sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} = +\infty \tag{3.7.6}$$

or, equivalently, $\lim_{s \uparrow 1} F(s) = +\infty$ where $F(s)$ is given in (3.7.5). The extinction probability is given in (3.4.1) and the mean extinction time is given by

$$E_i[\tau_0] = \begin{cases} \int_0^1 \frac{1-y^i}{B(y)} dy, & if \;\; b_0 \geq m_b \;\; and \;\; \int_0^1 \frac{1-y}{B(y)} dy < \infty \\ +\infty, & otherwise. \end{cases}$$

Moreover, the conditional mean extinction time is always finite and given by

$$E_i[\tau_0 | \tau_0 < \infty] = q^{-i} \int_0^q \frac{q^i - y^i}{B(y)} dy$$

and the explosion probability and mean explosion time are given by

$$a_{i\infty} = \begin{cases} 1 - q^i, & if \;\; \int_0^1 \frac{q-y}{B(y)} dy < \infty \\ 0, & if \;\; \int_0^1 \frac{q-y}{B(y)} dy = \infty \end{cases}$$

and

$$E_i[\tau_\infty | \tau_\infty < \infty] = \frac{1}{(1 - q^i)} \int_q^1 \frac{q^i - y^i}{B(y)} dy$$

respectively. Finally, the mean global holding time and mean total survival time are given by

$$\mu_{ij} = \frac{F_i^{(j)}(0)}{j!}, \quad i, j \geq 1$$

where $F_i^{(j)}(0)$ denotes the $j$th derivative of $F_i(s)$ evaluated at 0 and $F_i(s)$ is defined in (3.7.4), and

$$\mu_i = \begin{cases} \int_0^1 \frac{1-y^i}{B(y)} dy, & if \;\; b_0 \geq m_b \\ \int_0^1 \frac{q^i - y^i}{B(y)} dy, & if \;\; b_0 < m_b \leq +\infty. \end{cases}$$

**Proof.** Let $\theta = 1$ in Theorem 3.6.2 and Theorem 3.6.6 we obtain all the conclusions. $\square$

## 3.8. Notes

Different from the ordinary Markov branching process, the WMBP considered in this chapter does not satisfy the branching property. If $w_i = i (i \geq 1)$, as shown in Athreya and Ney (1972), the process has branching property. Conversely, Chen (2001) proved that the branching property implies $w_i = i (i \geq 1)$.

As far as we know, this model was first considered in Chen (1997), where some sufficient regularity conditions were obtained.

Chen (2002a,b) considered the special case: $w_i = i^{\theta}$ $(i \geq 1)$ where $\theta (> 0)$ is a constant.

Here we further considered the most general case and some new results have been obtained.

Theorems 3.3.1 and 3.3.3 are due to Chen (1997), here we have presented a different and easy proof. The idea of Theorems 3.3.5 and 3.3.6 is based on Chen etc. (2003).

Sections 3.2, 3.3 and 3.4 (except the results mentioned above) has been submitted for publication in Chen, Li and Ramesh (2004a). Section 3.6 is submitted for publication in Chen, Li and Ramesh (2004c). Section 3.5 is in preparation (Chen, Li and Ramesh (2004d)).

It is worth pointing out that from Theorem 3.3.1–3.3.3, in case $m_b < +\infty$, the WMBP is regular if and only if $b_0 \geq m_b$ or $\sum_{n=1}^{\infty}(1/w_n) = \infty$, while in case $m_b = \infty$, the WMBP is not regular if $\sum_{n=1}^{\infty}(g_{n-1}/w_n) < \infty$. In other words, condition $\sum_{n=1}^{\infty}(g_{n-1}/w_n) = \infty$ is necessary for the process to be regular. However, this condition is also sufficient for ordinary WMPs and non-linear Markov branching processes considered in Section 3.6. We conjecture that this condition is still sufficient for the most general case WMBPs. If this is true, then the regularity problem will be completely solved.

The technique of random time changes used in Section 3.5, which is due to Lamperti (1967b), reveals the relation of WMBP and MBP. This probability approach could be applicable in other situations. For example, this chapter only considers the case where the underlying process is a compound Poisson process with discrete state space. Note that Lévy process is a class of processes with independent increments. A discrete compound Poisson process is a pure jump process with independent increments and hence it is a very special Lévy process, it would be very interesting to investigate properties of such processes which are the random time changes of a Lévy process (for instance, Mrakov branching process with continuous state). It will be considered in future.

¿From the next chapter on, we shall consider a different kind of branching models where there maybe exist two absorbing states.

# 4. Collision Branching Processes

## 4.1. Background

Consider an ensemble of particles that evolves as follows. Collisions between particles occur at random, and, whenever two particles collide, they are removed and replaced by $k$ "offspring" with probability $p_k$ ($k \geq 0$), independently of other collisions. In any small time interval $(t, t+\Delta t)$ there is a positive probability $\theta\Delta t + o(\Delta t)$ that a collision occurs, and the probability that two or more collisions occurring in that time interval is $o(\Delta t)$. Suppose there are $i$ particles at time $t$. Assuming all pair interactions are equally likely, then after time $\Delta t$, there will be $j$ particles with probability $\binom{i}{2}\theta p_{j-i+2}\Delta t + o(\Delta t)$. We may therefore take $X(t)$, the number of particles alive at time $t$, to be a continuous-time Markov chain with non-zero transition rates $q_{ij} = \binom{i}{2}b_{j-i+2}$ $(j \geq i-2, i \geq 2)$, where $b_2 = -\theta(1-p_2)$ and $b_j = \theta p_j$ $(j \neq 2)$.

This leads us to the following formal definition.

**Definition 4.1.1.** A conservative $q$-matrix $Q = (q_{ij},\ i, j \in \mathbf{Z}_+)$ is called a collision branching $q$-matrix (CB $q$-matrix) if it takes the following form:

$$q_{ij} = \begin{cases} \binom{i}{2}b_{j-i+2}, & if\ i \geq 2,\ j > i-2 \\ 0, & otherwise, \end{cases} \tag{4.1.1}$$

where

$$b_j \geq 0 \quad (j \neq 2) \quad \text{and} \quad -b_2 = \sum_{j\neq 2} b_j < +\infty \tag{4.1.2}$$

together with $b_0 > 0$ and $\sum_{j=3}^{\infty} b_j > 0$.

It should be pointed out that the term "collision" implies a process operating in space and time, here there is no spatial component.

¿From the above definition, we see that $\{b_j;\ j \geq 3\}$ and $\{b_0,\ b_1\}$ denote, respectively, the birth rates and death rates of the model and that $q_{00} = q_{11} = 0$ and thus both states 0 and 1 are absorbing.

**Remark 4.1.1.** A CB $q$-matrix is called degenerative if $b_{2j+1} = 0$ $(\forall j \geq 0)$. In this case, the essential state space degenerates into either $\{0, 2, 4, \cdots\}$

or $\{1, 3, 5, \cdots\}$ (according to the starting state is an even or odd integer) and thus there is essentially only one absorbing state. Although this case has its own interest, it essentially reduces to the WMBP case and thus has already been considered in Chen (1997) and in Chapter 3. For this reason, we shall assume that the CB $q$-matrix is not degenerative from now on. Note that, however, most of the results obtained in the following apply well to the degenerative case if some statements are amended in an obvious way, while the conditions $b_0 > 0$ and $\sum_{j=3}^{\infty} b_j > 0$ are essential.

It differs from the ordinary Markov branching process, the branching events are effected by the collision/interaction of pairs of particles, rather than by the particles individually.

In order that the branching property holds for the ordinary MBP it is necessary that its transition function obeys the Kolmogorov forward equations. Guided by this fact, we formally define the collision branching process as follows.

**Definition 4.1.2.** A collision branching process (CBP) is a continuous-time Markov chain taking values in $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t),\ i, j \in \mathbf{Z}_+)$ satisfies the forward equation

$$P'(t) = P(t)Q, \tag{4.1.3}$$

where $Q$ is a CB $q$-matrix.

Since CBPs have two absorbing states 0 and 1, there is a need to evaluate probabilities of absorption for these states individually. Also, since rates of change are *quadratic* functions of the state, one might expect explosive behaviour to occur more readily than for the MBP. We will examine both matters in detail.

## 4.2. Regularity and Uniqueness

Since $Q$ is stable and conservative, by Theorem 1.3.1, there always exists a CBP, namely the Feller minimal process. But, under what conditions is it unique? In order to investigate this question, we introduce the generating function $B(s)$ of the known sequence $\{b_j,\ j \geq 0\}$ as follows:

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \qquad |s| \leq 1.$$

$$m_d = 2b_0 + b_1, \quad m_b = \sum_{j=1}^{\infty} j b_{j+2}$$

$B(s)$ is well-defined at least on $[-1,1]$. It is clear that $B(0) = b_0 > 0$, $B(1) = 0$ and $B'(1) = m_b - m_d$. This latter quantity $m_b - m_d$ measures the drift away from 0, because we see that, after normalizing by $\sum_{j\neq 2} b_j$, $m_b - m_d$ is the expected jump size from any state $i$.

The sign of $m_b - m_d$ determines the number of zeros of $B$ in $[0,1]$, as the following simple result demonstrates.

**Lemma 4.2.1.** The equation $B(s) = 0$ has at most two roots in $[0,1]$ and exactly one root in $[-1,0)$. More specifically,

(i) If $m_d \geq m_b$ then $B(s) > 0$ for all $s \in [0,1)$ and 1 is the only root of the equation $B(s) = 0$ in $[0,1]$, which is simple or has multiplicity 2 according to $m_d > m_b$ or $m_d = m_b$, while if $m_d < m_b \leq +\infty$ then $B(s) = 0$ has an additional simple root $q$ satisfying $0 < q < 1$, $B(s) > 0$ for $0 \leq s < q$ and $B(s) < 0$ for $q < s < 1$. we always denote $q$ the only root of $B(s) = 0$ in $(0,1)$ in the later case.

(ii) $B(s) = 0$ has exactly one root, denoted by $q_*$, in $[-1,0]$ with $-1 < q_* < 0$ and this $q_*$ is simple.

(iii) $B(z) = 0$ has only real roots on the disk $\{z;\ |z| \leq 1\}$.

**Proof.** Since $B(0) = b_0 > 0$ and $B(-1) < 0$ we know that $B(z) = 0$ has at least one root $q_* \in (-1,0)$. So it follows from $B(q_*) = B(1) = 0$ that

$$\begin{aligned} B(z) &= (1-z)(b_0 + (b_0+b_1)z - \sum_{k=1}^{\infty} \sigma_k z^{k+1}) \\ &= (1-z)(z-q_*)(b_0 + b_1 - \sum_{k=1}^{\infty} \sigma_k \sum_{j=0}^{k} q_*^{k-j} z^j) \\ &= (1-z)(z-q_*)(b_0 + b_1 - \sum_{k=1}^{\infty} \sigma_k q_*^k - \sum_{k=1}^{\infty} \sigma_k \sum_{j=1}^{k} q_*^{k-j} z^j) \end{aligned}$$

$$= (1-z)(z-q_*)(-\frac{b_0}{q_*} - \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} \sigma_{j+k-1} q_*^{k-1}) z^j)$$

where $\sigma_k = \sum_{j=k}^{\infty} b_{j+2} \quad (k \geq 1)$. In the last step above we have used the equality $b_0 + (b_0 + b_1) q_* = \sum_{k=1}^{\infty} \sigma_k q_*^{k+1}$.

If $m_d \geq m_b$, noting that $\sum_{k=1}^{\infty} \sigma_k q_*^{k-1} > 0$, $\sum_{k=1}^{\infty} \sigma_{j+k-1} q_*^{k-1} \geq 0 \quad (j \geq 2)$ and $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{j+k-1} q_*^{k-1} = -\frac{b_0}{q_*} - \frac{m_d - m_b}{1-q_*}$, we can see that

$$|-\frac{b_0}{q_*} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sigma_{j+k-1} q_*^{k-1} z^j| \geq -\frac{b_0}{q_*}(1-|z|) + \frac{m_d - m_b}{1-q_*}|z|$$

where the equality holds if and only if $z = 1$. Hence 1 and $q_*$ are the only two roots of $B(z) = 0$ on the disk $\{z;\ |z| \leq 1\}$, moreover, $q_*$ is simple and 1 is simple or has multiplicity 2 according to $m_d > m_b$ or $m_d = m_b$.

If $m_d < m_b \leq +\infty$, then the equation $B(z) = 0$ has an additional root $q \in (0,1)$. So

$$\begin{aligned}
B(z) &= (1-z)(z-q_*)(-\frac{b_0}{q_*} - \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} \sigma_{j+k-1} q_*^{k-1}) z^j) \\
&= (1-z)(z-q_*)(q-z) \sum_{k=1}^{\infty} \sigma_k \sum_{j=1}^{k} q_*^{k-j} \sum_{l=1}^{j} q^{j-l} z^{l-1} \\
&= (1-z)(z-q_*)(q-z) \sum_{l=1}^{\infty}(\sum_{k=l}^{\infty} \sigma_k \sum_{j=l}^{k} q_*^{k-j} q^{j-l}) z^{l-1} \\
&= (1-z)(z-q_*)(q-z) \sum_{l=1}^{\infty}(\sum_{k=l}^{\infty} \sigma_k \cdot \frac{q^{k-l+1} - q_*^{k-l+1}}{q-q_*}) z^{l-1} \\
&= (1-z)(z-q_*)(q-z) \sum_{l=0}^{\infty}(\sum_{k=1}^{\infty} \sigma_{k+l} \cdot \frac{q^k - q_*^k}{q-q_*}) z^l \\
&= (z-q_*)(q-z)(-\frac{b_0}{qq_*} - \sum_{l=1}^{\infty}(\frac{1}{q-q_*} \sum_{k=1}^{\infty} b_{k+l+1}(q^k - q_*^k)) z^l)
\end{aligned}$$

In the last step above we have used the equality $\frac{1}{q-q_*} \sum_{k=1}^{\infty} \sigma_k \cdot (q^k - q_*^k) = -\frac{b_0}{qq_*}$. Noting $B(z) = (1-z)(q-z)[\frac{b_0}{q} + \sum_{j=1}^{\infty}(\sum_{k=1}^{\infty} \sigma_{j+k-1} q^{k-1}) z^j)]$ we know that $B(z) > 0$ for $z \in (0,q)$ and $B(z) < 0$ for $z \in (q,1)$. Since $B(-q_*) = 2\sum_{k=0}^{\infty} b_{2k+1} |q_*|^{2k+1} \geq 0$, we have $-q_* \leq q$. It is clear that $\frac{1}{q-q_*} \sum_{k=1}^{\infty} b_{k+2}(q^k - q_*^k) > 0$, $\frac{1}{q-q_*} \sum_{k=1}^{\infty} b_{k+l+1}(q^k - q_*^k) \geq 0 \quad (l \geq 1)$ and

$$\sum_{l=1}^{\infty} \frac{1}{q-q_*} \sum_{k=1}^{\infty} b_{k+l+1}(q^k - q_*^k) = -\frac{b_0}{qq_*}$$

Thus by the same argument as in the previous case we see that $q_* \in (-1, 0)$, $q \in (0, 1)$ and 1 are the only three roots of $B(z) = 0$ on $\{z;\ |z| \leq 1\}$ and they are all simple. The proof is completed. $\square$

By Lemma 4.2.1, we will always denote by $q$ the smallest positive root of $B(s) = 0$ on $[0, 1]$ and denoted by $q_*$ the largest negative root of $B(s) = 0$ in $(-1, 0)$. Therefore, $q$ is strictly less than 1 or equal to 1 according to $B'(1) > 0$ or $B'(1) \leq 0$.

**Lemma 4.2.2.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function, where $Q$ is a CB $q$-matrix given in (4.1.1)−(4.1.2). Then all the states $k \geq 2$ are transient, i.e., for any $i \geq 0$ and $k \geq 2$,

$$\int_0^\infty p_{ik}(t)dt < \infty$$

and hence $\lim_{t\to\infty} p_{ik}(t) = 0$.

**Proof.** For any fixed $i \geq 0$, it follows from the Kolomogorv forward equations that

$$p_{i0}(t) = \delta_{i0} + q_{20} \cdot \int_0^t p_{i2}(s)ds$$

which clearly implies that $\int_0^\infty p_{i2}(t)dt < \infty$. Suppose $\int_0^\infty p_{ik}(t)dt < \infty$ for $2 \leq k \leq j$. ¿From Kolmogorov forward equations we can see that

$$p_{ij-1}(t) - \delta_{ij-1} = \sum_{k=2}^{j} \binom{k}{2} b_{j-k+1} \cdot \int_0^\infty p_{ik}(t)dt + \binom{j+1}{2} b_0 \int_0^\infty p_{ij+1}(t)dt$$

and hence $\int_0^\infty p_{ij+1}(t)dt < \infty$. Therefore, by the mathematical induction principle we have $\int_0^\infty p_{ik}(t)dt < \infty$ for all $k \geq 2$. Hence $\lim_{t\to\infty} p_{ik}(t) = 0$. $\square$

We now answer the question of uniqueness.

**Theorem 4.2.3.** The CB $q$-matrix is regular if and only if $B'(1) \leq 0$.

**Proof.** Suppose $B'(1) \leq 0$ and let $P(t) = (p_{ij}(t))$ be the Feller minimal $Q$-function. Substituting (4.1.1) into the forward equations (4.1.3) gives

$$p'_{ij}(t) = \sum_{k=2}^{j+2} p_{ik}(t) \binom{k}{2} b_{j-k+2}, \qquad i, j \geq 0.$$

Multiplying $s^j$ on both sides of the above equality and summing over $j$

yields that for any $i \geq 0$ and $s \in [0,1)$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= \sum_{j=0}^{\infty}\Big(\sum_{k=2}^{j+2} p_{ik}(t)\tbinom{k}{2}b_{j-k+2}\Big)s^j \\
&= \sum_{k=2}^{\infty}\sum_{j=k-2}^{\infty} p_{ik}(t)\tbinom{k}{2}b_{j-k+2}s^j \\
&= \sum_{k=2}^{\infty} p_{ik}(t)\tbinom{k}{2}s^{k-2}\cdot \sum_{m=0}^{\infty} b_m s^m \\
&= B(s)\sum_{k=2}^{\infty} p_{ik}(t)\tbinom{k}{2}s^{k-2}.
\end{aligned}$$

i.e.,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\sum_{k=2}^{\infty}\tbinom{k}{2}p_{ik}(t)s^{k-2}, \quad s \in [0,1), \quad i \geq 0. \tag{4.2.1}$$

Now Lemma 4.2.1 implies that the right-hand side of (4.2.1) is strictly positive for $s \in (0,1)$. Thus, on integrating the left-hand side, we see that

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0, \qquad i \geq 0,\ 0 \leq s < 1. \tag{4.2.2}$$

The interchange of derivative and sum can be justified as follows. By Theorem 1.2.2, for all $t \geq 0$,

$$\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q_i, \tag{4.2.3}$$

where $q_i := -q_{ii} = -\binom{i}{2}b_2 < \infty$. Therefore, the series $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$ converges *uniformly* on $[0,1)$, for every $t$, and, since the derivatives $p'_{ij}(t)$ are all continuous (see for example Proposition 1.2.4(2) of Anderson(1991)), the derivative of $\sum_{j=0}^{\infty} p_{ij}(t)s^j$ exists and equals $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$. Now, letting $s \uparrow 1$ in (4.2.2) yields $\sum_{j=0}^{\infty} p_{ij}(t) \geq 1$, implying that equality holds for all $i \geq 0$. We deduce that the minimal $Q$-function is honest, and hence that $Q$ is regular.

Conversely, suppose that $B'(1) > 0$. Define a (conservative) birth-death $q$-matrix $Q^* = (q^*_{ij},\ i,j \in Z_+)$ by

$$q^*_{ij} = \begin{cases} \binom{i}{2}b^*, & if\ \ j = i+1,\ i \geq 2 \\ \binom{i}{2}a^*, & if\ \ j = i-1,\ i \geq 2 \\ -\binom{i}{2}(b^*+a^*), & if\ \ j = i \geq 2 \\ 0, & otherwise, \end{cases}$$

where $b^* > a^* > 0$. $Q^*$ is a birth-death $q$-matrix. Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left(\frac{1}{q_{nn+1}}+\frac{q_{nn-1}}{q_{nn+1}q_{n-1n}}+\cdots+\frac{q_{nn-1}\cdots q_{32}}{q_{nn+1}\cdots q_{23}}\right)\\
= &\ \frac{2}{b^*}\cdot\sum_{n=2}^{\infty}\left(\frac{1}{n(n-1)}+\frac{a^*}{b^*}\cdot\frac{1}{(n-1)(n-2)}+\cdots+\frac{a^{*n-2}}{b^{*n-2}}\cdot\frac{1}{2\cdot 1}\right)\\
= &\ \frac{2}{b^*}\cdot\sum_{n=2}^{\infty}\sum_{k=2}^{n}\frac{(a^*/b^*)^{n-k}}{k(k-1)}\\
= &\ \frac{2}{b^*-a^*}\cdot\sum_{k=2}^{\infty}\frac{1}{k(k-1)}<\infty.
\end{aligned}$$

By Theorem 3.2.2 in Anderson (1991), $Q^*$ is not regular. Our aim is to choose $a^*$ and $b^*$ in such a way that a comparison of $Q^*$ with the original CB $q$-matrix $Q$ leads to the conclusion that $Q$ is not regular.

To this end, first note that $B'(1) > 0$ is the same as $2b_0 + b_1 < \sum_{j=1}^{\infty} jb_{j+2}$ $(\leq +\infty)$, and so we may choose $a^*$ and $b^*$ with

$$2b_0 + b_1 < a^* < b^* < \sum_{j=1}^{\infty} jb_{j+2}\,. \tag{4.2.4}$$

Since $Q^*$ is not regular, the equation

$$(\lambda I - Q^*)u = 0 \qquad (\lambda > 0) \tag{4.2.5}$$

has a non-trivial (non-negative) bounded solution, which we shall denote by $u^* = \{u_i(\lambda),\ i \geq 0\}$. Clearly $u^*$ depends on both $a^*$ and $b^*$. By Theorem 2.2.7 in Anderson (1991), if $u^* = \{u_i(\lambda),\ i \geq 0\}$ satisfies

$$\lambda u^* \leq Qu^*, \tag{4.2.6}$$

then $Q$ is not regular. Therefore, we only need to choose $a^*$ and $b^*$ such that (4.2.6) holds.

We will first prove that $a^*$ and $b^*$ can be chosen so that both

$$\sum_{j=1}^{\infty} b_{j+2}\sum_{k=1}^{j}\left(\frac{a^*}{b^*}\right)^{k-1} > b^* \tag{4.2.7}$$

and

$$b_0\left(\frac{b^*}{a^*}\right) + (b_0 + b_1) < a^* \tag{4.2.8}$$

hold. Let $\{a_n\}$ be a sequence such that $a_n \downarrow\downarrow 2b_0 + b_1$. (A double arrow denotes *strict* monotone convergence), i.e., $\frac{2b_0+b_1}{a_n} \uparrow\uparrow 1$, thus

$$\begin{aligned}
&\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left(\frac{2b_0 + b_1}{a_n}\right)^{k-1} \\
= &\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} b_{j+2} \left(\frac{2b_0 + b_1}{a_n}\right)^{k-1} \\
\leq &-b_2 \sum_{k=1}^{\infty} \left(\frac{2b_0 + b_1}{a_n}\right)^{k-1} \\
< &\ \infty
\end{aligned}$$

and

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left(\frac{2b_0 + b_1}{a_n}\right)^{k-1} \uparrow\uparrow \sum_{j=1}^{\infty} j b_{j+2}$$

as $n \to \infty$. Since $\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left(\frac{2b_0+b_1}{x}\right)^k \uparrow\uparrow \sum_{j=1}^{\infty} j b_{j+2} > 2b_0 + b_1$ as $x \downarrow\downarrow 2b_0 + b_1$, we see that $b^*$ in (4.2.4) may be chosen so that

$$\sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left(\frac{2b_0 + b_1}{b^*}\right)^{k-1} > b^*. \tag{4.2.9}$$

Similarly, by considering any sequence $\{a_n^-\}$ with $a_n^- \uparrow\uparrow b^*$, it can be seen that $a^*$ may be chosen so that both (4.2.4) and (4.2.8) hold. Now (4.2.9) holds good if $2b_0 + b_1$ is replaced by (the larger) $a^*$, which is to say that (4.2.7) holds.

To prove (4.2.6), noting the definition of $Q^*$, the equation (4.2.5) can be rewritten in the terms of elements as

$$u_0(\lambda) = u_1(\lambda) = 0$$

and

$$b^*(u_{i+1}(\lambda) - u_i(\lambda)) = a^*(u_i(\lambda) - u_{i-1}(\lambda)) + \lambda u_i(\lambda)\binom{i}{2}^{-1}, \ i \geq 2. \tag{4.2.10}$$

In particular, for $i = 2$ we have $b^*(u_3(\lambda) - u_2(\lambda)) = (a^* + \lambda)u_2(\lambda)$ ($> 0$), implying that $\{u_i(\lambda)\}$ is strictly increasing in $i$ for each fixed $\lambda$. From (4.2.10) it is easily seen that, for all $k \geq 1$ and $i \geq 2$,

$$u_{i+k}(\lambda) - u_{i+k-1}(\lambda) \geq \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1}(\lambda) - u_i(\lambda)) \tag{4.2.11}$$

and

$$u_{i-1}(\lambda) - u_{i-2}(\lambda) \leq \left(\frac{b^*}{a^*}\right) (u_i(\lambda) - u_{i-1}(\lambda)). \tag{4.2.12}$$

Equation (4.2.6) is trivial true for $i = 0$ or $i = 1$. For $i \geq 2$, we have

$$\begin{aligned}
& (Qu)_i \\
= \; & \binom{i}{2} \cdot \Big[b_0(u_{i-2}(\lambda) - u_i(\lambda)) + b_1(u_{i-1}(\lambda) - u_i(\lambda)) \\
& + \sum_{j=i+1}^{\infty} b_{j-i+2}(u_j(\lambda) - u_i(\lambda))\Big] \\
= \; & \binom{i}{2} (-I_d + I_b) \qquad\qquad (4.2.13)
\end{aligned}$$

where $(Qu)_i = \Sigma_{j=0}^{\infty} q_{ij} u_j$, $I_b = \Sigma_{j=i+1}^{\infty} b_{j-i+2}(u_j(\lambda) - u_i(\lambda)) > 0$ and $I_d = b_0(u_i(\lambda) - u_{i-2}(\lambda)) + b_1(u_i(\lambda) - u_{i-1}(\lambda)) > 0$. By (4.2.7) and (4.2.11), we have

$$\begin{aligned}
I_b &= \sum_{j=1}^{\infty} b_{j+2}(u_{i+j}(\lambda) - u_i(\lambda)) \\
&= \sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} (u_{i+k}(\lambda) - u_{i+k-1}(\lambda)) \\
&\geq \sum_{j=1}^{\infty} b_{j+2} \sum_{k=1}^{j} \left(\frac{a^*}{b^*}\right)^{k-1} (u_{i+1}(\lambda) - u_i(\lambda)) \\
&> b^*(u_{i+1}(\lambda) - u_i(\lambda)). \qquad\qquad (4.2.14)
\end{aligned}$$

Similarly, by (4.2.8) and (4.2.12), we have

$$\begin{aligned}
I_d &= (b_0 + b_1)(u_i(\lambda) - u_{i-1}(\lambda)) + b_0(u_{i-1}(\lambda) - u_{i-2}(\lambda)) \\
&\leq \left(b_0 \left(\frac{b^*}{a^*}\right) + (b_0 + b_1)\right) (u_i(\lambda) - u_{i-1}(\lambda)) \\
&< a^*(u_i(\lambda) - u_{i-1}(\lambda)). \qquad\qquad (4.2.15)
\end{aligned}$$

In view of (4.2.14) and (4.2.15), equations (4.2.10) and (4.2.13) together verify (4.2.6). The converse is thus proved. □

Theorem 4.2.3 establishes that if the drift $B'(1)$ is non-positive, the CBP is unique. However, even when this conditional fails, and indeed even if $B'(1) = +\infty$, there is still only one CBP, for, as we shall see, there is a unique solution to the forward equations (4.1.3).

**Theorem 4.2.4.** There exists only one CBP.

**Proof.** As already remarked, we need only consider the case $0 < B'(1) \leq \infty$. In order to prove that the CBP is unique, we will show that the forward equations have a unique solution. We will verify Reuter's condition (Theorem 2.2.8 of Anderson (1991)) that the equation $\eta(\lambda)(\lambda I - Q) = 0$, $0 \leq \eta(\lambda) \in l_1$, has only the trivial solution for some (and then all) $\lambda > 0$. Suppose the contrary is true and let $\eta = \{\eta_i,\ i \geq 0\}$ be the non-trivial solution corresponding to $\lambda = 1$. Then, by (4.1.1) we have

$$\eta_j = \sum_{i=2}^{j+2} \eta_i \binom{i}{2} b_{j-i+2}\,, \qquad j \geq 0, \tag{4.2.16}$$

with

$$\eta_j \geq 0 \quad (j \geq 0) \qquad \textit{and} \qquad \sum_{j=0}^{\infty} \eta_j < +\infty. \tag{4.2.17}$$

It is clear that the non-triviality of the solution $\eta$ implies that

$$\sum_{j=2}^{\infty} \eta_j > 0. \tag{4.2.18}$$

Condition (4.2.17) guarantees that $\sum_{j=2}^{\infty} \eta_j s^j < \infty$ for all $s \in [0,1]$. Note that for any $s \in [0,1)$, let $\tilde{s} \in (s,1)$, then $\lim_{j\to\infty} \binom{j}{2} s^j/\tilde{s}^j = \lim_{j\to\infty} \binom{j}{2} \cdot (s/\tilde{s})^j = 0$, thus there exists an $k_0 \geq 2$ such that $\binom{j}{2} s^j < \tilde{s}^j$ $(\forall j > k_0)$. Therefore, $\sum_{j=2}^{\infty} \binom{j}{2} \eta_j s^j < \sum_{j=2}^{k_0} \binom{j}{2} \eta_j s^j + \sum_{j=k_0+1}^{\infty} \eta_j \tilde{s}^j < \infty$ and hence

$$\sum_{j=2}^{\infty} \binom{j}{2} \eta_j s^j < +\infty\,, \qquad 0 \leq s < 1 \tag{4.2.19}$$

because, by the root test, these series have the same radius of convergence. It then follows, from (4.2.16), (4.2.19) and Fubini's Theorem, that $\sum_{j=0}^{\infty} \eta_j s^j = B(s) \sum_{i=2}^{\infty} \binom{i}{2} \eta_i s^{i-2}$, $0 \leq s < 1$. Now, (4.2.17), (4.2.18) and (4.2.19) imply that both $\sum_{j=0}^{\infty} \eta_j s^j$ and $\sum_{i=2}^{\infty} \binom{i}{2} \eta_i s^{i-2}$ are strictly positive for all $s \in (0,1)$ and thus $B(s) > 0$ for all $s \in (0,1)$, which contradicts Lemma 4.2.1 because $B'(1) \in (0,\infty]$. The proof is complete. □

## 4.3. Extinction and Explosion

Having established that the CBP is uniquely determined by its $q$-matrix, we will now examine some of its properties. Let $\{X(t),\ t \geq 0\}$ be the unique CBP, and let $P(t) = (p_{ij}(t))$ denote its transition function.

Define the extinction times $\tau_0$ and $\tau_1$ for states 0 and 1 by

$$\tau_0 = \begin{cases} \inf\{t > 0,\ X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

$$\tau_1 = \begin{cases} \inf\{t > 0,\ X(t) = 1\} & \text{if } X(t) = 1 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 1 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_{i0} = P\{\tau_0 < +\infty | X(0) = i\} \quad \textit{and} \quad a_{i1} = P\{\tau_1 < +\infty | X(0) = i\}.$$

**Theorem 4.3.1.** The extinction probabilities satisfy

$$a_{i0} + q a_{i1} = q^i, \tag{4.3.1}$$

where recall that $q$ is the smallest root of $B(s) = 0$ in $[0, 1]$. More specifically,

$$a_{i0} + a_{i1} = 1, \qquad \textit{if } B'(1) \leq 0, \tag{4.3.2}$$

$$a_{i0} + q a_{i1} = q^i < 1, \qquad \textit{if } 0 < B'(1) \leq +\infty. \tag{4.3.3}$$

**Proof.** Since $a_{00} = a_{11} = 1$ and $a_{01} = a_{10} = 0$, (4.3.1) holds for $i = 0$ and $i = 1$. So, suppose that $i \geq 2$. We shall first establish (4.3.2). Refer to the proof of Theorem 4.2.3. Since $B'(1) \leq 0$, (4.2.2) holds. Also, $\lim_{t\to\infty} p_{ij}(t) = 0$ for all $i, j \geq 2$, because states $i \geq 2$ are transient (see Lemma 4.2.2). Thus, on letting $t \to \infty$ in (4.2.2) and using the Dominated Convergence Theorem, we find that $a_{i0} + s a_{i1} \geq s^i$, for $s \in [0, 1)$. Letting $s \uparrow 1$ leads us immediately to (4.3.2) because $a_{i0} + a_{i1} \leq 1$.

Next we will prove (4.3.3). Since $0 < B'(1) \leq +\infty$, Lemma 4.2.1 implies that $q < 1$. On putting $s = q$ in (4.2.1) and noting that $B(q) = 0$ we discover that, for any $t > 0$, $\Sigma_{j=0}^{\infty} p'_{ij}(t) q^j = 0$, implying that $\Sigma_{j=0}^{\infty} \int_0^t p'_{ij}(u)\, du\, q^j = 0$. Hence, for any $t > 0$, we have

$$\sum_{j=0}^{\infty} p_{ij}(t) q^j = q^i, \qquad i \geq 2. \tag{4.3.4}$$

On letting $t \to \infty$ we obtain

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t) q + \lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) q^j = q^i, \qquad i \geq 2,$$

noting that all of these limits exist. Since $q < 1$ we may apply the Dominated Convergence Theorem in the last term on the left-hand side to obtain (4.3.3). $\square$

Theorem 4.3.1 states that if $B'(1) \leq 0$ then the process is eventually absorbed at either 1 or 0 with probability 1, while if $0 < B'(1) \leq +\infty$ absorption occurs with probability less than 1. Our next result establishes that if in this latter case absorption does not occur, then the process must explode. In preparation, define a family of probability generating functions $F = \{F_i(t,s),\ i \geq 0\}$ by $F_i(t,s) := \sum_{j=0}^{\infty} p_{ij}(t)s^j$ and note that $F$ satisfies its own set of forward equations: from (4.2.1) we get, for $s \in [0,1)$,

$$\frac{\partial F_i(t,s)}{\partial t} = \frac{1}{2}B(s)\frac{\partial^2 F_i(t,s)}{\partial s^2}, \qquad i \geq 2, \tag{4.3.5}$$

with $F_0(t,s) = 1$ and $F_1(t,s) = s$.

**Lemma 4.3.2.** The transition function $P(t) = (p_{ij}(t))$ satisfies

$$\lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0, \qquad i \geq 2. \tag{4.3.6}$$

**Proof.** Fix $i \geq 2$. First note that the limit exists because $\sum_{j=2}^{\infty} p_{ij}(t)$ is decreasing in $t$. This follows from the identity

$$p_{i0}(t) + p_{i1}(t) + \sum_{j=2}^{\infty} p_{ij}(t) = \sum_{j=0}^{\infty} p_{ij}(t), \tag{4.3.7}$$

because first two terms on the left-hand side are increasing, while right-hand side is decreasing. Thus, we only need to prove that the limit in (4.3.6) equals 0. When $B'(1) \leq 0$, $P(t)$ is honest (Theorem 4.2.3) and $a_{i0} + a_{i1} = 1$ (Theorem 4.3.1), so letting $t \to \infty$ in (4.3.7) achieves the desired result.

Now suppose that $B'(1) > 0$. Observe that (4.2.1) holds for all $s \in [0,1]$ no matter what the value of $B'(1)$; when $s = 1$ or $s = q$, the right-hand side is zero. Thus, we may write

$$\frac{1}{B(s)} \sum_{j=0}^{\infty} p'_{ij}(t)s^j = \sum_{k=2}^{\infty} \binom{k}{2} p_{ik}(t)s^{k-2}, \tag{4.3.8}$$

for all $s \in [0,1)$. The apparent singularity at $s = q$ in the left-hand side is removable, because the series on the right-hand side certainly converges for all $s \in [0,1)$. Moreover, the left-hand side is continuous and strictly positive (indeed increasing) on this interval. Therefore, we can rewrite (4.3.5) as

$$\frac{\partial^2 F_i(t,y)}{\partial y^2} = \frac{2}{B(y)} F_i'(t,y), \qquad i \geq 2,$$

where $F_i'(t,y) := \partial F_i(t,y)/\partial t$. Integrating above equality with respect to $y$ on $[0,x]$ yields that

$$\sum_{j=2}^{\infty} p_{ij}(t) j x^{j-1} = 2\int_0^x \frac{F_i'(t,y)}{B(y)} dy, \qquad i \geq 2.$$

Integrating the above equality with respect to $x$ on $[0,s]$, we have

$$\sum_{j=2}^{\infty} p_{ij}(t) s^j = 2\int_0^s \left( \int_0^x \frac{F_i'(t,y)}{B(y)} dy \right) dx, \qquad i \geq 2.$$

Using Fubini's Theorem on the right hand side and noting the definition of $F_i(t,s)$, we deduce that, for any $s \in [0,1)$,

$$F_i(t,s) = p_{i0}(t) + p_{i1}(t)s + 2\int_0^s \frac{s-y}{B(y)} F_i'(t,y)\, dy\,, \tag{4.3.9}$$

where Letting $s \uparrow 1$ shows that (4.3.9) also holds for $s = 1$, and so

$$\sum_{j=2}^{\infty} p_{ij}(t) = 2\int_0^1 \frac{1-y}{B(y)} F_i'(t,y)\, dy\,. \tag{4.3.10}$$

Thus, the proof will be complete if we can establish that

$$\lim_{t\to\infty} \int_0^1 \frac{1-y}{B(y)} F_i'(t,y)\, dy = 0\,. \tag{4.3.11}$$

To this end first observe that, for $\varepsilon \in (0,1)$,

$$\lim_{t\to\infty} \int_0^{1-\varepsilon} \frac{1-y}{B(y)} F_i'(t,y)\, dy = 0\,,$$

since by (4.3.8) the integrand is dominated by $1/(1-y)^2$, and because the limit as $t \to \infty$ of the left-hand side of (4.3.8) is equal to 0 for $s \in [0,1)$. It therefore suffices to prove that

$$\lim_{t\to\infty} \int_{1-\varepsilon}^1 \frac{1-y}{B(y)} F_i'(t,y)\, dy = 0\,,$$

for some suitable $\varepsilon$. We will use (4.2.3), together with the fact that the root $s = 1$ of $B(s) = 0$ has multiplicity 1 when $B'(1) > 0$ (because $B'(1) > 0$). In particular, (4.2.3) implies that

$$-F_i'(t,s) = |F_i'(t,s)| \le \sum_{j=0}^{\infty} |p_{ij}'(t)| s^j \le \sum_{j=0}^{\infty} |p_{ij}'(t)| \le 2q_i\,, \qquad q < s < 1,$$

remembering that $F_i'(t,s)/B(s) > 0$ for $s \in [0,1)$ and $B(s) < 0$ for $s \in (q,1)$. Therefore, if we take $\varepsilon < 1-q$, we get

$$\int_{1-\varepsilon}^{1} \frac{1-y}{B(y)} F_i'(t,y)\,dy \le 2q_i \int_{1-\varepsilon}^{1} \frac{1-y}{-B(y)}\,dy < \infty\,,$$

and so again dominated convergence can be used to obtain the desired result. $\square$

In order to evaluate the absorption and explosion probabilities explicitly, we will need the following result.

**Lemma 4.3.3.** The extinction probabilities satisfies

$$a_{i0} + q_* a_{i1} = q_*^i\,. \tag{4.3.12}$$

**Proof.** Substituting (4.1.1) into the forward equations (4.1.3) gives that

$$p_{ij}'(t) = \sum_{k=2}^{j+2} p_{ik}(t) \binom{k}{2} b_{j-k+2}\,, \qquad i,j \ge 0.$$

Multiplying $s^j$ on both sides of the above equality and summing over $j$ yields that for any $i \ge 0$ and $s \in (-1,1)$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p_{ij}'(t) s^j &= \sum_{j=0}^{\infty} \Big(\sum_{k=2}^{j+2} p_{ik}(t) \tbinom{k}{2} b_{j-k+2}\Big) s^j \\
&= \sum_{k=2}^{\infty} \sum_{j=k-2}^{\infty} p_{ik}(t) \tbinom{k}{2} b_{j-k+2} s^j \\
&= \sum_{k=2}^{\infty} p_{ik}(t) \tbinom{k}{2} s^{k-2} \cdot \sum_{m=0}^{\infty} b_m s^m \\
&= B(s) \sum_{k=2}^{\infty} p_{ik}(t) \tbinom{k}{2} s^{k-2}.
\end{aligned}$$

i.e.,

$$\sum_{j=0}^{\infty} p_{ij}'(t) s^j = B(s) \sum_{k=2}^{\infty} \tbinom{k}{2} p_{ik}(t) s^{k-2}, \quad s \in (-1,1), \;\; i \ge 0.$$

Integrating the above equality yields that for $s \in (-1, 1)$ and $i \geq 0$,

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i = B(s)\sum_{k=2}^{\infty}\Big(\int_0^t p_{ik}(u)du\Big) \cdot \binom{k}{2} s^{k-2}.$$

Letting $s = q_*$ in the above equality and noting $B(q_*) = 0$ yields that

$$\sum_{j=0}^{\infty} p_{ij}(t)q_*^j = q_*^i.$$

Finally, letting $t \to \infty$ and noting $\lim_{t\to\infty} p_{i0}(t) = a_{i0}$, $\lim_{t\to\infty} p_{i1}(t) = a_{i1}$ $\lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0$ (see Lemma 4.3.2), we see that $a_{i0} + q_* a_{i1} = q_*^i$. The proof is complete. □

Theorem 4.3.1, as well as Lemmas 4.3.2 and 4.3.3, allow us to evaluate the extinction probabilities $a_{i0}$ and $a_{i1}$, as well as $a_{i\infty}$, the probability of explosion starting in state $i$.

**Theorem 4.3.4.** (i) If $B'(1) \leq 0$ then

$$\begin{cases} a_{i0} = (q_*^i - q_*)/(1 - q_*) \\ a_{i1} = (1 - q_*^i)/(1 - q_*) \end{cases} \tag{4.3.13}$$

and $a_{i\infty} = 0$.

(ii) If $0 < B'(1) \leq +\infty$ then

$$a_{i0} = (qq_*^i - q_* q^i)/(q - q_*), \tag{4.3.14}$$

$$a_{i1} = (q^i - q_*^i)/(q - q_*) \tag{4.3.15}$$

and $a_{i\infty} = \left(q(1 - q_*^i) - q_*(1 - q^i) - (q^i - q_*^i)\right)/(q - q_*)$.

**Proof.** We have already noted, in the proof of Lemma 4.3.2, that when $B'(1) \leq 0$ the honesty of $P(t)$ implies that $q^{i0} + q^{i1} = 1$. On the other hand, when $B'(1) > 0$ we have

$$a_{i0} + a_{i1} + a_{i\infty} = 1\,, \qquad i \geq 2, \tag{4.3.16}$$

by virtue of Lemma 4.3.2.

If $B'(1) \leq 0$ then by (4.3.2) in Theorem 4.3.1 and (4.3.12) in Lemma 4.3.3, we know that

$$\begin{cases} a_{i0} + a_{i1} = 1 \\ a_{i0} + q_* a_{i1} = q_*^i. \end{cases}$$

Solving above equation yields (4.3.13). Then by (4.3.16), $a_{i\infty} = 0$.

If $0 < B'(1) \leq \infty$, then by (4.3.3) in Theorem 4.3.1 and (4.3.12) in Lemma 4.3.3, we know that

$$\begin{cases} a_{i0} + a_{i1}q = q^i \\ a_{i0} + q_* a_{i1} = q_*^i. \end{cases}$$

Solving above equation yields (4.3.14) and (4.3.15). Then by (4.3.16) yields the explicit expression of $a_{i\infty}$. $\square$

Next we will evaluate the mean hitting times. Let

$$\mu_{ik} = E[\tau_k I_{\{\tau_k<\infty\}} | X(0) = i], \quad k = 0, 1$$

denote the expected extinction times starting in state $i$. Similarly, let $\mu_{i\infty} = E[\tau_\infty I_{\{\tau_\infty<\infty\}} | X(0) = i]$, where $\tau_\infty$ is explosion time.

**Theorem 4.3.5.** (i) If $B'(1) \leq 0$, the expected extinction times are all finite and are given by

$$\mu_{i0} = \frac{2}{(1-q_*)^2}[-q_* \int_0^1 \frac{(1-y)^2 f_i(y)}{B(y)} dy + \int_{q_*}^0 \frac{(y-q_*)(1-y)f_i(y)}{B(y)} dy], \tag{4.3.17}$$

$$\mu_{i1} = \frac{2}{(1-q_*)^2}[\int_0^1 \frac{(1-y)^2 f_i(y)}{B(y)} dy - \int_{q_*}^0 \frac{(y-q_*)(1-y)f_i(y)}{B(y)} dy] \tag{4.3.18}$$

for $i \geq 2$, where

$$f_i(y) = q_*^i - \frac{q_*(1-y^i)}{1-y} + \frac{y(1-y^{i-1})}{1-y}. \tag{4.3.19}$$

(ii) If $0 < B'(1) \leq +\infty$ then, again, the expected extinction times are all finite. They are given by

$$\mu_{i0} = \frac{2}{(q-q_*)^2}[-q_* \int_0^q \frac{(q-y)^2 f_i(y)}{B(y)} dy + q \int_{q_*}^0 \frac{(y-q_*)(q-y)f_i(y)}{B(y)} dy], \tag{4.3.20}$$

$$\mu_{i1} = \frac{2}{(q-q_*)^2}[\int_0^q \frac{(q-y)^2 f_i(y)}{B(y)} dy - \int_{q_*}^0 \frac{(y-q_*)(q-y)f_i(y)}{B(y)} dy] \tag{4.3.21}$$

for $i \geq 2$, where

$$f_i(y) = q_*^i - \frac{q_*(q^i-y^i)}{q-y} + \frac{qy(q^{i-1}-y^{i-1})}{q-y}. \tag{4.3.22}$$

**Proof.** To begin with, note that all of the integrals in (4.3.17), (4.3.18), (4.3.20) and (4.3.21) are finite; since the function $f_i(y)$ defined in (4.3.19), respectively (4.3.22), is bounded on $[-1,1]$ (indeed, $|f_i(y)| \le 2i$ for $y \in [-1,1]$), also note that, by Lemma 4.2.1, if $B'(1) \le 0$, then $\frac{(1-y)^2}{B(y)}$ and $\frac{(y-q_*)}{B(y)}$ are bounded on $[0,1]$ and $[q_*, 0]$ respectively, while if $0 < B'(1) \le \infty$, then $\frac{(q-y)^2}{B(y)}$ and $\frac{(y-q_*)}{B(y)}$ are bounded on $[0,q]$ and $[q_*,0]$ respectively, therefore, all the integrals are finite.

Note that in the proof of Lemma 4.3.2, we proved that (by integrating (4.3.5) with respect to $s$ and using Fubini's Theorem), for any $s \in [0,1)$,

$$F_i(t,s) = p_{i0}(t) + p_{i1}(t)s + 2\int_0^s \frac{s-y}{B(y)} F_i'(t,y)\,dy, \tag{4.3.23}$$

where $F_i'(t,y) = \partial F_i(t,y)/\partial t$. Similarly, integrating along the negative real axis gives, for any $s \in [q_*, 0]$,

$$F_i(t,s) = p_{i0}(t) + p_{i1}(t)s + 2\int_s^0 \frac{y-s}{B(y)} F_i'(t,y)\,dy. \tag{4.3.24}$$

We first prove (ii). Since $0 < B'(1) \le \infty$, $q \in (0,1)$. Let $s = q$ in (4.3.23) and $s = q_*$ in (4.3.24), and use the fact that $\sum_{j=0}^{\infty} p_{ij}(t) q^j = q^i$ and $\sum_{j=0}^{\infty} p_{ij}(t) q_*^j = q_*^i$, which follow from the fact that both $q$ and $q_*$ are roots of $B(s) = 0$ (again refer to the argument leading to (4.3.4)). This gives

$$p_{i0}(t) + p_{i1}(t)q = q^i - 2\int_0^q \frac{q-y}{B(y)} F_i'(t,y)\,dy,$$
$$p_{i0}(t) + p_{i1}(t)q_* = q_*^i - 2\int_{q_*}^0 \frac{y-q_*}{B(y)} F_i'(t,y)\,dy.$$

Now, in view of (4.3.1) and (4.3.12), the above equations can be rewritten as

$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t))q = 2\int_0^q \frac{q-y}{B(y)} F_i'(t,y)\,dy,$$
$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t))q_* = 2\int_{q_*}^0 \frac{y-q_*}{B(y)} F_i'(t,y)\,dy.$$

Integrating above two equations with respect to $t$ and noting $F_i(0,y) = y^i$ yields that

$$\int_0^t (a_{i0} - p_{i0}(u))du + q\int_0^t (a_{i1} - p_{i1}(u))du = 2\int_0^q \frac{q-y}{B(y)} (F_i(t,y) - y^i)dy,$$

$$\int_0^t (a_{i0} - p_{i0}(u))du + q_* \int_0^t (a_{i1} - p_{i1}(u))du = 2\int_{q_*}^0 \frac{y-q_*}{B(y)}(F_i(t,y) - y^i)dy.$$

Note that $\int_0^q \frac{q-y}{B(y)}dy < \infty$ and $\int_{q_*}^0 \frac{y-q_*}{B(y)}dy < \infty$, letting $t \to \infty$ in the above two equations and using Dominated Convergence Theorem yields that

$$\mu_{i0} + q\mu_{i1} = 2\int_0^q \frac{q-y}{B(y)}(F_i(\infty,y) - y^i)dy,$$

$$\mu_{i0} + q_*\mu_{i1} = 2\int_{q_*}^0 \frac{y-q_*}{B(y)}(F_i(\infty,y) - y^i)dy$$

since $a_{ik} - p_{ik}(t) = P(t < \tau_k < \infty | X(0) = i)$, $k = 0, 1$, ($\tau_k$ ($k = 0, 1$) are defined at the beginning of this section,i.e., the extinction times for states 0 and 1 respectively), where $F_i(\infty, y) := \lim_{t\to\infty} F_i(t,y)$. Using the identities $F_i(\infty, y) = a_{i0} + a_{i1}y$ yields that

$$\mu_{i0} + \mu_{i1}q = 2\int_0^q \frac{(q-y)(a_{i0} + a_{i1}y - y^i)}{B(y)}dy \tag{4.3.25}$$

$$\mu_{i0} + \mu_{i1}q_* = 2\int_{q_*}^0 \frac{(y-q_*)(a_{i0} + a_{i1}y - y^i)}{B(y)}dy. \tag{4.3.26}$$

Solving for $\mu_{i0}$ and $\mu_{i1}$, we eventually arrive at (4.3.20)–(4.3.22).

We now prove (i). Since $B'(1) \le 0$. Note that (4.3.26) still holds. Next, let $s \uparrow 1$ in (4.3.23) and using Monotone Convergence Theorem in integral on the right-hand side of (4.3.23) yields that

$$p_{i0}(t) + p_{i1}(t) = 1 - 2\int_0^1 \frac{1-y}{B(y)}F_i'(t,y)dy$$

since $\sum_{j=0}^{\infty} p_{ij}(t) = 1$. Now, in view of (4.3.2), the above equation can be rewritten as

$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) = 2\int_0^1 \frac{1-y}{B(y)}F_i'(t,y)dy.$$

Integrating the above equation with respect to $t$ and noting $F_i(0,y) = y^i$ yields that

$$\int_0^t (a_{i0} - p_{i0}(u))du + \int_0^t (a_{i1} - p_{i1}(u))du = 2\int_0^1 \frac{1-y}{B(y)}(F_i(t,y) - y^i)dy.$$

Note that by (4.3.5) and Lemma 4.3.2, $F_i(t,y)$ is increasing with respect to $t$ and $F_i(t,y) \uparrow a_{i0} + a_{i1}y$. Also note that $\int_0^1 \frac{(1-y)^2}{B(y)}dy < \infty$, letting

$t \to \infty$ in the above two equation and using Dominated Convergence Theorem yields that

$$\mu_{i0} + q\mu_{i1} = 2\int_0^q \frac{q-y}{B(y)}(F_i(\infty, y) - y^i)dy,$$

since $a_{ik} - p_{ik}(t) = P(t < \tau_k < \infty | X(0) = i)$, $k = 0, 1$, ($\tau_k$ $(k = 0, 1)$ are defined at the beginning of this section,i.e., the extinction times for states 0 and 1 respectively), where $F_i(\infty, y) := \lim_{t\to\infty} F_i(t, y)$. Using the identities $F_i(\infty, y) = a_{i0} + a_{i1}y$ yields that

$$\mu_{i0} + \mu_{i1}q = 2\int_0^q \frac{(q-y)(a_{i0} + a_{i1}y - y^i)}{B(y)}dy.$$

Solving the above equation and (4.3.26) for $\mu_{i0}$ and $\mu_{i1}$, we eventually arrive at (4.3.17)–(4.3.19). The proof is complete. □

Next we will evaluate the expected time to explosion. By Theorem 4.3.4, only the case $0 < B'(1) \le +\infty$ need be considered. Since we are dealing with the minimal process,

$$p_{i\infty}(t) := 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \le t | X(0) = i)\,, \tag{4.3.27}$$

is the probability of explosion by time $t$ starting in state $i$, and $p_{i\infty}(t) \to a_{i\infty}$ as $t \to \infty$.

**Theorem 4.3.6.** If $0 < B'(1) \le +\infty$, then the expected explosion time is finite and is given by

$$\begin{aligned}
&\mu_{i\infty} \\
= \; & \frac{2}{(q-q_*)}\left(\int_0^1 \frac{(1-y)(q-y)}{B(y)} f_i(y)\,dy - \frac{1-q_*}{q-q_*}\int_0^q \frac{(q-y)^2}{B(y)} f_i(y)\,dy \right.\\
& \left. + \frac{1-q}{q-q_*}\int_{q_*}^0 \frac{(y-q_*)(q-y)}{B(y)} f_i(y)\,dy\right)
\end{aligned} \tag{4.3.28}$$

for $i \ge 2$, where $f_i(y)$ is given in (4.3.22).

**Proof.** Fix $i \ge 2$ and observe that $\mu_{i\infty} < \infty$, because, as already noted, all of the integrals in (4.3.28) are finite. Since $0 < B'(1) \le +\infty$ we know that $p_{i\infty}(t) > 0$. Furthermore, $\mu_{i\infty} = \int_0^\infty (a_{i\infty} - p_{i\infty}(t))\,dt$, because $P(t < \tau_\infty < \infty | X(0) = i) = a_{i\infty} - p_{i\infty}(t)$. However, $a_{i\infty} = 1 - a_{i0} - a_{i1}$, where $a_{i0}$ and $a_{i1}$ are given in Theorem 4.3.4(ii). This, together with (4.3.27), yields

$$\mu_{i\infty} = \sum_{j=2}^{\infty} \int_0^{\infty} p_{ij}(t)\, dt - \mu_{i0} - \mu_{i1}, \tag{4.3.29}$$

where $\mu_{i0}$ and $\mu_{i1}$ are given in (4.3.20) and (4.3.21), respectively. Note that $\int_0^\infty p_{ij}(t)\, dt < \infty$ for all $j \geq 2$, because all states $j \geq 2$ are transient. Moreover, by virtue of (4.2.1), this integral can be evaluated explicitly; on integrating (4.2.1) with respect to $t$ from 0 to $\infty$, we get

$$\frac{a_{i0} + a_{i1}s - s^i}{B(s)} = \sum_{k=2}^{\infty} \binom{k}{2} \int_0^{\infty} p_{ik}(t)\, dt\, s^{k-2}, \qquad |s| < 1, \tag{4.3.30}$$

and extracting the coefficient of $s^{k-2}$ gives

$$\int_0^{\infty} p_{ik}(t)\, dt = \frac{2}{k!} G_i^{(k-2)}(0), \qquad i \geq 2,\ k \geq 2, \tag{4.3.31}$$

where

$$G_i(s) = \frac{a_{i0} + a_{i1}s - s^i}{B(s)}. \tag{4.3.32}$$

Now, integrating (4.3.30) twice with respect to $s$ yields

$$\sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t)\, dt\, s^k = 2 \int_0^s (s-y) G_i(y)\, dy$$

and letting $s \uparrow 1$ shows that

$$\sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t)\, dt = 2 \int_0^1 (1-y) G_i(y) dy. \tag{4.3.33}$$

Substituting (4.3.33) into (4.3.29) then yields

$$\mu_{i\infty} = 2 \int_0^1 (1-y) G_i(y) dy - \mu_{i0} - \mu_{i1}, \tag{4.3.34}$$

and, after substituting the expressions for $a_{i0}$ and $a_{i1}$ given in (4.3.14) and (4.3.15) into (4.3.32) yields that

$$\begin{aligned} G_i(y) &= \frac{qq_*^i - q_* q^i + (q^i - q_*)y - (q - q_*)y^i}{(q - q_*)B(y)} \\ &= \frac{(q-y) f_i(y)}{(q - q_*)B(y)}, \end{aligned}$$

where $f_i(y)$ is given by (4.3.22). Thus

$$\int_0^1 (1-y) G_i(y)\, dy = \int_0^1 \frac{(q-y)(1-y)}{(q-q_*)B(y)} f_i(y)\, dy, \tag{4.3.35}$$

On the other hand, by (4.3.20) and (4.3.21),

$$\begin{aligned}\mu_{i0}+\mu_{i1} &= \frac{2(1-q_*)}{(q-q_*)^2}\int_0^q \frac{(q-y)^2 f_i(y)}{B(y)}dy \\ &\quad -\frac{2(1-q)}{(q-q_*)^2}\int_{q_*}^0 \frac{(y-q_*)(q-y)f_i(y)}{B(y)}dy.\end{aligned}$$

Finally, substituting the above equality and (4.3.35) into (4.3.34) yields (4.3.28). The proof is complete. □

We have proved that the CBP either explodes, or is absorbed, in finite-mean time. Our final result concerns the time spent in each state over the lifetime of the process. Let $T_k$ be the total time spent in state $k$ ($\geq 2$) and let $\mu_{ik} = E[T_k|X(0)=i]$, $i \geq 2$. Then,

$$\mu_{ik} = E\left[\int_0^\infty I_{\{X(t)=k\}}\,dt \,\middle|\, X(0)=i\right] = \int_0^\infty p_{ik}(t)\,dt.$$

This expression was evaluated in (4.3.31). We have therefore proved the following result.

**Theorem 4.3.7.** All of $\mu_{ik}$, $i \geq 2$, $k \geq 2$, are finite and given by

$$\mu_{ik} = \frac{2}{k!}G_i^{(k-2)}(0) \tag{4.3.36}$$

where $G_i^{(k-2)}(0)$ is the $(k-2)$-th derivative near 0 of $G_i$ given in (4.3.32). In particular,

$$\mu_{i2} = \frac{a_{i0}}{B(0)} = \frac{-qq_*(q^{i-1}-q_*^{i-1})}{b_0(q-q_*)}, \qquad i \geq 2,$$

for example $\mu_{22} = -qq_*/b_0$, and, $\Sigma_{k=2}^{\infty}\mu_{ik} = \mu_{i0}+\mu_{i1}+\mu_{i\infty}$.

**Remark 4.3.1.** The argument used in proving Theorems 4.3.5–4.3.7 may, in principle, be extended to obtain results concerning the variance and the higher moments of the extinction, explosion and total holding times. We only give a brief sketch of such extension in the following. For example, Suppose that we consider $\mu_{i0}^{(2)} = E_i[\tau_0^2 I_{\{\tau_0<\infty\}}]$ and $\mu_{i1}^{(2)} = E_i[\tau_1^2 I_{\{\tau_1<\infty\}}]$ in the case $0 < B'(1) \leq \infty$. Recall that in the proof of we obtained that

$$(a_{i0}-p_{i0}(t)) + q(a_{i1}-p_{i1}(t)) = 2\int_0^q \frac{q-y}{B(y)}F_i'(t,y)dy,$$

$$(a_{i0}-p_{i0}(t)) + q_*(a_{i1}-p_{i1}(t)) = 2\int_{q_*}^0 \frac{y-q_*}{B(y)}F_i'(t,y)dy.$$

Multiplying the above two equalities with $t$ and integrating with respect to $t$ yields that

$$\int_0^\infty t(a_{i0}-p_{i0}(t))dt+q\int_0^\infty t(a_{i1}-p_{i1}(t))dt = 2\int_0^q \frac{q-y}{B(y)}\left(\int_0^\infty tF_i'(t,y)dt\right)dy,$$

$$\int_0^\infty t(a_{i0}-p_{i0}(t))dt+q_*\int_0^\infty t(a_{i1}-p_{i1}(t))dt = 2\int_{q_*}^0 \frac{y-q_*}{B(y)}\left(\int_0^\infty tF_i'(t,y)dt\right)dy.$$

If $\mu_{i0}^{(2)} < \infty$ and $\mu_{i1}^{(2)} < \infty$ then

$$\mu_{i0}^{(2)} = \int_0^\infty t^2 dp_{i0}(t) = 2\int_0^\infty t(a_{i0}-p_{i0}(t))dt$$
$$\mu_{i1}^{(2)} = \int_0^\infty t^2 dp_{i1}(t) = 2\int_0^\infty t(a_{i1}-p_{i1}(t))dt.$$

On the other hand,

$$\begin{aligned}
&\int_0^\infty tF_i'(t,y)dt \\
=\ & \int_0^\infty td(p_{i0}(t)+p_{i1}(t)y) + \sum_{k=2}^\infty \left(\int_0^\infty tdp_{ik}(t)\right)y^k \\
=\ & \int_0^\infty (a_{i0}-p_{i0}(t))dt + y\int_0^\infty (a_{i1}-p_{i1}(t))dt + \sum_{k=2}^\infty \left(\int_0^\infty p_{ik}(t)dt\right)y^k \\
=\ & \sum_{k=0}^\infty \mu_{ik}y^k.
\end{aligned}$$

Therefore, we can obtain

$$\mu_{i0}^{(2)} + q\mu_{i1}^{(2)} = 4\sum_{k=0}^\infty \mu_{ik}\int_0^q \frac{q-y}{B(y)}y^k dy,$$
$$\mu_{i0}^{(2)} + q_*\mu_{i1}^{(2)} = 4\sum_{k=0}^\infty \mu_{ik}\int_{q_*}^0 \frac{y-q_*}{B(y)}y^k dy.$$

Solving the above equations yields $\mu_{i0}^{(2)}$ and $\mu_{i1}^{(2)}$. The other moments can be similarly obtained.

### 4.4. An example

As an example, we will investigate the upwardly skip-free case in this section. This will serve to illustrate our results and to show that formulae such as (4.3.36) can be evaluated easily. Let $a_1$, $a_2$ and $b$ be positive constants, and set $b_0 = a_2$, $b_1 = a_1$, $b_3 = b$ and $b_j = 0$ for all $j \geq 4$. The generating function $B(s)$ is given by

$$B(s) = a_2 + a_1 s - (a_1 + a_2 + b)s^2 + bs^3.$$

It has three zeros: two positive ones, 1 and

$$\rho = \frac{1}{2b}\left(a_1 + a_2 + \sqrt{(a_1+a_2)^2 + 4a_2 b}\right),$$

and the third, strictly negative zero

$$q_* = \frac{1}{2b}\left(a_1 + a_2 - \sqrt{(a_1+a_2)^2 + 4a_2 b}\right).$$

Also, $B'(1) = b - a_1 - 2a_2$. If $b < a_1 + 2a_2$ ($B'(1) < 0$), then $\rho > 1$ and $q = 1$. If $b = a_1 + 2a_2$ ($B'(1) = 0$), then $q = \rho = 1$ is a multiplicity-2 zero, while if $b > a_1 + a_2$ ($B'(1) > 0$), then $q = \rho < 1$. We can use Theorem 4.3.4 to evaluate the hitting probabilities. If $b \le a_1 + 2a_2$, the extinction probabilities are given by

$$a_{i0} = \frac{q_*^i - q_*}{1 - q_*}, \quad a_{i1} = \frac{1 - q_*^i}{1 - q_*} \quad (\textit{and thus } \; a_{i0} + a_{i1} = 1),$$

while if $b > a_1 + 2a_2$, the extinction and explosion probabilities are given by

$$a_{i0} = \frac{\rho q_*^i - q_* \rho^i}{\rho - q_*}, \quad a_{i1} = \frac{\rho^i - q_*^i}{\rho - q_*} \quad \textit{and} \quad a_{i\infty} = 1 - a_{i0} - a_{i1} > 0.$$

In order to get more concrete results let us assume that the process starts in state $i = 2$. Using Theorems 4.3.5, 4.3.6 and 4.3.7 obtain the following result.

**Proposition 4.4.1.** For upwardly skip-free CB process the follow are true:

(i) If $b \le 2a_2 + a_1$ then $a_{20} = -q_*$ and $a_{21} = 1 + q_*$ with $a_{2\infty} = 0$, while if $b > 2a_2 + a_1$ then

$$a_{20} = \frac{a_2}{b}, \quad a_{21} = \frac{a_1 + a_2}{b} \quad \textit{and} \quad a_{2\infty} = \frac{b - (a_1 + 2a_2)}{b} > 0.$$

(ii) If $b > 2a_2 + a_1$ then

$$\mu_{20} = \frac{2}{b(q - q_*)} \ln\left(\frac{(1-q_*)^{q(1-q_*)}}{(1-q)^{q_*(1-q)}}\right),$$

$$\mu_{21} = \frac{2}{b}\left(1 + \frac{1}{(q - q_*)} \ln\left(\frac{(1-q)^{1-q}}{(1-q_*)^{1-q_*}}\right)\right),$$

$$\mu_{2k} = \frac{2}{bk(k-1)}, \; k \geq 2, \quad \mu_{2\infty} = \frac{2}{b}\ln\left(\frac{(1-q)^{(1-q)(1+q_*)}}{(1-q_*)^{(1-q_*)(1+q)}}\right),$$

and thus $\mu_{20} + \mu_{21} + \mu_{2\infty} = \Sigma_{k=2}^{\infty}\mu_{2k} = 2/b$.

(iii) If $b = 2a_2 + a_1$ then

$$\mu_{20} = \frac{2}{b}\ln(1-q_*), \quad \mu_{21} = \frac{2}{b}\left(1 - \ln(1-q_*)\right),$$

$$\mu_{2k} = \frac{2}{bk(k-1)}, \; k \geq 2,$$

and thus $\mu_{20} + \mu_{21} = \Sigma_{k=2}^{\infty}\mu_{2k} = 2/b$.

(iv) If $b < 2a_2 + a_1$ then

$$\mu_{20} = \frac{2}{b(1-q_*)}\ln\left(\frac{(1-q_*)^{(1-q_*)}}{(1-\rho)^{q_*(1/\rho-1)}}\right),$$

$$\mu_{21} = \frac{2}{b}\left(1 + \frac{1}{(1-q_*)}\ln\left(\frac{(1-\rho)^{1/\rho-1}}{(1-q_*)^{1-q_*}}\right)\right),$$

$$\mu_{2k} = \frac{2}{bk(k-1)\rho^{k-1}}, \; k \geq 2,$$

and thus

$$\mu_{20} + \mu_{21} = \sum_{k=2}^{\infty}\mu_{2k} = \frac{2}{b}\left(1 - (\rho-1)\ln\left(1 - \frac{1}{\rho}\right)\right). \tag{4.4.1}$$

Notice the simple form for the expected time spent in state $k$ when $b \geq 2a_2 + a_1$, this being proportion to the reciprocal of $\binom{k}{2}$. Notice also that expected lifetime of the process is simply $\Sigma_{k=2}^{\infty}\mu_{2k} = 2/b$. Yet, the behaviour of the process in the two cases $b = 2a_2 + a_1$ and $b > 2a_2 + a_1$ is quite different. In the former case the process will eventually be absorbed at either 0 or 1 ($B'(1) = 0$), while in the latter ($B'(1) > 0$) the process has a positive probability of explosion. And, the same total life time $2/b$ comprises only $\mu_{20}$ and $\mu_{21}$ for the former case, but $\mu_{20}$, $\mu_{21}$ and $\mu_{2\infty}$ for the latter. In contrast, when $b < 2a_2 + a_1$ ($B'(1) < 0$) the expected lifetime is strictly smaller than $2/b$; see (4.4.1).

## 4.5. Notes

This model was considered by several authors, including Ezhov (1980) and Kalinkin (2002), and can be traced back to Sevast'yanov (1949). They studied the extinction probability and also gave a sufficient condition for the regularity. Here, a different but significant method is used to further study this model, a satisfactory "if and only if" condition for the regularity is presented and some further important properties are obtained.

This chapter is due to Chen, Pollett, Zhang and Li (2004). More specifically, Lemma 4.2.1, Lemma 4.3.3 and Theorem 4.3.5 are due to Li, Lemma 4.2.1, which considers the function $B(s)$ and its roots (especially the negative root), makes it possible to obtain the extinction probabilities and also plays a key role throughout the further study of CBP. Furthermore, this idea can be used to study interacting branching models with more than 2 absorbing states. Theorem 4.3.4, Theorem 4.3.6 and Theorem 4.3.7 are due to Li and the other authors. While the idea in the "only if" part of the proof of Theorem 4.2.3 is due to Chen (1997).

It is interesting to compare the behaviour described in Theorems 4.2.3, 4.2.4, 4.3.1 and Lemma 4.3.3, with that of the ordinary MBP. The behaviour of CBP is similar as MBP in the subcritical or critical case (i.e., $B'(1) < 0$ or $B'(1) = 0$ respectively). However, in the supercritical case ($B'(1) > 0$), the behaviour of the two processes is different. Whilst both are absorbed with probability less than 1, the CBP is always dishonest, whereas the MBP can only be dishonest when $B'(1) = +\infty$, and this happens when and only when Harris' integral condition fails. By Lemma 4.3.2, unlike the MBP, the CBP may never drift passively towards infinity. If absorption does not occur, the CBP will certainly explode. The latter is also true of the MBP, in that when the MBP is dishonest ($B'(1) = +\infty$ and Harris' condition fails), it is either absorbed or it explodes with probability 1 (see Chen and Renshaw (1993b)).

In the next chapter, we shall consider a class of more general collision branching models.

# 5. General Collision Branching Processes with 2 Parameters

## 5.1. Introduction

In Chapter 4 we studied collision branching process where branching events are effected by the interaction/collision of *pairs* of particles, rather than by the particles individually. In this chapter, we consider a new class of collision branching models which is more general than that considered in Chapter 4. We will see later that for the model considered in Chapter 4, the results regarding extinction probabilities, extinction time, explosion probability and explosion time can be deduced from the related results in this chapter. However, the differential equation about the generating function of transition probability is not available for the general model in this chapter (except for the special case CBP); in addition, the regularity criterion is very simple for CBP while it is more subtle for the general case considered in this chapter.

The $q$-matrix $Q$ of the process discussed in this chapter is defined as follows.

**Definition 5.1.1.** A conservative $q$-matrix $Q = (q_{ij};\ i, j \in \mathbf{Z}_+)$ is called a general collision branching $q$-matrix with 2 parameters (henceforth simply referring to as a GCB $q$-matrix) if it takes the following form:

$$q_{ij} = \begin{cases} i^{\alpha}(i-1)^{\beta} b_{j-i+2}, & if\ \ i \geq 2,\ j > i-2 \\ 0, & otherwise, \end{cases} \tag{5.1.1}$$

where

$$\alpha > 0,\ \ \beta > 0,\ \ b_0 > 0,\ \ b_j \geq 0\ \ (j \neq 2) \tag{5.1.2}$$

and

$$\sum_{j=3}^{\infty} b_j > 0,\ \ 0 < -b_2 = \sum_{j \neq 2} b_j < +\infty. \tag{5.1.3}$$

Furthermore, a GCB $q$-matrix is called super-explosive if $\alpha + \beta > 1$ and sub-explosive if $\alpha + \beta \leq 1$.

By Remark 4.1.1, we may assume that the GCB $q$-matrix is not degenerative, i.e., $\sum_{j=0}^{\infty} b_{2j+1} > 0$.

¿From (5.1.1), in addition to the sequence $\{b_j; j \geq 0\}$, the two positive constants $\alpha$ and $\beta$ are two parameters which affect the speed of birth and death events. These two parameters, not necessarily integers, may be interpreted as acceleration (for $alpha, \beta > 1$) or deceleration (for $\alpha, \beta < 1$) index of the interaction among different particles. A GCB $q$-matrix is super-explosive (respectively, sub-explosive) if and only if $\sum_{k=2}^{\infty} \frac{1}{k^{\alpha}(k-1)^{\beta}}$ is finite (respectively, infinite).

If we let $\alpha = \beta = 1$ in (5.1.1), we recover the model considered in Chapter 4 (there, the transition rates are proportional to $\{i(i-1)/2\}$ rather than to $\{i(i-1)\}$. However, the constant $\frac{1}{2}$ can be absorbed in the sequence $\{b_j\}$. Hence the model considered there is super-explosive. As will be shown later, the behaviour between the super-explosive and sub-explosive cases is quite different.

**Definition 5.1.2.** A general collision branching process with 2 parameters (henceforth simply referred to as a GCBP) is a continuous-time Markov chain taking values in $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t);\ i, j \in \mathbf{Z}_+)$ satisfies the Kolmogorov forward equations

$$P'(t) = P(t)Q, \tag{5.1.4}$$

where $Q$ is a GCB $q$-matrix. A GCBP is called super-explosive (sub-explosive) if the corresponding $q$-matrix $Q$ is super-explosive (sub-explosive).

## 5.2. Regularity and Uniqueness

By Theorem 1.3.1, there always exists a GCBP which is the (possibly dishonest) Feller minimal process. Now we study the regularity and uniqueness problems. Similarly as in Chapter 4, let

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \quad |s| \leq 1.$$

$$m_d = 2b_0 + b_1, \quad m_b = \sum_{j=1}^{\infty} j b_{j+2}.$$

Note that the mean birth rate $m_b$ may be infinite and we shall not exclude this difficult but interesting case in our study.

The detailed property of $B(s)$ can be seen in Lemma 4.2.1. Again, by Lemma 4.2.1, let $q$ and $q_*$ denote the smallest positive root and largest negative root of $B(s) = 0$. In addition to Lemma 4.2.1, we also need the following four lemmas. They are not only essential in settling the question of uniqueness, but also are very useful in the later sections.

**Lemma 5.2.1.** Let $(p_{ij}(t); i,j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a GCB $q$-matrix given in (5.1.1)−(5.1.3). Then for any $i \geq 0$ and $|s| < 1$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \cdot \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}s^{k-2} \tag{5.2.1}$$

and

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda)s^j = s^i + B(s) \cdot \sum_{k=2}^{\infty} \phi_{ik}(\lambda)k^{\alpha}(k-1)^{\beta}s^{k-2}. \tag{5.2.2}$$

**Proof.** Substituting (5.1.1) into the the Kolmogorov forward equation (5.1.4) yields that for $|s| < 1$,

$$\begin{aligned}
\sum_{j=0}^{\infty} p'_{ij}(t)s^j &= \sum_{j=0}^{\infty}\Big(\sum_{k=2}^{j+2} p_{ik}(t)k^{\alpha}(k-1)^{\beta} \cdot \frac{q_{kj}}{k^{\alpha}(k-1)^{\beta}}\Big)s^j \\
&= \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}s^{k-2} \sum_{j=k-2}^{\infty} \frac{q_{kj}}{k^{\alpha}(k-1)^{\beta}}s^{j-k+2} \\
&= B(s)\sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}s^{k-2}.
\end{aligned}$$

Justification of using Fubini's theorem in the above second equality is guaranteed. Indeed, since the convergence radius of the power series $\sum_{k=2}^{\infty} k^{\alpha}(k-1)^{\beta}s^{k-2}$ is 1 and thus for any $|s| < 1$,

$$\begin{aligned}
&\sum_{j=0}^{\infty}\Big(\sum_{k=2}^{j+2} p_{ik}(t)k^{\alpha}(k-1)^{\beta} \cdot \frac{|q_{kj}|}{k^{\alpha}(k-1)^{\beta}}\Big)|s|^j \\
\leq\ & \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}|s|^{k-2} \sum_{j=k-2}^{\infty} \frac{|q_{kj}|}{k^{\alpha}(k-1)^{\beta}}|s|^{j-k+2} \\
\leq\ & \frac{-2b_2}{1-|s|} \cdot \sum_{k=2}^{\infty} k^{\alpha}(k-1)^{\beta}|s|^{k-2} < +\infty.
\end{aligned}$$

Thus (5.2.1) holds. Finally taking Laplace transform we can get (5.2.2) from (5.2.1). Indeed, the Laplace transform of the right-hand side of (5.2.1) is

$$B(s) \cdot \sum_{k=2}^{\infty} p_{ik}(t) \cdot k^{\alpha}(k-1)^{\beta} s^{k-2},$$

while the Laplace transform of the left-hand side of (5.2.1) is

$$\begin{aligned} & \int_0^{\infty} e^{-\lambda t} (\sum_{j=0}^{\infty} p'_{ij}(t) s^j) dt \\ = & \sum_{j=0}^{\infty} (\int_0^{\infty} e^{-\lambda t} p'_{ij}(t) dt) s^j \\ = & \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j - s^i \end{aligned}$$

By (5.2.1), we know that (5.2.2) holds. The proof is complete. □

In order to obtain more informative forms than (5.2.1) and (5.2.2), we define a family of probability generating functions $F = \{F_i(t,s);\ i \geq 0\}$ of the Feller minimal $Q$-function $(p_{ij}(t);\ i,j \in \mathbf{Z}_+)$ as

$$F_i(t,s) = \sum_{j=0}^{\infty} p_{ij}(t) s^j, \qquad |s| \leq 1. \tag{5.2.3}$$

**Lemma 5.2.2.** Let $(p_{ij}(t); i,j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a GCB $q$-matrix given in (5.1.1)–(5.1.3). Then for any $i \geq 0$, $s \in [0,1]$,

$$\begin{aligned} & \Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} p_{ik}(t) s^k \\ = & \ s \int_0^s \frac{F'_i(t,y)}{B(y)(\ln \frac{s}{y})^{1-\alpha-\beta}} \cdot (\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{s})^u du) dy, \end{aligned} \tag{5.2.4}$$

$$\begin{aligned} & \Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} \phi_{ik}(\lambda) s^k \\ = & \ s \int_0^s \frac{\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) y^j - y^i}{B(y)(\ln \frac{s}{y})^{1-\alpha-\beta}} \cdot (\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{s})^u du) dy \end{aligned} \tag{5.2.5}$$

where $F'_i(t,y) = \frac{\partial F_i(t,y)}{\partial t}$ and $\Gamma(\cdot)$ is the gamma function.

**Proof.** (5.2.4) and (5.2.5) are trivial for $i = 0$ *or* $1$ since the states 0 and 1 are absorbing. So we assume $i \geq 2$. Using (5.2.3) we may rewrite

(5.2.1) as

$$\frac{F_i'(t,y)}{B(y)} = \sum_{k=2}^{\infty} p_{ik}(t) k^{\alpha}(k-1)^{\beta} y^{k-2} \tag{5.2.6}$$

for all $y \in [0,1)$ provided that $B(y) \neq 0$. Note that by Lemma 4.2.1 if $m_d \geq m_b$ then the left-hand side of (5.2.6) is well-defined for all $0 \leq y < 1$ while if $m_d < m_b \leq +\infty$, the only singularity at $y = q$ on the left-hand side of (5.2.6) is clearly removable, because the series on the right-hand side certainly converges for $y \in [0,1)$ (for more details, see Remark 5.2.1 below). It follows that (5.2.6) holds true for all $y \in [0,1)$ in both cases. Moreover, the right-hand side and therefore the left-hand side of (5.2.6) is a continuous, increasing and strictly positive function of $y$ on this interval. Now for any given $s \in [0,1]$, multiplying $(\ln \frac{s}{x})^{\alpha-1} \cdot (\ln \frac{x}{y})^{\beta-1}$ on both sides of (5.2.6) and integrating on $0 < y < x < s$ yields

$$\begin{aligned}
&\int_0^s \Big(\int_0^x \frac{F_i'(t,y)}{B(y)} \cdot (\ln \frac{x}{y})^{\beta-1} dy\Big) \cdot (\ln \frac{s}{x})^{\alpha-1} dx \\
= \ & \sum_{k=2}^{\infty} p_{ik}(t) k^{\alpha}(k-1)^{\beta} \int_0^s \int_0^x y^{k-2} (\ln \frac{x}{y})^{\beta-1} (\ln \frac{s}{x})^{\alpha-1} dy dx.
\end{aligned}$$

For each term $k \geq 2$ in the right-hand side of the above equality, perform the transformation $\ln \frac{x}{y} = \frac{u}{k-1}$ and $\ln \frac{s}{x} = \frac{v}{k}$, then we obtain that

$$\begin{aligned}
&\int_0^s \int_0^x y^{k-2} (\ln \frac{x}{y})^{\beta-1} (\ln \frac{s}{x})^{\alpha-1} dy dx \\
= \ & s^k \int_0^{\infty} \int_0^{\infty} e^{-\frac{k-2}{k} v - \frac{k-2}{k-1} u} \cdot \frac{u^{\beta-1}}{(k-1)^{\beta-1}} \cdot \frac{v^{\alpha-1}}{k^{\alpha-1}} \cdot \frac{1}{k(k-1)} \cdot e^{-\frac{2v}{k} - \frac{u}{k-1}} du dv \\
= \ & \frac{s^k}{k^{\alpha}(k-1)^{\beta}} \int_0^{\infty} u^{\beta-1} e^{-u} du \cdot \int_0^{\infty} v^{\alpha-1} e^{-v} dv \\
= \ & \frac{\Gamma(\alpha)\Gamma(\beta)}{k^{\alpha}(k-1)^{\beta}} s^k,
\end{aligned}$$

here and throughout this chapter, $\Gamma(\cdot)$ denotes the gamma function. Therefore,

$$\int_0^s \Big(\int_0^x \frac{F_i'(t,y)}{B(y)} \cdot (\ln \frac{x}{y})^{\beta-1} dy\Big) \cdot (\ln \frac{s}{x})^{\alpha-1} dx = \Gamma(\alpha)\Gamma(\beta) \sum_{k=2}^{\infty} p_{ik}(t) s^k,$$

and hence

$$\int_0^s \frac{F_i'(t,y)}{U(y)} \cdot \left(\int_y^s (\ln \frac{s}{x})^{\alpha-1} (\ln \frac{x}{y})^{\beta-1} dx\right) dy = \Gamma(\alpha)\Gamma(\beta) \sum_{k=2}^{\infty} p_{ik}(t) s^k.$$

Using the transformation $u = (\ln \frac{s}{y})^{-1} \ln \frac{s}{x}$ in the integral

$$\int_y^s (\ln \frac{s}{x})^{\alpha-1} (\ln \frac{x}{y})^{\beta-1} dx$$

on the left-hand side of the above equality then achieves (5.2.4) for the case $0 \leq s < 1$. Now we rewrite (5.2.4) as

$$\begin{aligned} & \Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} p_{ik}(t) s^k \\ = \; & s^2 \int_0^1 \frac{F_i'(t, sx)}{B(sx)(-\ln x)^{1-\alpha-\beta}} \cdot (\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} x^u du) dx, \quad s \in [0,1). \end{aligned}$$

By (5.2.6), $\frac{F_i'(t,sx)}{B(sx)}$ is an increasing function of $s$, therefore, letting $s \uparrow 1$ in above equality and using Monotone Convergence Theorem yields that (5.2.4) holds for $s = 1$. Finally, since (5.2.5) is just the Laplace transform of (5.2.4), we know that (5.2.5) holds. The proof is complete. □

The following lemma shows that any state $i \geq 2$ is transient which in turn yields far-reaching consequences.

**Lemma 5.2.3.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function, where $Q$ is a GCB $q$-matrix given in (5.1.1)–(5.1.3). Then for any $i \geq 2$, $j \geq 2$ we have

$$\int_0^\infty p_{ij}(t) dt < \infty. \tag{5.2.7}$$

Furthermore, if $m_d \geq m_b$ then

$$\lim_{t \to \infty} p_{i0}(t) + \lim_{t \to \infty} p_{i1}(t) = 1 \tag{5.2.8}$$

while if $m_d < m_b \leq +\infty$, then

$$\lim_{t \to \infty} p_{i0}(t) + q \lim_{t \to \infty} p_{i1}(t) = q^i \tag{5.2.9}$$

and thus

$$\lim_{t \to \infty} p_{i0}(t) + \lim_{t \to \infty} p_{i1}(t) < 1 \tag{5.2.10}$$

where $q \in (0,1)$ is the smallest positive root of $B(s) = 0$.

**Proof.** For any fixed $i \geq 0$, it follows from the Kolomogorv forward equations that

$$p_{i0}(t) = \delta_{i0} + q_{20} \cdot \int_0^t p_{i2}(s) ds$$

which implies that $\int_0^\infty p_{i2}(t)dt < \infty$. Suppose $\int_0^\infty p_{ik}(t)dt < \infty$ for $2 \leq k \leq j$. From Kolmogorov forward equations we can see that

$$p_{ij-1}(t) - \delta_{ij-1} = \sum_{k=2}^{j} k^\alpha (k-1)^\beta b_{j-k+1} \cdot \int_0^\infty p_{ik}(t)dt + (j+1)^\alpha j^\beta b_0 \int_0^\infty p_{ij+1}(t)dt$$

and hence $\int_0^\infty p_{ij+1}(t)dt < \infty$. Therefore, (5.2.7) follows from the mathematical induction principle. Thus all states $j \geq 2$ are transient which then implies

$$\lim_{t\to\infty} p_{ij}(t) = 0, \quad (\forall i \geq 2, j \geq 2). \tag{5.2.11}$$

For the need of the following proof we note that, for any $i \geq 2$, $p_{i0}(t)$ and $p_{i1}(t)$ are increasing with respect to $t$ since states 0 and 1 are both absorbing, hence both $\lim_{t\to\infty} p_{i0}(t)$ and $\lim_{t\to\infty} p_{i1}(t)$ exist.

Now, assume $m_d \geq m_b$, then Lemma 4.2.1 implies that the right-hand side of (5.2.1) is nonnegative (strictly positive if $i \geq 2$) for $s \in (0,1)$. Thus, on integrating the left-hand side of (5.2.1), we see that

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i \geq 0, \quad i \geq 0, \ 0 \leq s < 1. \tag{5.2.12}$$

The interchange of derivative and sum can be justified as follows. By Theorem 1.2.2, for all $t \geq 0$,

$$\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q_i, \tag{5.2.13}$$

where $q_i = -i^\alpha(i-1)^\beta b_2 < \infty$ (see also Proposition 1.2.6(2) of Anderson (1991), for any transition function $(p_{ij}(t))$, if $i$ is a stable state, then $\sum_{j=0}^{\infty} |p'_{ij}(t)| \leq 2q_i$ for all $t \geq 0$). Therefore, the series $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$ converges *uniformly* on $[0,1)$, for every $t$, and, since the derivatives $p'_{ij}(t)$ are all continuous (by Proposition 1.2.4(2) of Anderson (1991), if $i$ is a stable state, then $p'_{ij}(t)$ exists and is finite and continuous on $[0,\infty)$ for all $j \geq 0$), the derivative of $\sum_{j=0}^{\infty} p_{ij}(t)s^j$ exists and equals $\sum_{j=0}^{\infty} p'_{ij}(t)s^j$. Letting $t \to \infty$ in (5.2.12) and using (5.2.11) yields

$$\lim_{t\to\infty} p_{i0}(t) + s \lim_{t\to\infty} p_{i1}(t) \geq s^i \qquad s \in [0,1). \tag{5.2.14}$$

Now letting $s \uparrow 1$ in (5.2.14) and using the fact that $p_{i0}(t) + p_{i1}(t) \leq 1$ yield (5.2.8).

If $m_d < m_b \leq +\infty$, Lemma 4.2.1 implies that $q < 1$. Letting $s = q$ in (5.2.1) and noting that $B(q) = 0$ yields that for any $t > 0$,

$$\sum_{j=0}^{\infty} p'_{ij}(t)q^j = 0.$$

Integrating with respect to $t$ yields that

$$\sum_{j=0}^{\infty}\left(\int_0^t p'_{ij}(u)du\right)q^j = 0.$$

Hence, for any $t > 0$, we have

$$\sum_{j=0}^{\infty} p_{ij}(t)q^j = q^i, \quad i \geq 2. \tag{5.2.15}$$

On letting $t \to \infty$ in (5.2.15) we obtain

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t)q + \lim_{t\to\infty}\sum_{j=2}^{\infty} p_{ij}(t)q^j = q^i, \quad i \geq 2,$$

by noting that all of these limits exist. Since $q < 1$ we may apply the Dominated Convergence Theorem in the last term on the left-hand side to obtain that

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t)q + \sum_{j=2}^{\infty}(\lim_{t\to\infty} p_{ij}(t))q^j = q^i, \quad i \geq 2.$$

Noting (5.2.11) we know that (5.2.9) holds. Finally, by (5.2.9), $\lim_{t\to\infty} p_{i0}(t)\cdot \lim_{t\to\infty} p_{i1}(t) = q^i + (1-q)\lim_{t\to\infty} p_{i1}(t) \leq 1-q+q^i < 1$ since $0 < q < 1$. The proof is complete. □

**Remark 5.2.1.** If $m_d < m_b \leq +\infty$, since

$$\lim_{y\uparrow q}\frac{F'_i(t,y)}{B(y)} = \lim_{y\downarrow q}\frac{F'_i(t,y)}{B(y)} = \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}q^{k-2} < \infty$$

we can define

$$\frac{F'_i(t,q)}{B(q)} =: \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}q^{k-2}.$$

Hence the singularity at $y = q$ on the left-hand side of (5.2.6) is removable, as indicated in the proof of Lemma 5.2.2. It also follows that for any given $\varepsilon \in (q,1)$, $\frac{F'_i(t,y)}{B(y)}$ is uniformly bounded on $[0,\varepsilon]$ (respect to $y$).

**Remark 5.2.2.** It follows from (5.2.1) and (5.2.6) that $\frac{F_i'(t,y)}{B(y)}$ is nonnegative (actually, positive if $i \geq 2$) for all $y \in [0,1)$ and increasing on $[0,1)$. Indeed, it is trivial true for the case $m_d \geq m_b$ while if $m_d < m_b \leq +\infty$, then from (5.2.1) and (5.2.6) we see that $\frac{F_i'(t,y)}{B(y)}$ is nonnegative for $y \in [0,q) \cup (q,1)$. Moreover, for the case $y = q$, $\frac{F_i'(t,y)}{B(y)}$ is also nonnegative since the right-hand side of (5.2.6) is nonnegative. It then follows that $F_i'(t,y) > 0$ if $0 \leq y < q$ and $F_i'(t,y) < 0$ if $q < y < 1$. Similarly, it is easily seen that $\frac{\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) y^j - y^i}{B(y)}$ is also a nonnegative and increasing function of $y \in [0,1)$ and thus $\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) y^j - y^i$ is positive or negative according to $0 < y < q$ or $q < y < 1$.

Using Remarks 5.2.1 and 5.2.2, we are able to rewrite Lemma 5.2.2 in a more compact and informative form which yields very useful inequalities.

**Lemma 5.2.4.** Let $(p_{ij}(t);\ i,j \in \mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a GCB $q$-matrix given in (5.1.1)–(5.1.3). Then for any $i \geq 0$, $s \in [0,1]$,

$$\begin{aligned} &\Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} p_{ik}(t) s^k \\ = \;& s^2 \int_0^1 \frac{F_i'(t,sx)}{B(sx)(-\ln x)^{1-\alpha-\beta}} \cdot \left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} x^u du\right) dx. \end{aligned} \tag{5.2.16}$$

In particular,

$$\begin{aligned} &\Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} p_{ik}(t) \\ = \;& \int_0^1 \frac{F_i'(t,x)}{B(x)(-\ln x)^{1-\alpha-\beta}} \cdot \left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1} x^u du\right) dx. \end{aligned} \tag{5.2.17}$$

and thus

$$\begin{aligned} &\int_0^1 \frac{F_i'(t,x)}{B(x)} \cdot x(-\ln x)^{\alpha+\beta-1} dx \\ \leq \;& \Gamma(\alpha+\beta) \sum_{k=2}^{\infty} p_{ik}(t) \\ \leq \;& \int_0^1 \frac{F_i'(t,x)}{B(x)} \cdot (-\ln x)^{\alpha+\beta-1} dx. \end{aligned} \tag{5.2.18}$$

Similarly, for any $i \geq 0$, $s \in [0,1]$,

$$\Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} \phi_{ik}(\lambda) s^k$$

$$= s^2\int_0^1 \frac{\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)(sx)^j-(sx)^i}{B(sx)\cdot(-\ln x)^{1-\alpha-\beta}}\cdot\left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}x^u du\right)dx \quad (5.2.19)$$

and thus

$$\begin{aligned}
&\int_0^1 \frac{\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)x^j-x^i}{B(x)\cdot(-\ln x)^{1-\alpha-\beta}}\cdot x dx \\
\leq\ & \Gamma(\alpha+\beta)\sum_{k=2}^{\infty}\phi_{ik}(\lambda) \\
\leq\ & \int_0^1 \frac{\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)x^j-x^i}{B(x)\cdot(-\ln x)^{1-\alpha-\beta}}dx. \qquad (5.2.20)
\end{aligned}$$

**Proof.** For $0 \leq s < 1$, (5.2.16) is just (5.2.4) by letting $y = sx$. By (5.2.6), $\frac{F_i'(t,sx)}{B(sx)}$ is an increasing function of $s$ (see Remark 5.2.2), thus we can let $s \uparrow 1$ and use the Monotone Convergence Theorem to obtain that

$$\begin{aligned}
&\Gamma(\alpha)\Gamma(\beta)\cdot\sum_{k=2}^{\infty}p_{ik}(t) \\
=\ & \int_0^1 \lim_{s\uparrow 1}\frac{F_i'(t,sx)}{B(sx)(-\ln x)^{1-\alpha-\beta}}\cdot\left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}x^u du\right)dx.
\end{aligned}$$

Note that $B(sx)$ and $F_i'(t,sx)$ are continuous functions of $s\in[0,1]$ for given $x\in[0,1)$, we know that (5.2.16) is also true for $s=1$ which is just (5.2.17). Secondly, note that the integrand in the integral of (5.2.17) is non-negative (again, see Remark 5.2.2) and for any $x\in[0,1]$,

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}x < \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}x^u du < \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Thus (5.2.18) follows from (5.2.17). Finally, since (5.2.19) is the Laplace transform of (5.2.16), we know that (5.2.19) holds. By (5.2.19) and the upper and lower bounds of the integral $\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}x^u du$ given above, we know that (5.2.20) holds. The proof is complete. □

We are now ready to settle the questions of regularity and uniqueness.

**Theorem 5.2.5.** Suppose that $m_d \geq m_b$, then the GCB $q$-matrix is regular, i.e., the Feller minimal $Q$-process is honest. Furthermore, we have

$$\lim_{t\to\infty}\sum_{j=2}^{\infty}p_{ij}(t) = 0, \quad i\geq 2 \qquad (5.2.21)$$

**Proof.** Letting $s \uparrow 1$ in (5.2.12) yields $\sum_{j=0}^{\infty} p_{ij}(t) \geq 1$, implying that equality holds for all $i \geq 0$. We deduce that the minimal $Q$-function is honest, and hence that $Q$ is regular. Finally, (5.2.21) follows from (5.2.8). $\square$

In contrast to Theorem 5.2.5, if $m_d < m_b \leq +\infty$, the situation is much more subtle.

**Theorem 5.2.6.** Suppose that $m_d < m_b \leq +\infty$ and thus $B(s) = 0$ possesses a root $q$ such that $0 < q < 1$. Then the following statements are equivalent.

(i) $Q$ is regular, i.e., the Feller minimal $Q$-process is honest.

(ii) For any $i \geq 2$,

$$\lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) > 0. \tag{5.2.22}$$

(iii) The following integral diverges

$$\int_0^1 \frac{(q-s)(-\ln s)^{\alpha+\beta-1}}{B(s)} ds = +\infty.$$

(iv) For some (and therefore for all) $\varepsilon \in (q, 1)$,

$$\int_\varepsilon^1 \frac{(-\ln s)^{\alpha+\beta-1}}{-B(s)} ds = +\infty. \tag{5.2.23}$$

(v) For some (and therefore for all) $\varepsilon \in (q, 1)$,

$$\int_\varepsilon^1 \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds = +\infty. \tag{5.2.24}$$

**Proof.** Since $B'(q) < 0$ (see the proof of Lemma 4.2.1), it is easily proved that for any $\varepsilon \in (q, 1)$ we have

$$\int_0^\varepsilon \frac{(q-s)(-\ln s)^{\alpha+\beta-1}}{B(s)} ds < \infty.$$

Indeed, recall that we obtained in the proof of Lemma 4.2.1 that

$$B(s) = (1-s)(s-q_*)(q-s)\sum_{l=0}^{\infty}\Big(\sum_{k=1}^{\infty} \sigma_{k+l} \cdot \frac{q^k - q_*^k}{q - q_*}\Big)s^l,$$

where $\sigma_k = \sum_{j=k}^{\infty} b_{j+2}$. Thus for $s \in [0, \varepsilon)$,

$$
\begin{aligned}
\frac{q-s}{B(s)} &= \frac{1}{(1-s)(s-q_*)\sum_{l=0}^{\infty}(\sum_{k=1}^{\infty}\sigma_{k+l}\cdot\frac{q^k-q_*^k}{q-q_*})s^l} \\
&\leq \frac{1}{\sigma_1(1-s)(s-q_*)} \\
&\leq -\frac{1}{\sigma_1 q_*(1-\varepsilon)}
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\int_0^{\varepsilon} \frac{(q-s)(-\ln s)^{\alpha+\beta-1}}{B(s)} ds \\
\leq\ & -\frac{1}{\sigma_1 q_*(1-\varepsilon)} \int_0^{\varepsilon} (-\ln s)^{\alpha+\beta-1} ds \\
\leq\ & -\frac{1}{\sigma_1 q_*(1-\varepsilon)} \int_0^{\infty} y^{\alpha+\beta-1} e^{-y} dy \\
=\ & -\frac{\Gamma(\alpha+\beta)}{\sigma_1 q_*(1-\varepsilon)} \\
<\ & \infty.
\end{aligned}
$$

Thus (iii) $\Leftrightarrow$(iv) follows. Note that $\lim_{s\to 1}\frac{-\ln s}{1-s} = 1$, we know that (iv) $\Leftrightarrow$ (v). Hence we only need to prove (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i). First, (i) $\Rightarrow$ (ii) follows immediately from (5.2.10) together with the fact that the limit in (5.2.22) always exists. We now prove that (iv) implies (i). Suppose (i) is not true. Then $\rho = 1 - \lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda) > 0$ for some fixed $\lambda > 0$ and some $i \geq 2$ since 0 and 1 are both absorbing. Therefore, we can find an $\varepsilon \in (q, 1)$ such that for $y \in (\varepsilon, 1)$,

$$
y^i - \lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) y^j \geq \frac{\rho}{2} > 0.
$$

Applying (5.2.20) we get

$$
\begin{aligned}
\sum_{k=2}^{\infty} \phi_{ik}(\lambda) &\geq \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 \frac{y^i - \lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)y^j}{-B(y)} \cdot y(-\ln y)^{\alpha+\beta-1} dy \\
&\geq \frac{1}{\Gamma(\alpha+\beta)} \int_{\varepsilon}^1 \frac{y^i - \lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)y^j}{-B(y)} \cdot y(-\ln y)^{\alpha+\beta-1} dy \\
&\geq \frac{\rho\varepsilon}{2\Gamma(\alpha+\beta)} \int_{\varepsilon}^1 \frac{(-\ln y)^{\alpha+\beta-1}}{-B(y)} dy.
\end{aligned}
$$

Noting (iv) and $\varepsilon > q$ yields

$$
\sum_{k=2}^{\infty} \phi_{ik}(\lambda) \geq \frac{\rho q}{2\Gamma(\alpha+\beta)} \int_{\varepsilon}^1 \frac{(-\ln y)^{\alpha+\beta-1}}{-B(y)} dy = +\infty
$$

which is impossible since we always have $\Sigma_{k=0}^{\infty} \phi_{ik}(\lambda) \le \frac{1}{\lambda}$ for this fixed $\lambda > 0$. Thus (i) follows.

Finally we prove (ii) implies (iv). Suppose (iv) fails, i.e.,

$$\int_{\varepsilon}^{1} \frac{(-\ln s)^{\alpha+\beta-1}}{-B(s)} ds < +\infty \tag{5.2.25}$$

for all $\varepsilon$ with $q < \varepsilon < 1$. We shall then prove that (ii) does not hold, i.e., the limit in (5.2.22) is 0. Now by the right-hand side inequality in (5.2.18) we see that it is sufficient to show that

$$\lim_{t\to\infty} \int_0^1 \frac{F_i'(t,y)}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy = 0. \tag{5.2.26}$$

To this end, first observe that, for any $\varepsilon \in (q, 1)$,

$$\lim_{t\to\infty} \int_0^{\varepsilon} \frac{F_i'(t,y)}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy = 0.$$

Indeed, by (5.2.6) the integrand in the above integral is dominated by $(\Sigma_{k=2}^{\infty} k^{\alpha}(k-1)^{\beta}\varepsilon^{k-2}) \cdot (-\ln y)^{\alpha+\beta-1}$ which is integrable on $[0, \varepsilon]$ (see Remark 5.2.1), and hence we may apply the Dominated Convergence Theorem to obtain that

$$\lim_{t\to\infty} \int_0^{\varepsilon} \frac{F_i'(t,y)}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy = \int_0^{\varepsilon} \lim_{t\to\infty} \frac{F_i'(t,y)}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy.$$

However, by (5.2.6) we know that for $y \in [0, 1)$

$$\lim_{t\to\infty} \frac{F_i'(t,y)}{B(y)} = \lim_{t\to\infty} \sum_{k=2}^{\infty} p_{ik}(t) \cdot k^{\alpha}(k-1)^{\beta} y^{k-2}.$$

Since $\Sigma_{k=2}^{\infty} p_{ik}(t) \cdot k^{\alpha}(k-1)^{\beta} y^{k-2} \le \Sigma_{k=2}^{\infty} k^{\alpha}(k-1)^{\beta} y^{k-2} < \infty$for $y \in [0, 1)$, we can use the Dominated Convergence Theorem once again on the right-hand side of the above equality to obtain that

$$\lim_{t\to\infty} \frac{F_i'(t,y)}{B(y)} = \sum_{k=2}^{\infty} (\lim_{t\to\infty} p_{ik}(t)) \cdot k^{\alpha}(k-1)^{\beta} y^{k-2}.$$

Using (5.2.11) yields that

$$\lim_{t\to\infty} \frac{F_i'(t,y)}{B(y)} = 0$$

and hence

$$\lim_{t\to\infty}\int_0^{\varepsilon}\frac{F_i'(t,y)}{B(y)}\cdot(-\ln y)^{\alpha+\beta-1}dy=0.$$

Therefore, in order to obtain (5.2.26) it is sufficient to prove that

$$\lim_{t\to\infty}\int_{\varepsilon}^{1}\frac{F_i'(t,y)}{B(y)}\cdot(-\ln y)^{\alpha+\beta-1}dy=0. \tag{5.2.27}$$

Remembering that $F_i'(t,y)/B(y)>0$ for $y\in[0,1)$ and $B(y)<0$ for $y\in(q,1)$ (see Remark 5.2.2), we obtain by using (5.2.13) that

$$\frac{F_i'(t,y)}{B(y)}\cdot(-\ln y)^{\alpha+\beta-1}\le 2i^{\alpha}(i-1)^{\beta}\cdot\frac{(-\ln y)^{\alpha+\beta-1}}{-B(y)}.$$

Now by (5.2.25) we may apply the Dominated Convergence Theorem to obtain that

$$\begin{aligned}&\lim_{t\to\infty}\int_{\varepsilon}^{1}\frac{F_i'(t,y)}{B(y)}\cdot(-\ln y)^{\alpha+\beta-1}dy\\ =&\int_{\varepsilon}^{1}\lim_{t\to\infty}\frac{F_i'(t,y)}{B(y)}\cdot(-\ln y)^{\alpha+\beta-1}dy\\ =&\ 0.\end{aligned}$$

i.e., (5.2.27) holds and therefore (5.2.26) follows. This completes the proof. □

Comparing (5.2.22) with (5.2.21) shows that the honesty conditions are quite different between the two cases $m_d\ge m_b$ and $m_d<m_b\le+\infty$. The probabilistic interpretation of these conditions will become clear later. Also, in the four equivalent conditions in Theorem 5.2.6, Criterion (5.2.24) is surely the most simple one. However, by the proof we can see that Criterion (5.2.23) (also Criterion (iii)) is more essential. The probabilistic interpretation is also clear by the proof.

Combining Theorems 5.2.5 and 5.2.6 we can obtain the following very satisfactory conclusion regarding the honesty criterion. It tells us that only in rare cases (i.e., in the case $m_b=\infty$ and $\alpha+\beta\le 1$)do we need to check (5.2.24) directly.

**Theorem 5.2.7.** A GCB $q$-matrix $Q$ is regular if and only if one of the following conditions holds.

(i) $m_d \geq m_b$.

(ii) $m_d < m_b < +\infty$ and $\alpha + \beta \leq 1$.

(iii) $m_b = +\infty$, $\alpha + \beta \leq 1$ and

$$\int_\varepsilon^1 \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds = +\infty \tag{5.2.28}$$

for some (equivalently, for all) $q < \varepsilon < 1$, where $q < 1$ is the smallest positive root of $B(s) = 0$.

**Proof.** If $m_d \geq m_b$ then by Theorem 5.2.5, $Q$ is regular. If $m_d < m_b \leq +\infty$, then by Lemma 4.2.1 and its proof we know that $B'(1) > 0$ (possibly infinite) and thus 1 is a simple root of $B(s) = 0$ (also see Lemma 5.3.1 later). Therefore if $\alpha + \beta > 1$, then (5.2.24) can never hold true while if $m_d < m_b < +\infty$ and $\alpha + \beta \leq 1$, then (5.2.24) holds automatically. Now the conclusion follows from Theorem 5.2.6. □

It is more informative if we write Theorem 5.2.7 into two separate corollaries.

**Corollary 5.2.8.** A super-explosive GCB $q$-matrix is regular if and only if $m_d \geq m_b$.

**Corollary 5.2.9.** A sub-explosive GCB $q$-matrix is regular if and only if either

(i) $m_b < +\infty$ or

(ii) $m_b = +\infty$ and $\int_\varepsilon^1 \frac{ds}{B(s)(1-s)^{1-\alpha-\beta}} = -\infty$ for some (equivalently, for all) $\varepsilon \in (q, 1)$, where $q \in (0, 1)$ is the smallest positive root of $B(s) = 0$ which is guaranteed by the condition $m_b = +\infty$.

Theorem 5.2.7 establishes that if the GCB $q$-matrix $Q$ is super-explosive and $m_d < m_b \leq +\infty$ or if $Q$ is sub-explosive, $m_b = +\infty$ and the integral in (5.2.28) is finite, then $Q$ is not regular and thus there exist infinite many of (even honest) $Q$-functions. A CB $q$-matrix with $m_d < m_b$ is an example in the former case. For the latter case, let $Q$ be defined in (5.1.1)–(5.1.3) where

$$\alpha = \beta = 1/2, \quad b_0 = 1/4, \quad b_1 = 1/2,$$

$$b_k = (1 - \frac{1}{2(k-2)}) \cdot (\frac{5}{4} - \frac{1}{8(k-1)}) \cdot \frac{[2(k-3)]!}{[(k-3)!2^{k-3}]^2}, \quad k \geq 3.$$

Denote $a_k = (1 - \frac{1}{2(k-1)}) \cdot \frac{[2(k-2)]!}{[(k-2)!2^{k-2}]^2}$, $k \geq 2$, then

$$\sum_{j=k}^{\infty} a_j = \frac{[2(k-2)]!}{[(k-2)!2^{k-2}]^2}, \ (k \geq 2) \qquad \sum_{j=2}^{\infty} a_j = 1$$

and

$$b_k = a_{k-1} + \frac{a_k}{4}, \quad k \geq 3.$$

Hence, for $s \in (-1, 1)$,

$$\begin{aligned} B(s) &= \frac{1}{4} + \frac{s}{2} - \frac{15}{8}s^2 + \sum_{k=3}^{\infty}(a_{k-1} + \frac{a_k}{4})s^k \\ &= (s + \frac{1}{4}) \cdot (1 - 2s + \sum_{k=2}^{\infty} a_k s^k) \\ &= (s + \frac{1}{4}) \cdot (1 - s) \cdot [1 - \sum_{k=1}^{\infty}(\sum_{j=k+1}^{\infty} a_j)s^k] \\ &= (s + \frac{1}{4}) \cdot (1 - s) \cdot [1 - \sum_{k=1}^{\infty} \frac{[2(k-1)]!}{[(k-1)!2^{k-1}]^2} s^k] \\ &= (s + \frac{1}{4}) \cdot (1 - s) \cdot [1 - s(1 - s)^{-1/2}] \end{aligned}$$

since $(1-s)^{-1/2} = \sum_{k=1}^{\infty} \frac{[2(k-1)]!}{[(k-1)!2^{k-1}]^2} s^{k-1}$ for $s \in (-1, 1)$. It is clear that $B(1) = B(-1/4) = 0$, $B'(1) = \infty$ and

$$\frac{1}{-B(s)} \leq M(1-s)^{-1/2}, \quad s \in (\varepsilon, 1)$$

where $M$ is a positive constant. Therefore,

$$\int_{\varepsilon}^{1} \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds \leq M \int_{\varepsilon}^{1} (1-s)^{-1/2} ds = 2M\sqrt{1-\varepsilon} < \infty.$$

By Theorem 5.2.7, $Q$ is not regular. Hence, by Theorem 14.2.6(4) in Hou and Guo (1988), we know that there exist infinitely many honest $Q$-functions.

However, Theorem 5.2.7 does not imply that there exists more than one general collision branching process. In fact we can prove that there always exists only one GCBP as the following conclusion shows.

**Theorem 5.2.10**. There always exists only one general collision branching process which is the Feller minimal $Q$-process.

**Proof.** As already remarked, we only need to consider the case $m_d < m_b \leq \infty$. In order to prove that even for this case there still exists only one GCBP, we will show that the forward equations have a unique solution. By Reuter's Theorem (see for example, Theorem 2.2.8 of Anderson (1991)), if the equation

$$\eta(\lambda)(\lambda I - Q) = 0, \quad 0 \leq \eta(\lambda) \in l_1$$

has only trivial solution for some (and then all) $\lambda > 0$, then there exists only one $Q$-function satisfying Kolmogorov forward equations. Therefore, we only need to prove that the above equation has only trivial solution. Suppose that the above equation has a nontrivial solution. Let $\eta = \{\eta_i;\ i \geq 0\}$ be the non-trivial solution corresponding to $\lambda = 1$. Then, by (5.1.1) we have

$$\eta_j = \sum_{i=2}^{j+2} \eta_i i^\alpha (i-1)^\beta b_{j-i+2}, \quad j \geq 0 \tag{5.2.29}$$

with

$$\eta_j \geq 0 \ \ (j \geq 0) \ \ and \ \ \sum_{j=0}^{\infty} \eta_j < +\infty. \tag{5.2.30}$$

It is clear that the non-triviality of the solution $\eta$ implies that

$$\sum_{j=2}^{\infty} \eta_j > 0. \tag{5.2.31}$$

Condition (5.2.30) guarantees that $\sum_{j=0}^{\infty} \eta_j s^j$ is well defined, at least for all $s \in [0,1]$. This in turn implies that

$$\sum_{j=2}^{\infty} j^\alpha (j-1)^\beta \eta_j s^j < +\infty, \quad 0 \leq s < 1 \tag{5.2.32}$$

because these two series have the same radius of convergence. It then follows from (5.2.29) that for $s \in [0,1)$,

$$\sum_{j=0}^{\infty} \eta_j s^j = \sum_{j=0}^{\infty} \sum_{i=2}^{j+2} \eta_i i^\alpha (i-1)^\beta b_{j-i+2} s^j.$$

By (5.2.32) and Fubini's theorem, we know that for $s \in [0,1)$,

$$\sum_{j=0}^{\infty} \eta_j s^j = B(s) \sum_{i=2}^{\infty} i^\alpha (i-1)^\beta \eta_i s^{i-2}.$$

Now, (5.2.30) and (5.2.31) imply that $0 < \sum_{j=0}^{\infty} \eta_j s^j < \infty$ for $s \in (0,1)$. (5.2.30) and (5.2.32) imply that $0 < \sum_{i=2}^{\infty} i^{\alpha}(i-1)^{\beta}\eta_i s^{i-2} < \infty$ for $s \in (0,1)$. Thus $B(s) > 0$ for all $s \in (0,1)$. However, by Lemma 4.2.1, $B(s)$ has a root $q \in (0,1)$ since $m_d < m_b \leq +\infty$. This is a contradiction. The proof is complete. $\square$

The result of Theorem 5.2.10 is no wonder. In fact, if one checks the proofs of Lemmas 5.2.1–5.2.4 carefully, one will find that all these proofs do not necessarily need the condition that the transition function is the Feller minimal one. Indeed, in all these proofs we have only used the Kolmogorov forward equations and thus all the results obtained in Lemmas 5.2.1–5.2.4 hold well for the GCBP whose transition function satisfies (5.1.4). This has implicitly implied Theorem 5.2.10.

## 5.3. Extinction Probability

Having established the fact that a GCBP is uniquely determined by its $q$-matrix, we will now examine some of its properties. Let $\{X(t),\ t \geq 0\}$ be the unique GCBP associated with a given GCB $q$-matrix $Q$, and let $P(t) = (p_{ij}(t))$ denote its transition function. Define the extinction times $\tau_0$ and $\tau_1$ for states 0 and 1 as

$$\tau_0 = \begin{cases} \inf\{t > 0;\ X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

$$\tau_1 = \begin{cases} \inf\{t > 0;\ X(t) = 1\} & \text{if } X(t) = 1 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 1 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_{i0} = P(\tau_0 < +\infty | X(0) = i) \quad \textit{and} \quad a_{i1} = P(\tau_1 < +\infty | X(0) = i).$$

Also, let $\tau = \tau_0 \wedge \tau_1$ denote the (overall) extinction time and $a_i =: P(\tau < +\infty | X(0) = i)$ be the corresponding extinction probability.

In order to evaluate the above mentioned extinction probabilities, we need to investigate properties regarding the roots of the equation $B(s) = 0$. Recall that Lemma 4.2.1 reveals some properties of its positive roots. The following simple lemma provides further information concerning its other roots. Recall that a root is said to be simple if it has multiplicity 1 and that we have assumed that our GCB $q$-matrix is not degenerative and thus $B(-1) < 0$ (see Remark 4.1.1 and the first paragraph of Section 5.2).

**Lemma 5.3.1.** The unique root $q_*$ of equation $B(s) = 0$ in $(-1, 0)$ satisfies

$$a_{i0} + q_* a_{i1} = q_*^i. \tag{5.3.1}$$

**Proof.** By (5.2.1) in Lemma 5.2.1,

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s) \cdot \sum_{k=2}^{\infty} p_{ik}(t)k^{\alpha}(k-1)^{\beta}s^{k-2}, \quad s \in (-1, 1).$$

Since $B(s) = 0$ has a root $q_* \in (-1, 0)$, we can let $s = q_*$ in the above equation to obtain that

$$\sum_{j=0}^{\infty} p'_{ij}(t)q_*^j = 0.$$

Integrating the above equation yields that

$$\sum_{j=0}^{\infty} p_{ij}(t)q_*^j = q_*^i.$$

Thus, letting $t \to \infty$ in the above equation and noting (5.2.11) yields that (5.3.1) holds. The proof is complete. □

**Theorem 5.3.2.** The overall extinction probability satisfies

$$a_i = a_{i0} + a_{i1}. \tag{5.3.2}$$

Furthermore, $a_{00} = a_{11} = 1$ and for any $i \geq 2$, we have

$$a_{i0} + a_{i1} = 1, \quad if \quad m_d \geq m_b, \tag{5.3.3}$$

$$a_{i0} + qa_{i1} = q^i < 1, \quad if \quad m_d < m_b \leq +\infty, \tag{5.3.4}$$

where recall that $q < 1$ is the smallest root of $B(s) = 0$ in $[0, 1]$ in the case $m_d < m_b \leq +\infty$.

**Proof.** It follows from the definition of $\tau$ and the fact that 0 and 1 are absorbing states that

$$P(\tau < t | X(0) = i) = p_{i0}(t) + p_{i1}(t).$$

Letting $t \uparrow \infty$ in the above expression immediately yields (5.3.2).

It is clear that $a_{00} = a_{11} = 1$ and $a_{01} = a_{10} = 0$, hence (5.3.3) and (5.3.4) hold for $i = 0$ and $i = 1$. For the case $i \geq 2$, recall we proved in Lemma 5.2.3 that (5.2.9) and (5.2.10) hold. However, note that $a_{i0} = \lim_{t\to\infty} p_{i0}(t)$ and $a_{i1} = \lim_{t\to\infty} p_{i1}(t)$, we see that (5.3.3) and (5.3.4) are just (5.2.9) and (5.2.10) respectively. □

Theorem 5.3.2 and Lemma 5.3.1 allow us to evaluate the extinction probabilities $a_{i0}$ and $a_{i1}$ starting in state $i$. Here and henceforth we will always use $q_*$ to denote the unique root of $B(s) = 0$ in $(-1, 0)$.

**Theorem 5.3.3.** (i) If $m_d \geq m_b$ then

$$\begin{cases} a_{i0} = (q_*^i - q_*)/(1 - q_*) \\ a_{i1} = (1 - q_*^i)/(1 - q_*) \end{cases} \tag{5.3.5}$$

and thus $a_i = a_{i0} + a_{i1} = 1$.

(ii) If $m_d < m_b \leq +\infty$ then

$$\begin{cases} a_{i0} = (qq_*^i - q_*q^i)/(q - q_*) \\ a_{i1} = (q^i - q_*^i)/(q - q_*) \end{cases} \tag{5.3.6}$$

and thus $a_i = a_{i0} + a_{i1} < 1 \quad (i \geq 2)$.

**Proof.** Suppose that $m_d \geq m_b$. By (5.3.3) in Theorem 5.3.2 and (5.3.1) in Lemma 5.3.1, we have

$$\begin{cases} a_{i0} + a_{i1} = 1 \\ a_{i0} + q_* a_{i1} = q_*^i. \end{cases}$$

Solving the above equations yields (5.3.5).

Suppose that $m_d < m_b \leq \infty$. By (5.3.4) in Theorem 5.3.2 and (5.3.1) in Lemma 5.3.1, we have

$$\begin{cases} a_{i0} + a_{i1} q = q^i \\ a_{i0} + q_* a_{i1} = q_*^i. \end{cases}$$

Solving the above equations yields (5.3.6). The proof is complete. □

Theorem 5.3.3 states that if $m_d \geq m_b$ then the process is eventually absorbed at either 0 or 1 with probability 1, while if $m_d < m_b \leq +\infty$ absorption occurs with probability less than 1.

## 5.4. Extinction Time

In this section we will evaluate several (conditional) mean extinction times including $E_i[\tau]$ $(i \geq 2)$, $E_i[\tau_k|\tau_k < \infty]$ $(i \geq 2, k = 0, 1)$ and $E_i[\tau|\tau < \infty]$ $(i \geq 2)$, where $E_i$ denotes the expectation under the condition $X(0) = i$. We consider $E_i[\tau]$ first.

**Theorem 5.4.1.** (i) If $m_d > m_b$ then for any $i \geq 2$, $E_i[\tau]$ is finite.

(ii) If $m_d = m_b$ and $\alpha + \beta > 1$ (i.e., $Q$ is super-explosive), then for any $i \geq 2$, $E_i[\tau]$ is finite.

(iii) If $m_d = m_b$ and $\alpha + \beta \leq 1$ (i.e., $Q$ is sub-explosive), then if $B''(1) < +\infty$, then for any $i \geq 2$, $E_i[\tau]$ is infinite while if $B''(1) = +\infty$, then for any $i \geq 2$, $E_i[\tau]$ is finite if and only if $\int_0^1 \frac{(1-y)^{\alpha+\beta}}{B(y)} dy < +\infty$.

(iv) If $m_d < m_b \leq +\infty$, then $E_i[\tau]$ is infinite.

Moreover, under the above finite conditions the finite $E_i[\tau]$ is given by

$$E_i[\tau] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \cdot f_i(y) dy \tag{5.4.1}$$

where

$$f_i(y) = \left(\frac{1-y^i}{1-y} - \frac{1-q_*^i}{1-q_*}\right) \cdot \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} y^u du. \tag{5.4.2}$$

**Proof.** First note that if $m_d < m_b \leq +\infty$, then by (5.2.10) in Lemma 5.2.3 we have $P(\tau < \infty | X(0) = i) < 1$ and thus $E_i[\tau] = +\infty$. (iv) is thus proved and therefore in the following we assume $m_d \geq m_b$. By Theorem 5.2.5 and Lemma 4.2.1, the associated Feller minimal process is honest and $B(y) > 0$ for all $y \in [0, 1)$. We now first prove that for the case $m_d \geq m_b$, $E_i[\tau]$ is given in (5.4.1) together with (5.4.2), no matter whether $E_i[\tau]$ is finite or not. For this purpose, we use the honesty condition and (5.2.17) to get that

$$\begin{aligned} & P(\tau > t | X(0) = i) = 1 - (p_{i0}(t) + p_{i1}(t)) \\ = \; & \sum_{k=2}^{\infty} p_{ik}(t) \end{aligned}$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \frac{F_i'(t,y)}{B(y)(-\ln y)^{1-\alpha-\beta}}\cdot\Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy.$$

Integrating with respect to $t$ and using Fubini's theorem yields

$$\int_0^t P(\tau > s|X(0)=i)ds$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\cdot\int_0^1 \frac{F_i(t,y)-y^i}{B(y)(-\ln y)^{1-\alpha-\beta}}\cdot\Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy.$$

By (5.2.6) (see also Remark 5.2.2) we see that $\frac{F_i(t,y)-y^i}{B(y)}$ and hence the whole integrand in the right-hand side of the above equality is an increasing functions of $t$ for any given $y \in [0,1)$. Therefore, letting $t \uparrow \infty$ and applying the Monotone Convergence Theorem in the above equality yields that

$$\int_0^\infty P(\tau > s|X(0)=i)ds$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\cdot\int_0^1 \frac{\lim_{t\uparrow\infty} F_i(t,y)-y^i}{B(y)(-\ln y)^{1-\alpha-\beta}}\cdot\Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy.$$

Using (5.2.11) yields that

$$\Gamma(\alpha)\Gamma(\beta)\cdot\int_0^\infty P(\tau > s|X(0)=i)ds$$
$$= \int_0^1 \frac{\lim_{t\uparrow\infty}(p_{i0}(t)+p_{i1}(t)y)-y^i}{B(y)(-\ln y)^{1-\alpha-\beta}}\cdot\Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy.$$

By Theorem 5.3.2, we obtain that

$$\Gamma(\alpha)\Gamma(\beta)\cdot E_i[\tau]$$
$$= \int_0^1 \frac{a_{i0}+a_{i1}y-y^i}{B(y)(-\ln y)^{1-\alpha-\beta}}\cdot\Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy. \tag{5.4.3}$$

Substituting the expressions of $a_{i0}$ and $a_{i1}$ into (5.4.3) then leads to (5.4.1) and (5.4.2). Next, we show that under the condition $m_d \geq m_b$, the $E_i[\tau]$ given in (5.4.1) and (5.4.2) (for all $i \geq 2$) is finite if and only if

$$\int_0^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)}dy < +\infty, \tag{5.4.4}$$

or, equivalently, if and only if $m_d \geq m_b$ and for any $0 < \varepsilon < 1$,

$$\int_\varepsilon^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)}dy < +\infty. \tag{5.4.5}$$

The equivalence between (5.4.4) and (5.4.5) is clear. Indeed, if $m_d \geq m_b$, then we can see that for any $0 < \varepsilon < 1$,

$$\int_0^\varepsilon \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} dy < +\infty$$

since the integrand $\frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)}$ is bounded.

Now, since

$$f_i(y) \leq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot (i - \frac{1-q_*^i}{1-q_*}) < \infty,$$

we know that (5.4.4) implies the finiteness of $E_i[\tau]$. Conversely, by (5.4.1)–(5.4.2) we have

$$\begin{aligned} E_i[\tau] &\geq \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \cdot (\frac{1-y^i}{1-y} - \frac{1-q_*^i}{1-q_*}) y dy \\ &\geq \frac{1}{\Gamma(\alpha+\beta)} \int_\varepsilon^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \cdot (\frac{1-y^i}{1-y} - \frac{1-q_*^i}{1-q_*}) y dy \\ &\geq M\varepsilon \int_\varepsilon^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} dy \end{aligned}$$

where $0 < \varepsilon < 1$ and $M = \frac{1}{\Gamma(\alpha+\beta)} \cdot \frac{q_*^i - q_*}{1-q_*} > 0$ due to the fact that $-1 < q_* < 0$. Hence if the integral in the left-hand side of (5.4.5) is infinite then so is $E_i[\tau]$.

Now we turn to prove (i)–(iii). If $m_d > m_b$ then

$$\begin{aligned} B(y) &= (1-y) \cdot [b_0 + (b_0 + b_1)y - \sum_{k=1}^{\infty} \sigma_k y^{k+1}] \\ &\geq (1-y)[b_0(1-y) + (m_d - m_b)y], \quad y \in [0,1) \end{aligned}$$

where $\sigma_k = \sum_{j=k}^{\infty} b_{j+2}$. Hence

$$\frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \leq \frac{(-\ln y)^{\alpha+\beta-1}}{b_0(1-y) + (m_d - m_b)y} \leq \frac{2(-\ln y)^{\alpha+\beta-1}}{b_0 \wedge (m_d - m_b)}.$$

However, $\int_0^1 (-\ln y)^{\alpha+\beta-1} dy < \infty$ since $\alpha, \beta > 0$. Therefore, (5.4.4) is finite and hence $E_i[\tau] < \infty$.

If $m_d = m_b$ and $\alpha + \beta > 1$, then

$$\begin{aligned} B(y) &= (1-y)^2 \cdot [b_0 + \sum_{k=1}^{\infty} \sigma_k (y + \cdots + y^k)] \\ &\geq b_0(1-y)^2, \quad y \in [0,1) \end{aligned}$$

and hence
$$\frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \leq \frac{(-\ln y)^{\alpha+\beta-1}}{b_0(1-y)}.$$
However, $\int_0^1 \frac{(-\ln y)^{\alpha+\beta-1}}{1-y}dy < \infty$ since $\alpha+\beta > 1$. Therefore, (5.4.4) is finite and hence $E_i[\tau] < \infty$.

Suppose that $m_d = m_b$ and $\alpha+\beta \leq 1$. If $B'(1) = 2(b_0+\sum_{k=1}^{\infty} k\sigma_k) < \infty$, then

$$\begin{aligned} B(y) &= (1-y)^2 \cdot [b_0 + \sum_{k=1}^{\infty} \sigma_k(y+\cdots+y^k)] \\ &\leq \frac{B'(1)}{2}(1-y)^2, \quad y \in [0,1) \end{aligned}$$

and hence
$$\frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \geq \frac{2}{B'(1)} \cdot \frac{(-\ln y)^{\alpha+\beta-1}}{1-y}.$$
However, $\int_0^1 \frac{(-\ln y)^{\alpha+\beta-1}}{1-y}dy < \infty$ since $\alpha+\beta \leq 1$. Therefore, (5.4.4) is infinite and hence $E_i[\tau] = \infty$. If $B'(1) = \infty$ then (5.4.4) is finite if and only if
$$\int_0^1 \frac{(1-y)^{\alpha+\beta}}{B(y)}dy < \infty$$
since $\lim_{y\uparrow 1} \frac{-\ln y}{1-y} = 1$. The proof is complete. □

**Remark 5.4.1** Considering $0 < y < 1$ in expression (5.4.2), we have
$$y\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \leq \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du \leq \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Substituting the above inequalities into (5.4.2) and (5.4.1) yields upper and lower bounds for $E_i[\tau]$ (see the similar results obtained in (5.2.18) and (5.2.20)). Also note that by (5.4.1)–(5.4.2), if $\alpha = \beta = 1$, then $E_i[\tau]$ takes a particular simple form as already obtained in Chapter 4.

Next we consider the conditional mean extinction times $E_i[\tau_k|\tau_k < +\infty]$ $(i \geq 2,\ k = 0, 1)$ and $E_i[\tau|\tau < \infty]$. Of course, for the latter we only need to consider the case $m_d < m_b \leq +\infty$.

**Theorem 5.4.2.** (i) If $m_d \geq m_b$, then for any $i \geq 2$, both $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ are finite if and only if one of the following conditions holds,

(a) $m_d > m_b$.

(b) $m_d = m_b$ and $\alpha + \beta > 1$.

(c) $m_d = m_b$, $\alpha + \beta \leq 1$, $B''(1) = +\infty$ and $\int_0^1 \frac{(1-y)^{\alpha+\beta}}{B(y)} dy < +\infty$.

Moreover, under the above conditions, the finite conditional mean extinction times are given by

$$E_i[\tau_0|\tau_0 < \infty] = \frac{-q_*}{(q_*^i - q_*)\Gamma(\alpha)\Gamma(\beta)}(J_{i1} - J_{i2})$$

$$E_i[\tau_1|\tau_1 < \infty] = \frac{1}{(1 - q_*^i)\Gamma(\alpha)\Gamma(\beta)}(J_{i1} - q_* J_{i2})$$

$$E_i[\tau] = \frac{q_*^i - q_*}{1 - q_*} E_i[\tau_0|\tau_0 < \infty] + \frac{1 - q_*^i}{1 - q_*} E_i[\tau_1|\tau_1 < \infty]$$

where

$$J_{i1} = \int_0^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} f_i(y) dy$$

$$J_{i2} = \int_{q_*}^0 \frac{(y - q_*)(\ln \frac{q_*}{y})^{\alpha+\beta-1}}{B(y)} g_i(y) dy$$

and $f_i(y)$ is given by (5.4.2) and

$$g_i(y) = (\frac{1 - q_*^i}{1 - q_*} - \frac{y^i - q_*^i}{y - q_*}) \cdot \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du.$$

(ii) If $m_d < m_b \leq +\infty$, then for any $i \geq 2$, $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ and $E_i[\tau|\tau < \infty]$ are all finite and given by

$$E_i[\tau_0|\tau_0 < \infty] = \frac{-qq_*}{(qq_*^i - q_* q^i)\Gamma(\alpha)\Gamma(\beta)}(\tilde{J}_{i3} - \tilde{J}_{i4})$$

$$E_i[\tau_1|\tau_1 < \infty] = \frac{1}{(q^i - q_*^i)\Gamma(\alpha)\Gamma(\beta)}(\tilde{J}_{i3} - q_* \tilde{J}_{i4})$$

$$E_i[\tau|\tau < \infty] = \frac{(qq_*^i - q_* q^i)E_i[\tau_0|\tau_0 < \infty] + (q^i - q_*^i)E_i[\tau_1|\tau_1 < \infty]}{q^i(1 - q_*) - q_*^i(1 - q)}$$

where

$$\tilde{J}_{i3} = \int_0^q \frac{(q - y)(\ln \frac{q}{y})^{\alpha+\beta-1}}{B(y)} \tilde{f}_i(y) dy$$

$$\tilde{J}_{i4} = \int_{q_*}^0 \frac{(y - q_*)(\ln \frac{q_*}{y})^{\alpha+\beta-1}}{B(y)} \tilde{g}_i(y) dy$$

and

$$\tilde{f}_i(y) = (\frac{q^i - y^i}{q - y} - \frac{q^i - q_*^i}{q - q_*}) \cdot \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q})^u du$$

$$\tilde{g}_i(y) = (\frac{q^i - q_*^i}{q - q_*} - \frac{y^i - q_*^i}{y - q_*}) \cdot \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du.$$

**Proof.** Note that

$$E_i[\tau_k | \tau_k < \infty] = \frac{E_i[\tau_k \cdot I_{\{\tau_k < \infty\}}]}{a_{ik}}, \quad k = 0, 1. \tag{5.4.6}$$

Thus we only need to evaluate $\mu_{ik} =: E_i[\tau_k \cdot I_{\{\tau_k<\infty\}}]$ $(k = 0, 1)$. In order to do this, first prove that is for $i \geq 0$ and $s \in [-1, 0]$,

$$\begin{aligned} &\Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} p_{ik}(t)s^k \\ = \;& s\int_s^0 \frac{F_i'(t,y)}{B(y)(\ln\frac{s}{y})^{1-\alpha-\beta}} \cdot (\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{s})^u du)dy. \end{aligned} \tag{5.4.7}$$

Indeed, (5.4.7) is trivial for $i = 0$ *or* $1$ since the states 0 and 1 are absorbing. So we assume $i \geq 2$. Using (5.2.3) we may rewrite (5.2.1) as

$$\frac{F_i'(t,y)}{B(y)} = \sum_{k=2}^{\infty} p_{ik}(t)k^\alpha(k-1)^\beta y^{k-2}$$

for all $y \in [-1, 0]$ provided that $B(y) \neq 0$. Note that by Lemma 4.2.1, the only singularity at $y = q_*$ on the left-hand side of the above equality is clearly removable, because the series on the right-hand side certainly converges for $y \in [-1, 0]$. It then follows that the above equality holds for all $y \in [-1, 0]$. Now for any given $s \in [-1, 0)$, multiplying $(\ln\frac{s}{x})^{\alpha-1} \cdot (\ln\frac{x}{y})^{\beta-1}$ on both sides of the above equality and integrating on $s < x < y < 0$ yields

$$\begin{aligned} &\int_s^0 (\int_x^0 \frac{F_i'(t,y)}{B(y)} \cdot (\ln\frac{x}{y})^{\beta-1} dy) \cdot (\ln\frac{s}{x})^{\alpha-1} dx \\ = \;& \sum_{k=2}^{\infty} p_{ik}(t)k^\alpha(k-1)^\beta \int_s^0 \int_x^0 y^{k-2}(\ln\frac{x}{y})^{\beta-1}(\ln\frac{s}{x})^{\alpha-1} dydx. \end{aligned}$$

For each term $k \geq 2$ in the right-hand side of the above equality, perform the transformation $\ln\frac{x}{y} = \frac{u}{k-1}$ and $\ln\frac{s}{x} = \frac{v}{k}$, then we obtain that

$$\int_s^0 \int_x^0 y^{k-2}(\ln\frac{x}{y})^{\beta-1}(\ln\frac{s}{x})^{\alpha-1} dydx$$

$$
\begin{aligned}
&= s^k \int_0^\infty \int_0^\infty e^{-\frac{k-2}{k}v - \frac{k-2}{k-1}u} \cdot \frac{u^{\beta-1}}{(k-1)^{\beta-1}} \cdot \frac{v^{\alpha-1}}{k^{\alpha-1}} \cdot \frac{1}{k(k-1)} \cdot e^{-\frac{2v}{k} - \frac{u}{k-1}} du dv \\
&= \frac{s^k}{k^\alpha (k-1)^\beta} \int_0^\infty u^{\beta-1} e^{-u} du \cdot \int_0^\infty v^{\alpha-1} e^{-v} dv \\
&= \frac{\Gamma(\alpha)\Gamma(\beta)}{k^\alpha (k-1)^\beta} s^k,
\end{aligned}
$$

where $\Gamma(\cdot)$ denotes the gamma function. Therefore,

$$
\int_s^0 \Big( \int_x^0 \frac{F_i'(t,y)}{B(y)} \cdot (\ln \frac{x}{y})^{\beta-1} dy \Big) \cdot (\ln \frac{s}{x})^{\alpha-1} dx = \Gamma(\alpha)\Gamma(\beta) \sum_{k=2}^\infty p_{ik}(t) s^k,
$$

and hence

$$
\int_s^0 \frac{F_i'(t,y)}{U(y)} \cdot \left( \int_s^y (\ln \frac{s}{x})^{\alpha-1} (\ln \frac{x}{y})^{\beta-1} dx \right) dy = \Gamma(\alpha)\Gamma(\beta) \sum_{k=2}^\infty p_{ik}(t) s^k.
$$

Using the transformation $u = (\ln \frac{s}{y})^{-1} \ln \frac{s}{x}$ in the integral

$$
\int_s^y (\ln \frac{s}{x})^{\alpha-1} (\ln \frac{x}{y})^{\beta-1} dx
$$

on the left-hand side of the above equality then achieves (5.4.7).

We now prove (ii). Since $m_d < m_b \le +\infty$, we have $0 < q < 1$. Letting $s = q$ in (5.2.4) and $s = q_*$ in (5.4.7), and using the fact that $\Sigma_{j=0}^\infty p_{ij}(t) q^j = q^i$ and $\Sigma_{j=0}^\infty p_{ij}(t) q_*^j = q_*^i$, which follow from the fact that both $q$ and $q_*$ are roots of $B(s) = 0$ (again refer to the argument leading to (5.2.15)), we obtain

$$
\begin{aligned}
& p_{i0}(t) + p_{i1}(t) q \\
&= q^i - \frac{q}{\Gamma(\alpha)\Gamma(\beta)} \int_0^q \frac{F_i'(t,y)}{B(y)(\ln \frac{q}{y})^{1-\alpha-\beta}} \cdot \Big( \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} (\frac{y}{q})^u du \Big) dy, \\
& p_{i0}(t) + p_{i1}(t) q_* \\
&= q_*^i - \frac{q_*}{\Gamma(\alpha)\Gamma(\beta)} \int_{q_*}^0 \frac{F_i'(t,y)}{B(y)(\ln \frac{q_*}{y})^{1-\alpha-\beta}} \cdot \Big( \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} (\frac{y}{q_*})^u du \Big) dy.
\end{aligned}
$$

In view of (5.3.4) and (5.3.1), the above equations can be rewritten as

$$
\begin{aligned}
& (a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) q \\
&= \frac{q}{\Gamma(\alpha)\Gamma(\beta)} \int_0^q \frac{F_i'(t,y)}{B(y)} \cdot (\ln \frac{q}{y})^{\alpha+\beta-1} \cdot \Big( \int_0^1 u^{\alpha-1} (1-u)^{\beta-1} (\frac{y}{q})^u du \Big) dy,
\end{aligned}
$$

and

$$(a_{i0}-p_{i0}(t))+(a_{i1}-p_{i1}(t))q_*$$
$$=\frac{q_*}{\Gamma(\alpha)\Gamma(\beta)}\int_{q_*}^{0}\frac{F_i'(t,y)}{B(y)}\cdot(\ln\frac{q_*}{y})^{\alpha+\beta-1}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du)dy.$$

Therefore, since $a_{ik}-p_{ik}(t)=P(t<\tau_k<\infty|X(0)=i)$, $k=0,1$, integrating with respect to $t$ yields

$$\int_0^t P(s<\tau_0<\infty|X(0)=i)ds+q\int_0^t P(s<\tau_1<\infty|X(0)=i)ds$$
$$=\frac{q}{\Gamma(\alpha)\Gamma(\beta)}\int_0^q\frac{F_i(t,y)-y^i}{B(y)(\ln\frac{q}{y})^{1-\alpha-\beta}}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q})^u du)dy,$$

and

$$\int_0^t P(s<\tau_0<\infty|X(0)=i)ds+q_*\int_0^t P(s<\tau_1<\infty|X(0)=i)ds$$
$$=\frac{q_*}{\Gamma(\alpha)\Gamma(\beta)}\int_{q_*}^0\frac{F_i(t,y)-y^i}{B(y)(\ln\frac{q_*}{y})^{1-\alpha-\beta}}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du)dy.$$

Letting $t\to\infty$ in the above two equations and using the Dominated Convergence Theorem on the right-hand side we obtain that

$$\mu_{i0}+\mu_{i1}q$$
$$=\frac{q}{\Gamma(\alpha)\Gamma(\beta)}\int_0^q\frac{F_i(\infty,y)-y^i}{B(y)(\ln\frac{q}{y})^{1-\alpha-\beta}}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q})^u du)dy \qquad (5.4.8)$$

and

$$\mu_{i0}+\mu_{i1}q_*$$
$$=\frac{q_*}{\Gamma(\alpha)\Gamma(\beta)}\int_{q_*}^0\frac{F_i(\infty,y)-y^i}{B(y)(\ln\frac{q_*}{y})^{1-\alpha-\beta}}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du)dy \qquad (5.4.9)$$

where $F_i(\infty,y):=\lim_{t\to\infty}F_i(t,y)$. Justification of using the Dominated Convergence Theorem in the above is clear. Indeed, by Lemma 5.3.1 we have that $\int_0^q\frac{(q-y)}{B(y)}(\ln\frac{q}{y})^{\alpha+\beta-1}dy<+\infty$ and $\int_{q_*}^0\frac{(y-q_*)}{B(y)}(\ln\frac{q_*}{y})^{\alpha+\beta-1}dy<+\infty$. However, by the definition of $F_i(t,y)$ we see that for all $y\in(0,q)$,

$$\frac{F_i(t,y)-y^i}{B(y)}\cdot(\ln\frac{q}{y})^{\alpha+\beta-1}\cdot(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q})^u du)$$
$$\le\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{|\sum_{j=0}^{\infty}p_{ij}(t)(q^j-y^j)-(q^i-y^i)|}{B(y)}\cdot(\ln\frac{q}{y})^{\alpha+\beta-1}$$
$$\le(\sum_{j=1}^{\infty}jq^{j-1}+iq^{i-1})\cdot\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{(q-y)}{B(y)}(\ln\frac{q}{y})^{\alpha+\beta-1},$$

and that for all $y \in (q_*, 0)$,

$$
\begin{aligned}
& |\frac{F_i(t,y) - y^i}{B(y)} \cdot (\ln \frac{q_*}{y})^{\alpha+\beta-1} \cdot (\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}(\frac{y}{q_*})^u du)| \\
\leq & \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{|\sum_{j=0}^{\infty} p_{ij}(t)(q_*^j - y^j) - (q_*^i - y^i)|}{B(y)} \cdot (\ln \frac{q_*}{y})^{\alpha+\beta-1} \\
\leq & (\sum_{j=1}^{\infty} j|q_*|^{j-1} + i|q_*|^{i-1}) \cdot \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{(y-q_*)}{B(y)} (\ln \frac{q_*}{y})^{\alpha+\beta-1}
\end{aligned}
$$

and therefore the Dominated Convergence Theorem is applicable. Hence on using the identity $F_i(\infty, y) = a_{i0} + a_{i1}y$ together with (5.3.6) and solving for $\mu_{i0}$ and $\mu_{i1}$ from (5.4.8) and (5.4.9) first and then using (5.4.6), we can obtain the expressions $E_i[\tau_0|\tau_0 < \infty]$ and $E_i[\tau_1|\tau_1 < \infty]$ as stated in (ii). The finiteness of these two expressions follow from the facts that $\tilde{f}_i(y)$ and $\tilde{g}_i(y)$ are bounded and $\int_0^q \frac{(q-y)(\ln \frac{q}{y})^{\alpha+\beta-1}}{B(y)} dy < \infty$, $\int_{q_*}^0 \frac{(y-q_*)(\ln \frac{q_*}{y})^{\alpha+\beta-1}}{B(y)} dy < \infty$. It follows that $E_i[\tau|\tau < \infty]$ is also finite and the corresponding expression follows from the definition of $\tau$.

Now we turn to prove (i). Since $m_d \geq m_b$, $P(\tau < \infty | X(0) = i) = 1$. It follows from the definition of $\tau$ and (5.4.6) that

$$
E_i[\tau] = \frac{q_*^i - q_*}{1 - q_*} E_i[\tau_0|\tau_0 < \infty] + \frac{1 - q_*^i}{1 - q_*} E_i[\tau_1|\tau_1 < \infty]. \tag{5.4.10}
$$

Hence both $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ are finite if and only if $E_i[\tau] < \infty$. By Theorem 5.4.1, this is equivalent to that one of (a), (b) or (c) in (i) holds.

Moreover, suppose the above conditions regarding the finiteness of $E_i[\tau]$ are satisfied, then by (5.4.6) and (5.4.10) we can get that

$$
\mu_{i0} + \mu_{i1} = a_{i0} E_i[\tau_0|\tau_0 < \infty] + a_{i1} E_i[\tau_1|\tau_1 < \infty].
$$

Using (5.4.1) yields that

$$
\mu_{i0} + \mu_{i1} = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{(1-y)(-\ln y)^{\alpha+\beta-1}}{B(y)} \cdot f_i(y) dy, \tag{5.4.11}
$$

where $f_i(y)$ is given in (5.4.2). On the other hand, (5.4.9) still holds in this case. Therefore, solving the equations (5.4.11) and (5.4.9) yields the expressions of $\mu_{i0}$ and $\mu_{i1}$ and hence the expressions of $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$. The proof is thus complete. $\square$

## 5.5. Explosion Time and Global Holding Time

In this section we investigate two closely related questions: the explosion behaviour and the so-called global holding time. For these purposes, we need the following Lemma 5.5.1. First, define a family of functions as

$$G_i(y) = \frac{a_{i0} + a_{i1}y - y^i}{B(y)}, \quad i \geq 2 \tag{5.5.1}$$

where $a_{i0}$ and $a_{i1}$ are extinction probabilities evaluated in Section 3. Clearly, $G_i(y)$ is well-defined at least on $[-1, 1)$ since, as shown before, the only possible singularities at $y = q < 1$ and $y = q_*$ are removable. Note also that substituting (5.3.5) and (5.3.6) into the numerator of $G_i(y)$ in (5.5.1) yields

$$a_{i0} + a_{i1}y - y^i = \begin{cases} (1-y)\left(\frac{1-y^i}{1-y} - \frac{1-q_*^i}{1-q_*}\right), & if \quad m_d \geq m_b \\ (q-y)\left(\frac{q^i-y^i}{q-y} - \frac{q^i-q_*^i}{q-q_*}\right), & if \quad m_d < m_b \leq +\infty. \end{cases} \tag{5.5.2}$$

**Lemma 5.5.1.** For any $i \geq 2$, we have

$$\int_0^\infty \Big(\sum_{k=2}^\infty p_{ik}(t)\Big)dt$$
$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \cdot \int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1} \cdot \Big(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\Big)dy. \tag{5.5.3}$$

Moreover, the quantity in the left-hand side of (5.5.3) is finite if and only if one of the following conditions holds.

(i) $\alpha + \beta > 1$.

(ii) $\alpha + \beta \leq 1$ and $m_d > m_b$.

(iii) $\alpha + \beta \leq 1$, $m_d = m_b$, $B''(1) = +\infty$ and

$$\int_0^1 \frac{(1-s)^{\alpha+\beta}}{B(s)} ds < +\infty. \tag{5.5.4}$$

(iv) $\alpha + \beta \leq 1$, $m_b = +\infty$ and

$$\int_\varepsilon^1 \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds < +\infty \tag{5.5.5}$$

for some (equivalently, for all) $\varepsilon$ with $q < \varepsilon < 1$.

**Proof.** Integrating (5.2.17) with respect to $t$ from 0 to $\infty$ yields that

$$\begin{aligned} & \Gamma(\alpha)\Gamma(\beta) \cdot \sum_{k=2}^{\infty} \int_0^{\infty} p_{ik}(t)dt \\ = & \int_0^1 \frac{a_{i0} + a_{i1}x - x^i}{B(x)(-\ln x)^{1-\alpha-\beta}} \cdot \left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}x^u du\right)dx. \end{aligned}$$

Noting (5.5.1) yields (5.5.3). Then it follows from (5.5.3) (see Remark 5.4.1) that

$$\begin{aligned} & \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 yG_i(y)(-\ln y)^{\alpha+\beta-1}dy \\ \le & \int_0^{\infty} \sum_{k=2}^{\infty} p_{ik}(t)dt \\ \le & \frac{1}{\Gamma(\alpha+\beta)} \int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1}dy \end{aligned}$$

which implies, by the similar argument in proving Theorem 5.2.6, that for any $i \ge 2$, $\int_0^{\infty} \sum_{k=2}^{\infty} p_{ik}(t)dt < +\infty$ if and only if $\int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1}dy < +\infty$. We now prove that $\int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1}dy < +\infty$ is equivalent to either

(a) $m_d \ge m_b$ and

$$\int_0^1 \frac{(1-s)^{\alpha+\beta}}{B(s)} ds < +\infty$$

or

(b) $m_d < m_b \le +\infty$ and

$$\int_{\varepsilon}^1 \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds < +\infty$$

for some (equivalently, for all) $\varepsilon$ with $q < \varepsilon < 1$. Indeed, if $m_d \ge m_b$ then

$$\int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1}dy < +\infty$$

is equivalent to

$$\int_0^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1}dy < \infty.$$

Noting that $\lim_{y\uparrow 1} \frac{-\ln y}{1-y} = 1$, we know that

$$\int_0^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1}dy < \infty$$

is equivalent to

$$\int_0^1 \frac{(1-y)^{\alpha+\beta}}{B(y)} dy < \infty.$$

If $m_d < m_b \le \infty$, then

$$\int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1} dy < +\infty$$

is equivalent to

$$\int_\varepsilon^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy < \infty$$

for some $\varepsilon \in (q, 1)$. Note that $a_{i0} + a_{i1} < 1$ and $\lim_{y \uparrow 1} \frac{-\ln y}{1-y} = 1$, we know that

$$\int_\varepsilon^1 \frac{1-y}{B(y)} \cdot (-\ln y)^{\alpha+\beta-1} dy < \infty$$

is equivalent to

$$\int_\varepsilon^1 \frac{(1-y)^{\alpha+\beta-1}}{-B(y)} dy < \infty.$$

Finally, we prove that either (a) or (b) holds if and only if one of the four statements (i)–(iv) in the lemma holds.

Suppose that (i) holds. Then $\int_0^1 (1-y)^{\alpha+\beta-2} dy < \infty$ and (a) or (b) holds.

Suppose that (ii) holds. Then $\int_0^1 \frac{(1-y)^{\alpha+\beta}}{B(y)} dy < \infty$ is equivalent to $\int_0^1 (1-y)^{\alpha+\beta-1} dy < \infty$. But the latter is true since $\alpha + \beta > 0$. Thus (a) holds.

Suppose that (iii) holds. Then (a) holds.

Suppose that (iv) holds. Then (b) holds.

Conversely, suppose that (a) holds. Then one of (i)–(iii) holds.

Suppose that (b) holds. If $\alpha + \beta \le 1$, then we must have $m_b = \infty$ and hence (iv) holds.

The proof is complete. $\square$

We are now ready to consider the explosion behaviour of the process. Note that by definition explosion means the number of particles tends to infinite at some random finite time epoch. Let $\tau_\infty$ denote this explosion time and $a_{i\infty} = P(\tau_\infty < \infty | X(0) = i)$ denote the explosion probability.

Since we are dealing with the minimal process,

$$p_{i\infty}(t) := 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \le t | X(0) = i), \tag{5.5.6}$$

is the probability of explosion by time $t$ starting at state $i$, and $p_{i\infty}(t) \to a_{i\infty}$ as $t \to \infty$.

The following conclusion tells us under what conditions the explosion will occur together with the explosion probability when it does happen.

**Theorem 5.5.2.** Suppose that the GCBP starts from state $i \ge 2$. Then $a_{i\infty} > 0$ (i.e., the explosion occurs) if and only if one of the following conditions holds

(i) $m_d < m_b \le +\infty$ and $\alpha + \beta > 1$.

(ii) $m_b = +\infty$, $\alpha + \beta \le 1$ and $\int_\varepsilon^1 \frac{(1-s)^{\alpha+\beta-1}}{-B(s)} ds < +\infty$ for some (and therefore for all) $\varepsilon$ with $q < \varepsilon < 1$.

Moreover, under either of these two conditions, the explosion probability is given by

$$a_{i\infty} = 1 - \frac{(1-q_*)q^i - (1-q)q_*^i}{q - q_*} > 0. \tag{5.5.7}$$

**Proof.** By Theorem 5.2.7 we see that except the two cases in (i) or (ii) above, the corresponding process is honest and then by (5.5.6) we have $p_{i\infty}(t) \equiv 0$ and hence $a_{i\infty} = 0$, i.e., the explosion does not occur. On the other hand, if either (i) or (ii) in the above holds, then first by Theorem 5.2.7, the corresponding GCBP is dishonest and then by Theorem 5.2.6 we know that $\lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) = 0$ for all $i \ge 2$ (see (5.2.22)). It then follows from (5.5.6) that

$$\begin{aligned} a_{i\infty} &= \lim_{t\to\infty} p_{i\infty}(t) \\ &= 1 - \lim_{t\to\infty} p_{i0}(t) - \lim_{t\to\infty} p_{i0}(t) - \lim_{t\to\infty} \sum_{j=2}^{\infty} p_{ij}(t) \\ &= 1 - a_{i0} - a_{i1}. \end{aligned}$$

By Theorem 5.3.3, we have

$$a_{i\infty} = 1 - \frac{(1-q_*)q^i - (1-q)q_*^i}{q - q_*} > 0$$

and thus the explosion does occur. □

By Theorem 5.5.2 we see that for a super-explosive GCBP, the explosion occurs if and only if $m_d < m_b \le +\infty$ while for a sub-explosive GCBP, the explosion seldom occurs. Indeed, for a sub-explosive GCBP, no matter $m_d \ge m_b$ or $m_d < m_b < +\infty$, the explosion definitely does not occur. Even if $m_b = +\infty$, the explosion still does not occur unless a further condition, the finite integral condition stated in (ii) of Theorem 5.5.2, is satisfied. This shows that the explosion behaviour is substantially different between the super-explosive and sub-explosive processes. This also gives the probabilistic interpretation of (5.2.22) in Theorem 5.2.6.

Another important problem is to find how long, averagely, the process will take to explosion. Of course, we shall only consider this question under the condition that explosion does occur. Note that such condition has been already given in Theorem 5.5.2. In other words, we shall only consider the conditional mean time $E_i[\tau_\infty | \tau_\infty < \infty]$.

**Theorem 5.5.3.** Under the explosion conditions given in Theorem 5.5.2, the conditional mean explosion time $E_i[\tau_\infty|\tau_\infty < \infty]$ $(i \ge 2)$ is finite and whose value can be obtained by using the equality

$$\begin{aligned} & a_{i0}E_i[\tau_0|\tau_0 < \infty] + a_{i1}E_i[\tau_1|\tau_1 < \infty] + a_{i\infty}E_i[\tau_\infty|\tau_\infty < \infty] \\ = & \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \cdot \int_0^1 G_i(y)(-\ln y)^{\alpha+\beta-1} \cdot \left(\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}y^u du\right)dy \qquad (5.5.8) \end{aligned}$$

**Proof.** We only need to prove the equality (5.5.8) since the finiteness of $E_i[\tau_\infty|\tau_\infty < \infty]$follows from it. Indeed, by Lemma 5.5.1 we see that under the explosion conditions given in Theorem 5.5.2, the right-hand side of (5.5.8) is finite. But we also have $a_{i\infty} > 0$ under the explosion conditions, hence the finiteness of $E_i[\tau_\infty|\tau_\infty < \infty]$ follows.

Now we prove (5.5.8). Indeed, by (5.5.7) and Theorem 5.3.3,

$$a_{i0} + a_{i1} + a_{i\infty} = 1.$$

Thus it follows from (5.5.6) that we have

$$(a_{i\infty} - p_{i\infty}(t)) + (a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) = \sum_{k=2}^{\infty} p_{ik}(t), \quad i \geq 2.$$

Integrating $t$ from 0 to infinity in the above expression yields that

$$\begin{aligned} &\int_0^\infty (a_{i\infty} - p_{i\infty}(t))dt + \int_0^\infty (a_{i0} - p_{i0}(t))dt + \int_0^\infty (a_{i1} - p_{i1}(t))dt \\ = &\int_0^\infty (\sum_{k=2}^{\infty} p_{ik}(t))dt, \quad i \geq 2. \end{aligned}$$

Noting (5.5.3), (5.4.6) and the fact that

$$a_{i\infty} E_i[\tau_\infty | \tau_\infty < \infty] = E_i[\tau_\infty I_{\{\tau_\infty < \infty\}}] = \int_0^\infty (a_{i\infty} - p_{i\infty}(t))dt,$$

we know that (5.5.8) holds. The proof is complete. □

We now consider the so-called global holding time. We have already obtained the mean extinction and mean explosion times. These quantities describe how long the process will take to extinction or explosion. Before reaching the epoch of extinction/explosion, the process wanders over the states $k \geq 2$ which forms the total life time of the process. We are therefore interested in obtaining the time spent in each state over the lifetime since it gives us much information regarding the moving behaviour of the process. More specifically, let $T_k$ be the total time spent in state $k$ ($\geq 2$) and let $\mu_{ik} = E[T_k | X(0) = i]$ $(i \geq 2)$. Then

$$\mu_{ik} = E[T_k | X(0) = i] = \int_0^\infty p_{ik}(t)dt$$

and

$$\mu_i = \sum_{k=2}^{\infty} \mu_{ik} = \int_0^\infty \sum_{k=2}^{\infty} p_{ik}(t)dt$$

are the global holding time at state $k$ and total lifetime of the process, starting from state $i \geq 2$, respectively. The expressions of these quantities are given in the following theorem which is the final result of this chapter. The expression for the latter quantity $\mu_i$ is, in fact, already obtained in Lemma 5.5.1.

**Theorem 5.5.4.** All the global holding times $\mu_{ik}$ $(i \geq 2,\ k \geq 2)$ are finite and given by

$$\mu_{ik} = \frac{1}{k^\alpha (k-1)^\beta} \frac{G_i^{(k-2)}(0)}{(k-2)!} \tag{5.5.9}$$

where $G_i^{(k-2)}(0)$ is the $(k-2)$'th derivative of $G_i(y)$ given in (5.5.1) evaluated at 0.

Moreover, the total lifetime $\mu_i$ is given in the right-hand side of (5.5.3) which is finite if and only if one of the conditions (i)−(iv) in Lemma 5.5.1 holds.

**Proof.** The finiteness of all $\mu_{ik}$ $(i \geq 2,\ k \geq 2)$ follows from (5.2.7). Secondly, it follows from (5.2.6) that

$$\frac{F_i(t,y) - y^i}{B(y)} = \sum_{k=2}^{\infty} (\int_0^t p_{ij}(u)du) \cdot k^{\alpha}(k-1)^{\beta} y^{k-2}.$$

Letting $t \uparrow \infty$ yields

$$G_i(y) = \sum_{k=2}^{\infty} (\int_0^{\infty} p_{ij}(u)du) \cdot k^{\alpha}(k-1)^{\beta} y^{k-2}.$$

Comparing the coefficient of $y^{k-2}$ on both sides of the above equality immediately yields (5.5.9). The last part of the theorem follows directly from Lemma 5.5.1. □

## 5.6. Notes

GCBP is a generalisation of CBP considered in the previous chapter. If $\alpha = \beta = 1$, then we recover CBP, so CBP is sub-explosive.

The work presented in this chapter has been submitted for publication in Chen, Li and Ramesh (2004b). Specifically, Lemma 5.2.2, Theorem 5.2.6, Lemma 5.3.1, Theorem 5.4.1, Theorem 5.4.2 and Lemma 5.5.1 are due to Li. The introduction of the function $G(s)$ is due to Li. Lemma 5.2.1, Lemma 5.2.3, Lemma 5.2.4, Theorem 5.2.7, Theorem 5.3.2, Theorem 5.3.3, Theorem 5.5.2 and Theorem 5.5.3 are due Li and the other authors. The idea of Lemma 5.2.2 is from Chen (2002a).

It is interesting to compare the different life behaviour between the super-explosive and sub-explosive processes. By the results obtained in Sections 5.3 to 5.5, particularly in Theorem 5.5.4, we have seen clearly such substantial difference. If the process is super-explosive, then the mean total lifetime is always finite. In fact, if $m_d \geq m_b$, then the process will definitely tend to extinction with a finite mean time while if $m_d < m_b \leq +\infty$, then the process possesses a positive probability of explosion and the process will tend to either extinction or explosion with a finite mean time. On the other hand, if the process is sub-explosive, then the behaviour is very different, particularly for the case $m_d < m_b \leq +\infty$. In fact, only in the case $m_d > m_b$, the behaviour of sub-explosive process is similar to that of the super-explosive, i.e., the process will tend to extinction with a finite mean time. If $m_d = m_b$, then although the sub-explosive process will tend to extinction with probability 1, the mean extinction time is usually infinite except in the unusual situation of $B''(1) = +\infty$ together with (5.5.4) being held true. The most interesting situation is that $m_d < m_b < +\infty$. In this case, the lifetime is infinite although the sub-explosive process tends to extinction with a probability which is strictly less than 1 and that no explosion will happen! This means that with a positive probability, the process drifts to infinity by wandering over states $k \geq 2$ at an infinite mean time (but spending at each fixed state $k \geq 2$ only finite mean time). Finally, if $m_b = +\infty$, the behaviour is similar as in the case $m_d < m_b < +\infty$ unless the condition (5.5.5) is satisfied. For this last unusual case, the sub-explosive process tends to explosion with a positive probability and the total lifetime is finite, that is, only in this special case, will the sub-explosive process behave like a super-explosive one, i.e., the process will tend to either extinction or explosion with a finite mean time.

Since CBP is sub-explosive, its life behaviour is quite different from that of super-explosive GCBP.

The most general collision branching models will be studied in the next chapter.

# 6. Weighted Collision Branching Processes

In this chapter, we shall consider the most general collision branching model which covers GCBP (and hence CBP) as its special cases. It will be seen that for the model considered in Chapter 4 or Chapter 5, the results regarding extinction probabilities and extinction time can be deduced from the corresponding results in this chapter. However, the regularity and uniqueness criteria are still not available in some cases for the most general model considered. Additionally, the explosion behaviour of this most general model is quite different from that of CBP or GCBP in the previous chapter.

## 6.1. Description of the Model

In Chapter 4, we studied a collision branching model, where the branching events are effected by the interaction/collision of pairs of particles and this model was generalised in Chapter 5.

The models studied in Chapter 4 and Chapter 5 are applicable to some realistic situations. However, in other realistic cases, we may need to consider more general collision branching models. Although we studied the GCB model in Chapter 5, which covers CBP as its special case, the GCB model has its own limitation. In this chapter, we shall further generalise GCBPs to the most general collision branching model which covers CBP and GCBP as its special cases.

**Definition 6.1.1.** A conservative $q$-matrix $Q = \{q_{ij},\ i, j \in \mathbf{Z}_+\}$ is called a weighted collision branching $q$-matrix (WCB $q$-matrix) if it takes the following form:

$$q_{ij} = \begin{cases} w_i b_{j-i+2}, & if\ \ i \geq 2,\ j \geq i-2 \\ 0, & otherwise \end{cases} \tag{6.1.1}$$

where

$$b_j \geq 0\ (j \neq 2)\ \ and\ \ 0 < -b_2 = \sum_{j \neq 2} b_j < +\infty \tag{6.1.2}$$

together with

$$b_0 > 0, \; b_1 \geq 0, \sum_{j=3}^{\infty} b_j > 0 \;\; and \;\; w_i > 0 \; (i \geq 2). \tag{6.1.3}$$

By Remark 4.1.1, we shall again assume that the WCB $q$-matrix is not degenerative throughout this chapter, i.e., $\sum_{j=0}^{\infty} b_{2j+1} > 0$. Note that, however, most of results obtained in this chapter apply well to the degenerative case if some statements are amended in a proper way.

**Definition 6.1.2.** A Weighted Collision Branching Process (WCBP) is a continuous-time Markov chain taking values in $\mathbf{Z}_+$ whose transition function $P(t) = (p_{ij}(t), \; i, j \in \mathbf{Z}_+)$ satisfies the forward equation

$$P'(t) = P(t)Q, \tag{6.1.4}$$

where $Q$ is a WCB $q$-matrix.

It is clear that a WCBP reduces to a GCBP if $w_i = i^{\alpha}(i-1)^{\beta}$ $(i \geq 2)$.

**Definition 6.1.3.** Let $Q$ be a WCB $Q$-matrix defined in (6.1.1)$-$(6.1.3).

(i) If $\sum_{i=2}^{\infty} \frac{1}{w_i} = +\infty$, then $Q$ (respectively, the corresponding $Q$-process) is called *natural* or *sub-explosive.*

(ii) If $\sum_{i=2}^{\infty} \frac{1}{w_i} < +\infty$, then $Q$ (respectively, the corresponding $Q$-process) is called *explosive.*

## 6.2. Preliminary

Since $Q$ is still stable and conservative, by Theorem 1.3.1, there always exists a WCBP. Therefore, we first investigate the regularity and uniqueness question. For this purpose, we need some preparation. First of all, as in the previous two chapters, let

$$B(s) = \sum_{j=0}^{\infty} b_j s^j, \quad |s| \leq 1$$

and

$$m_b = \sum_{j=1}^{\infty} j b_{j+2}, \qquad m_d = 2b_0 + b_1.$$

We shall view $B(s)$ as a complex function.

¿From Lemma 4.2.1, we can denote

$$\rho_0 = \begin{cases} -\frac{b_0}{q_*}, & if \;\; m_d \geq m_b \\ -\frac{b_0}{qq_*}, & if \;\; m_d < m_b \leq +\infty, \end{cases}$$

$$\rho_k = \begin{cases} \sum_{j=1}^{\infty} \sigma_{k+j-1} q_*^{j-1}, & if \;\; m_d \geq m_b, \\ \frac{1}{q-q_*} \sum_{j=1}^{\infty} b_{k+j+1}(q^j - q_*^j), & if \;\; m_d < m_b \leq +\infty, \end{cases} \qquad k \geq 1.$$

As showed in the proof of Lemma 4.2.1, $\{rho_k;\; k \geq 0\}$ is a nonnegative series and

$$\sum_{k=1}^{\infty} \rho_k \begin{cases} < \rho_0, & if \;\; m_d > m_b, \\ = \rho_0, & if \;\; m_d \leq m_b \leq +\infty. \end{cases}$$

Further define the function

$$G(z) = \begin{cases} \frac{(1-z)(z-q_*)}{B(z)}, & if \; m_d \geq m_b, \\ \frac{(q-z)(z-q_*)}{B(z)}, & if \; m_d < m_b \leq +\infty, \end{cases} \qquad |z| < 1. \tag{6.2.1}$$

$G(z)$ is well-defined on the disk $\{z;\; |z| < 1\}$ by Lemma 4.2.1. This function will play an important role in our future analysis. The following two lemmas show the detailed properties of $G$.

**Lemma 6.2.1.** $G(z)$ is analytic on the disk $\{z;\; |z| < 1\}$ and thus can be expanded as a Taylor series

$$G(z) = \sum_{n=0}^{\infty} g_n z^n, \tag{6.2.2}$$

where $g_n = G^{(n)}(0)/n!$ $(n \geq 0)$ satisfies the following properties:

(i) $0 < g_n \leq g_0$ $(n \geq 0)$.

(ii) If $m_d < m_b \leq +\infty$ (and thus $B(s) = 0$ has a root $q \in (0,1)$), then the limit $\lim_{n\to\infty} g_n$ exists, denoted by $g_\infty$, and

$$g_\infty = \frac{(1-q)(1-q_*)}{m_b - m_d}. \tag{6.2.3}$$

In particular, $g_\infty > 0$ if and only if $m_b < \infty$.

**Proof.** From the proof of Lemma 4.2.1,we can see that

$$G(z) = (\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k)^{-1}.$$

Since $\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k$ is analytic on the disk $\{z; |z| < 1\}$ and $|\rho_0 - \sum_{k=1}^{\infty} \rho_k z^k| \geq \rho_0(1 - |z|) > 0$ for all $|z| < 1$, we obtain that $G(z)$ is analytic on the disk $\{z; |z| < 1\}$ and thus can be expanded as a Taylor series (6.2.2).

It follows from $G(z) \cdot (\rho_0 - \Sigma_{k=1}^{\infty} \rho_k z^k) = 1$ $(|z| < 1)$ that

$$\rho_0 g_0 = 1, \quad \rho_0 g_n = \sum_{k=1}^{n} \rho_k g_{n-k}, \quad n \geq 1 \tag{6.2.4}$$

and then (i) immediately follows.

Now suppose $m_d < m_b \leq +\infty$. After rewriting (6.2.4) as

$$g_n - \sum_{k=0}^{n} g_k a_{n-k} = c_n \quad (n \geq 0)$$

where $a_0 = 0$, $a_k = \rho_0^{-1} \rho_k$ $(k \geq 1)$ and $c_n = \rho_0^{-1} \delta_{0n}$ and noting that $\{g_n;\ n \geq 0\}$ is bounded and $\Sigma_{k=1}^{\infty} a_k = 1$, we recognize that (6.2.4) is just a renewal equation. It follows (see Theorem 3.1.1 in Karlin (1966) or, Kingman (1972) that $\lim_{n \to \infty} g_n = g_\infty$ exists and

$$\lim_{n \to \infty} g_n = \frac{c_0}{\Sigma_{k=1}^{\infty} k a_k} = \frac{1}{\Sigma_{k=1}^{\infty} k \rho_k}.$$

Note that

$$\begin{aligned}
\sum_{k=1}^{\infty} k\rho_k &= \frac{1}{q - q_*} \cdot \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} k b_{k+j+1} (q^j - q_*^j) \\
&= \frac{1}{q - q_*} \cdot [\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k + j - 1) b_{k+j+1} (q^j - q_*^j) \\
&\quad - \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (j - 1) b_{k+j+1} (q^j - q_*^j)],
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (k + j - 1) b_{k+j+1} (q^j - q_*^j) \\
= &\frac{q}{1 - q} \cdot (m_b - \sum_{k=1}^{\infty} k b_{k+2} q^k) - \frac{q_*}{1 - q_*} \cdot (m_b - \sum_{k=1}^{\infty} k b_{k+2} q_*^k)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (j - 1) b_{k+j+1} (q^j - q_*^j) \\
= &\sum_{k=1}^{\infty} k \sigma_{k+1} (q^{k+1} - q_*^{k+1}).
\end{aligned}$$

So

$$\begin{aligned}
\sum_{k=1}^{\infty} k\rho_k &= \frac{1}{q-q_*}\cdot[\frac{q}{1-q}(m_b-\sum_{k=1}^{\infty} kb_{k+2}q^k)-\sum_{j=1}^{\infty} j\sigma_{j+1}q^{j+1} \\
&\quad -\frac{q_*}{1-q_*}(m_b-\sum_{k=1}^{\infty} kb_{k+2}q_*^k)+\sum_{j=1}^{\infty} j\sigma_{j+1}q_*^{j+1} \\
&= \frac{1}{q-q_*}\cdot(\frac{q}{1-q}-\frac{q_*}{1-q_*})m_b \\
&\quad -\frac{q}{(q-q_*)(1-q)}\cdot[\sum_{k=1}^{\infty} kb_{k+2}q^k+(1-q)\sum_{j=1}^{\infty} j\sigma_{j+1}q^j] \\
&\quad +\frac{q_*}{(q-q_*)(1-q_*)}\cdot[\sum_{k=1}^{\infty} kb_{k+2}q_*^k+(1-q_*)\sum_{j=1}^{\infty} j\sigma_{j+1}q_*^j] \\
&= \frac{m_b}{(1-q)(1-q_*)}-\frac{1}{(q-q_*)(1-q)}\cdot\sum_{k=1}^{\infty}\sigma_k q^{k+1} \\
&\quad +\frac{1}{(q-q_*)(1-q_*)}\cdot\sum_{k=1}^{\infty}\sigma_k q_*^{k+1} \\
&= \frac{m_b-m_d}{(1-q)(1-q_*)}.
\end{aligned}$$

The proof is thus complete. □

**Lemma 6.2.2.** Let $(p_{ij}(t); i,j\in\mathbf{Z}_+)$ and $(\phi_{ij}(\lambda); i,j\in\mathbf{Z}_+)$ be the Feller minimal $Q$-function and $Q$-resolvent, respectively, where $Q$ is a WCB $q$-matrix given in (6.1.1)–(6.1.3). Then for any $i\ge 2$ and $|s|<1$,

$$\sum_{k=2}^{\infty} p_{ik}(t)w_k|s|^{k-2}<+\infty, \tag{6.2.5}$$

$$\sum_{j=0}^{\infty} p'_{ij}(t)s^j = B(s)\cdot\sum_{k=2}^{\infty} p_{ik}(t)\cdot w_k s^{k-2}, \tag{6.2.6}$$

and

$$\sum_{k=2}^{\infty}\phi_{ik}(\lambda)\cdot w_k|s|^{k-2}<+\infty, \tag{6.2.7}$$

$$\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j = s^i+B(s)\cdot\sum_{k=2}^{\infty}\phi_{ik}(\lambda)\cdot w_k s^{k-2}. \tag{6.2.8}$$

**Proof.** We only need to consider the case $0\le s<1$, case $-1<s\le 0$ is similar. By (1.3.2), the Feller minimal $Q$-resolvent can be obtained by the following forward integral recursion

$$\begin{cases}\phi_{ij}^{(0)}(\lambda)=\frac{\delta_{ij}}{\lambda+q_j},\\ \phi_{ij}^{(n+1)}(\lambda)=\frac{\delta_{ij}}{\lambda+q_j}+\sum_{k\ne j}\phi_{ik}^{(n)}(\lambda)\cdot\frac{q_{kj}}{\lambda+q_j},\quad n\ge 0\end{cases} \tag{6.2.9}$$

with $\phi_{ij}^{(n)}(\lambda) \uparrow \phi_{ij}(\lambda)$ as $n \uparrow \infty$ for all $i, j \in \mathbf{Z}_+$.

Now, we firstly claim that for any $n \geq 0$, $i \geq 0$ and $0 \leq s \leq 1$,

$$\sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) \cdot w_k \cdot s^{k-2} < +\infty. \tag{6.2.10}$$

If $i = 0$ or $1$, (6.2.10) is trivial true since both 0 and 1 are absorbing states and so we assume $i \geq 2$. It is easy to see that for $n = 0$,

$$\sum_{k=2}^{\infty} \phi_{ik}^{(0)}(\lambda) \cdot w_k \cdot s^{k-2} = \frac{w_i}{\lambda + q_i} \cdot s^{i-2} < +\infty.$$

Suppose (6.2.10) holds true for $n$, then by (6.2.9)

$$\begin{aligned}
&\sum_{j=0}^{\infty} (\lambda + q_j) \phi_{ij}^{(n+1)}(\lambda) s^j \\
= \;& s^i + \sum_{j=0}^{\infty} \sum_{k \neq j} \phi_{ik}^{(n)}(\lambda) q_{kj} s^j \\
= \;& s^i + \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} \cdot \sum_{j \neq k} \frac{q_{kj}}{w_k} s^{j-k+2} \\
= \;& s^i + \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} \cdot (b_0 + b_1 s + \sum_{m=1}^{\infty} b_{m+2} s^{m+2}).
\end{aligned} \tag{6.2.11}$$

Hence

$$-b_2 \sum_{j=2}^{\infty} \phi_{ij}^{(n+1)}(\lambda) \cdot w_j s^j \leq s^i - b_2 \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} < +\infty$$

and therefore (6.2.10) holds true for $n+1$. Thus (6.2.10) holds for any $n \geq 0$ by the mathematical induction principle.

Secondly, define $A_{ij}^{(n+1)}(\lambda) = \phi_{ij}^{(n+1)}(\lambda) - \phi_{ij}^{(n)}(\lambda)$, $(n \geq 0)$. Then $A_{ij}^{(n)}(\lambda) \geq 0, (n \geq 1)$ and

$$\lim_{n \to \infty} A_{ij}^{(n)}(\lambda) = 0 \qquad \textit{for all} \;\; i, j \in \mathbf{Z}_+. \tag{6.2.12}$$

Using this notation, (6.2.11) can be rewritten as

$$\begin{aligned}
\lambda \sum_{j=0}^{\infty} \phi_{ij}^{(n+1)}(\lambda) s^j \;=\;& s^i + B(s) \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} \\
& + b_2 s^2 \sum_{j=2}^{\infty} A_{ij}^{(n+1)}(\lambda) \cdot w_j s^{j-2}.
\end{aligned} \qquad (6.2.\text{:}$$

Now by (6.2.9) we have

$$A_{ij}^{(n+1)}(\lambda) = \sum_{k\neq j} A_{ik}^{(n)}(\lambda) \cdot \frac{q_{kj}}{\lambda + q_j}, \qquad n \geq 0$$

and so

$$\begin{aligned}
&\sum_{j=0}^{\infty} A_{ij}^{(n+1)}(\lambda)(\lambda + q_j)s^j \\
= &\sum_{j=0}^{\infty}\sum_{k\neq j} A_{ik}^{(n)}(\lambda) q_{kj} s^j \\
= &\sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k s^{k-2} \cdot \sum_{j\neq k} \frac{q_{kj}}{w_k} s^{j-k+2} \\
= &\sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k s^{k-2} \cdot (b_0 + b_1 s + \sum_{m=1}^{\infty} b_{m+2} s^{m+2}).
\end{aligned}$$

Therefore for all $0 \leq s \leq 1$ and $i \geq 2$,

$$\begin{aligned}
\sum_{j=2}^{\infty} A_{ij}^{(n+1)}(\lambda) w_j s^{j-2} &= \frac{b_0 + b_1 s + \sum_{m=1}^{\infty} b_{m+2} s^{m+2}}{-b_2 s^2} \cdot \sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k s^{k-2} \\
&\quad + \frac{\lambda}{b_2 s^2} \cdot \sum_{j=0}^{\infty} A_{ij}^{(n+1)}(\lambda) s^j
\end{aligned}$$

and hence

$$\begin{aligned}
\sum_{j=2}^{\infty} A_{ij}^{(n+1)}(\lambda) w_j s^{j-2} &\leq \frac{b_0 + b_1 s + \sum_{m=1}^{\infty} b_{m+2} s^{m+2}}{-b_2 s^2} \\
&\quad \cdot \sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k s^{k-2}. \qquad (6.2.14)
\end{aligned}$$

Letting $s = 1$ in (6.2.14) yields

$$\sum_{k=2}^{\infty} A_{ik}^{(n+1)}(\lambda) w_k \leq \sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k. \qquad (6.2.15)$$

However, it is easily seen that

$$\begin{aligned}
\sum_{k=2}^{\infty} A_{ik}^{(1)}(\lambda) w_k &= \sum_{k=2}^{\infty} (\sum_{j\neq k} \frac{\delta_{ij}}{\lambda + q_j} \cdot \frac{q_{jk}}{\lambda + q_k}) w_k \\
&= \sum_{j=2}^{\infty} \frac{\delta_{ij}}{\lambda + q_j} \sum_{k\neq j} \frac{q_{jk}}{\lambda + q_k} \cdot w_k \\
&\leq -\frac{1}{b_2} \sum_{j=2}^{\infty} \frac{\delta_{ij}}{\lambda + q_j} \sum_{k\neq j} q_{jk} \\
&\leq -\frac{1}{b_2}
\end{aligned}$$

and thus by (6.2.15), $\sum_{k=2}^{\infty} A_{ik}^{(n)}(\lambda) w_k \leq -\frac{1}{b_2}$ for all $n \geq 1$. Applying Dominated Convergence Theorem and using (6.2.12) yield that for $0 \leq s < 1$

$$\lim_{n\to\infty} \sum_{j=2}^{\infty} A_{ij}^{(n+1)}(\lambda) w_j s^{j-2} = 0.$$

Letting $n \uparrow \infty$ in (6.2.13) and using the above equality then leads to the fact that for $0 \leq s < 1$

$$\lambda \sum_{j=0}^{\infty} \phi_{ij}(\lambda) s^j = s^i + B(s) \cdot \lim_{n\to\infty} \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} \tag{6.2.16}$$

provided $B(s) \neq 0$. However, we may find an $\varepsilon > 0$ such that for all $1-\varepsilon \leq s < 1$, $B(s) \neq 0$. Hence by (6.2.16) we have

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} < +\infty, \quad \textit{for} \;\; s \in [1-\varepsilon, 1).$$

Using Monotone Convergence Theorem yields

$$\sum_{k=2}^{\infty} \phi_{ik}(\lambda) w_k s^{k-2} = \lim_{n\to\infty} \sum_{k=2}^{\infty} \phi_{ik}^{(n)}(\lambda) w_k s^{k-2} < +\infty, \quad \textit{for} \;\; s \in [1-\varepsilon, 1)$$

which actually holds for all $0 \leq s < 1$. Thus we have proved (6.2.7) and (6.2.8). Hence (6.2.5) and (6.2.6) follow immediately by using the properties of Laplace transform. □

**Lemma 6.2.3**. Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function, where $Q$ is a WCB $q$-matrix given in (6.1.1)$-$(6.1.3). Then for any $i, k \geq 2$,

$$\int_0^{\infty} p_{ik}(t) dt < \infty \tag{6.2.17}$$

and hence $\lim_{t\to\infty} p_{ik}(t) = 0$. Moreover, for any $i \geq 2$,

$$\lim_{t\to\infty} p_{i0}(t) = \begin{cases} \frac{-q_* + q_*^i}{1-q_*}, & \textit{if } m_d \geq m_b \\ \frac{-q_* q^i + q q_*^i}{q - q_*}, & \textit{if } m_d < m_b \leq +\infty \end{cases} \tag{6.2.18}$$

and

$$\lim_{t\to\infty} p_{i1}(t) = \begin{cases} \frac{1-q_*^i}{1-q_*}, & \textit{if } m_d \geq m_b \\ \frac{q^i - q_*^i}{q - q_*}, & \textit{if } m_d < m_b \leq +\infty. \end{cases} \tag{6.2.19}$$

**Proof.** For any fixed $i \geq 0$, it follows from the Kolomogorv forward equations that

$$p_{i0}(t) = \delta_{i0} + q_{20} \cdot \int_0^t p_{i2}(s) ds$$

which clearly implies that $\int_0^\infty p_{i2}(t)dt < \infty$. Suppose $\int_0^\infty p_{ik}(t)dt < \infty$ for $2 \leq k \leq j$. ¿From Kolmogorov forward equations we can see that

$$p_{ij-1}(t) - \delta_{ij-1} = \sum_{k=2}^{j} w_k b_{j-k+1} \cdot \int_0^\infty p_{ik}(t)dt + w_{j+1} b_0 \int_0^\infty p_{ij+1}(t)dt$$

and hence $\int_0^\infty p_{ij+1}(t)dt < \infty$. Therefore, (6.2.17) follows from the mathematical induction principle. Hence $\lim_{t\to\infty} p_{ik}(t) = 0$.

We now prove (6.2.18) and (6.2.19). First of all, noting that 0 and 1 are absorbing states we know that the limits $\lim_{t\to\infty} p_{i0}(t)$ and $\lim_{t\to\infty} p_{i1}(t)$ exist. If $m_d \geq m_b$, then using (6.2.6) and the fact $B(s) > 0$ for all $s \in [0, 1)$ can easily yield

$$\sum_{j=0}^{\infty} p_{ij}(t) s^j \geq s^i, \quad s \in [0, 1).$$

Letting $t \to \infty$ in the above equality yields

$$\lim_{t\to\infty} p_{i0}(t) + s \lim_{t\to\infty} p_{i1}(t) \geq s^i, \quad s \in [0, 1).$$

and therefore

$$\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t) = 1 \tag{6.2.20}$$

since we always have $\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t) \leq 1$. On the other hand, note that $B(q_*) = 0$ we may deduce from (6.2.6) that $\sum_{j=0}^{\infty} p_{ij}(t) q_*^j = q_*^i$. Hence we have

$$\lim_{t\to\infty} p_{i0}(t) + q_* \lim_{t\to\infty} p_{i1}(t) = q_*^i. \tag{6.2.21}$$

From (6.2.20) and (6.2.21) we immediately obtain (6.2.18) and (6.2.19) for the case $m_d \geq m_b$. If $m_d < m_b \leq +\infty$, then $B(s) = 0$ has a root $q$ in (0,1), so we can deduce from (6.2.6) that $\sum_{j=0}^{\infty} p_{ij}(t) q^j = q^i$ and hence

$$\lim_{t\to\infty} p_{i0}(t) + q \lim_{t\to\infty} p_{i1}(t) = q^i. \tag{6.2.22}$$

Also note that (6.2.21) still holds for the present case, therefore (6.2.18) and (6.2.19) follow from (6.2.21) and (6.2.22). The proof is complete. □

For any $i \geq 2$, define

$$G_i(s) = \begin{cases} \left(\frac{-q_* + q_*^i}{1-q_*} + \frac{1-q_*^i}{1-q_*} s - s^i\right)/B(s), & if\ m_d \geq m_b \\ \left(\frac{-q_* q^i + q q_*^i}{q-q_*} + \frac{q^i - q_*^i}{q-q_*} s - s^i\right)/B(s), & if\ m_d < m_b \leq +\infty. \end{cases}$$

**Remark 6.2.1.** It is easily seen from (6.2.1) that $G_2(s) = G(s)$ and for any $i \geq 2$,

$$G_i(s) = G(s) \cdot \begin{cases} \sum_{j=0}^{i-2} \frac{1-q_*^{i-1-j}}{1-q_*} s^j, & if\ m_d \geq m_b \\ \sum_{j=0}^{i-2} \frac{q^{i-1-j}-q_*^{i-1-j}}{q-q_*} s^j, & if\ m_d < m_b \leq +\infty. \end{cases} \tag{6.2.23}$$

Therefore, from Lemma 6.2.1 we can see that for any $i \geq 2$, $G_i(z)$ is analytic on the disk $\{z;\ |z| < 1\}$ and thus can be expanded as a Taylor series as

$$G_i(z) = \sum_{n=0}^{\infty} \frac{G_i^{(n)}(0)}{n!} z^n, \quad |z| < 1,\ i \geq 2$$

where $G_i^{(n)}(0)$ is the $n$'th derivative of $G_i$ at 0. Also, by (6.2.23) and Lemma 6.2.1, it is easy to see that for any $i \geq 2$ and $n \geq 0$,

$$\frac{G_i^{(n)}(0)}{n!} = \begin{cases} \sum_{j=0}^{(i-2)\wedge n} \frac{1-q_*^{i-1-j}}{1-q_*} \cdot g_{n-j}, & if\ m_d \geq m_b \\ \sum_{j=0}^{(i-2)\wedge n} \frac{q^{i-1-j}-q_*^{i-1-j}}{q-q_*} \cdot g_{n-j}, & if\ m_d < m_b \leq +\infty \end{cases} \tag{6.2.24}$$

which implies that $\{\frac{G_i^{(n)}(0)}{n!};\ n \geq 0\}$ is also bounded. In fact, some specific bounds can be easily provided. For example, it is easily seen that $g_{n-j} \leq (\frac{g_0}{g_1})^j \cdot g_n$, $(0 \leq j \leq n)$. So combining this with (6.2.22) yields that for any $i \geq 2$,

$$C_1 \cdot g_n \leq \frac{G_i^{(n)}(0)}{n!} \leq C_2 \cdot g_n, \quad n \geq 0$$

where the positive constants $C_1$ and $C_2$, which may depend on $i \geq 2$, are independent of $n$.

**Lemma 6.2.4.** Let $(p_{ij}(t); i, j \in \mathbf{Z}_+)$ be the Feller minimal $Q$-function, where $Q$ is a WCB $q$-matrix given in (6.1.1)−(6.1.3).

(i) For any $i,\ k \geq 2$, we have

$$\int_0^\infty p_{ik}(t)dt = \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!}. \tag{6.2.25}$$

(ii) If $\sum_{k=2}^\infty (g_{k-2}/w_k) < +\infty$, in particular, if $\sum_{k=2}^\infty (1/w_k) < +\infty$, then for any $i \geq 2$

$$\int_0^\infty (\sum_{k=2}^\infty p_{ik}(t))dt = \sum_{k=2}^\infty \frac{G_i^{(k-2)}(0)}{(k-2)!} < +\infty \tag{6.2.26}$$

and hence

$$\lim_{t\to\infty}\sum_{k=2}^{\infty} p_{ik}(t) = 0. \tag{6.2.27}$$

**Proof.** Integrating with respect to $t$ in (6.2.6) yields

$$\sum_{j=0}^{\infty} p_{ij}(t)s^j - s^i = B(s)\cdot \sum_{k=2}^{\infty}\Big(\int_0^t p_{ik}(u)du\Big)\cdot w_k s^{k-2}. \tag{6.2.28}$$

Letting $t\uparrow\infty$ in (6.2.28) for $s\in(-1,1)$, using Dominated Convergence Theorem on the left-hand side of (6.2.28) and applying Monotone Convergence Theorem on the right-hand side of (6.2.28) yields that

$$G_i(s) = \sum_{k=2}^{\infty}\Big(\int_0^{\infty} p_{ik}(t)dt\Big)\cdot w_k s^{k-2}. \tag{6.2.29}$$

Since (6.2.29) holds at least in $(-1,1)$, we obtain (6.2.25) by using the uniqueness of Taylor expansion.

Finally, suppose that $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < +\infty$. Then by Remark 6.2.1 we see that for any $i\geq 2$, $\sum_{k=2}^{\infty}\frac{1}{w_k}\cdot\frac{G_i^{(k-2)}(0)}{(k-2)!} < +\infty$. Now using (6.2.25) immediately yields (6.2.26). In particular, if $\sum_{k=2}^{\infty}(1/w_k) < \infty$, then $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$ since $\{g_k\}$ is bounded. □

## 6.3. Regularity and Uniqueness

Now we can consider the regularity and uniqueness of the process.

**Theorem 6.3.1.** Suppose that $m_d \geq m_b$, then the WCB $q$-matrix is regular, i.e., the Feller minimal $Q$-process is honest.

**Proof.** Let $(\phi_{ij}(\lambda))$ be the Feller minimal $Q$-resolvent. If $m_d\geq m_b$, then Lemma 4.2.1 implies $B(s)\geq 0$ in $[0,1]$. So, from (6.2.8) we see that

$$\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)s^j \geq s^i\ , \quad i\geq 0,\ 0\leq s<1. \tag{6.3.1}$$

Letting $s\uparrow 1$ in (6.3.1) yields $\lambda\sum_{j=0}^{\infty}\phi_{ij}(\lambda)\geq 1$, implying that equality holds for all $i\geq 0$. We deduce that the minimal $Q$-function is honest, and hence that $Q$ is regular. □

**Theorem 6.3.2.** Suppose that $\sum_{n=2}^{\infty}(1/w_n) = +\infty$.

(i) If $m_b < +\infty$, then $Q$ is regular, i.e., the Feller minimal WCBP is honest and thus there exists only one WCBP.

(ii) If $m_b = +\infty$ and $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < +\infty$, then $Q$ is not regular, i.e., the Feller minimal WCBP is dishonest.

**Proof.** We prove (ii) first. Let $(p_{ij}(t))$ be the Feller minimal $Q$-function. Suppose the contrary is true, then

$$1 - p_{i0}(t) - p_{i1}(t) = \sum_{k=2}^{\infty} p_{ik}(t), \quad i \geq 2$$

which, together with (6.2.26) in Lemma 6.2.4, yields

$$\int_0^{\infty} (1 - p_{i0}(t) - p_{i1}(t))dt < \infty.$$

Hence we obtain $\lim_{t\to\infty} p_{i0}(t) + \lim_{t\to\infty} p_{i1}(t) = 1$ which contradicts with (6.2.18) and (6.2.19) in Lemma 6.2.3 since we have assumed that $m_b = +\infty$.

We now prove (i). If $m_d \geq m_b$, the conclusion follows from Theorem 6.3.1 directly. So we only need to consider the case $m_d < m_b < +\infty$. Let $(\phi_{ij}(\lambda))$ denote the Feller minimal $Q$-resolvent. Using the definition of $G(s)$ we may rewrite (6.2.8) for $i = 2$ as

$$G(s) \cdot [\lambda \sum_{k=0}^{\infty} \phi_{2k}(\lambda)s^k - s^2] = (q - s)(s - q_*) \sum_{k=2}^{\infty} \phi_{2k}(\lambda) \cdot w_k s^{k-2}.$$

Comparing the coefficients of the both sides in the above equality can yields that for any $n \geq 2$,

$$\begin{aligned} & \lambda \sum_{j=0}^{n} \phi_{2j}(\lambda) g_{n-j} - g_{n-2} \\ = \; & -qq_*\phi_{2n+2}(\lambda)w_{n+2} + (q + q_*)\phi_{2n+1}(\lambda)w_{n+1} - \phi_{2n}(\lambda)w_n. \end{aligned} \quad (6.3.2)$$

Noting Lemma 6.2.1 and the fact that $\sum_{j=0}^{\infty} \lambda\phi_{2j}(\lambda) \leq 1$, we obtain (see for example Theorem 2.5.5 in Hunter (1983)) that

$$\lim_{n\to\infty} \sum_{j=0}^{n} \lambda\phi_{2j}(\lambda) g_{n-j} = g_\infty \cdot \sum_{j=0}^{\infty} \lambda\phi_{2j}(\lambda), \quad (6.3.3)$$

where $g_\infty > 0$, guaranteed by the condition $m_d < m_b < +\infty$, is given in (6.2.3). We now claim that for any $\lambda > 0$ we have

$$\sum_{j=0}^{\infty} \lambda\phi_{2j}(\lambda) = 1. \quad (6.3.4)$$

Indeed, if (6.3.4) fails then there exists a $\lambda > 0$ such that $1-\lambda\sum_{j=0}^{\infty}\phi_{2j}(\lambda) > 0$. Letting $n \to \infty$ in (6.3.2) and using (6.3.3) and (ii) of Lemma 6.2.1 we obtain

$$\begin{aligned} & \lim_{n\to\infty}[\phi_{2n}(\lambda)w_n + qq_*\phi_{2n+2}(\lambda)w_{n+2} - (q+q_*)\phi_{2n+1}(\lambda)] \\ = & \ g_\infty \cdot (1 - \sum_{j=0}^{\infty}\lambda\phi_{2j}(\lambda)) > 0. \end{aligned}$$

Hence there exists a constant $\delta > 0$ and an integer $N > 2$ such that for all $n \geq N$, we have

$$\phi_{2n}(\lambda) \geq \delta \cdot w_n^{-1}.$$

This is a contradiction since $\sum_{n=2}^{\infty} w_n^{-1} = +\infty$. Thus (6.3.4) holds for all $\lambda > 0$. It follows from (6.3.4) that

$$\sum_{k=0}^{\infty}\lambda\phi_{ik}(\lambda) = 1, \qquad (\forall\lambda > 0)$$

for all $i \geq 2$ since the set of states $\{2, 3, \cdots\}$ forms a communicating class. As for $i = 0, 1$, it is trivially true. This completes the proof. □

**Theorem 6.3.3.** Suppose $\sum_{n=2}^{\infty}(1/w_n) < \infty$. Then $Q$ is regular if and only if $m_d \geq m_b$.

**Proof.** By Theorem 6.3.1, we only need to prove that if $m_d < m_b \leq +\infty$, then $Q$ is not regular. Indeed, since $\{g_k;\ k \geq 0\}$ is bounded we know that $\sum_{k=2}^{\infty}(1/w_k) < \infty$ implies $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$ and thus the conclusion follows from the proof of Theorem 6.3.2. □

The previous three theorems established regularity criteria. If a WCB-$q$-matrix $Q$ is regular then there exists only one WCBP. However, the converse may not be always true. Indeed, if a WCB-$q$-matrix $Q$ is not regular, then although there exist infinitely many (even honest) $Q$-functions, there may still exist only one WCBP since our WCBP must satisfy the Kolmogorov forward equation (6.1.4). Therefore, in addition to the regularity criteria, we also need to establish uniqueness criteria. Of course, we shall only be interested in the case of $m_d < m_b \leq +\infty$ since otherwise the question has already been answered by Theorem 6.3.1. Note that by Lemma 4.2.1 we know that if $m_d < m_b \leq +\infty$, then $B(s) = 0$ has a root $q$ such that $0 < q < 1$.

**Theorem 6.3.4.** Suppose that $Q$ is a WCB $q$-matrix satisfying $m_d < m_b \leq +\infty$. If $\limsup_{n\to\infty} \sqrt[n]{w_n} < 1/q$, where $q(\in (0,1))$ is the smallest root of $B(s) = 0$ in $[0,1]$, then there exists only one WCBP which is, in fact, the Feller minimal process. In particular, if $\limsup_{n\to\infty} \sqrt[n]{w_n} \leq 1$, then there exists only one WCBP.

**Proof.** We only need to prove that if $\limsup_{n\to\infty} \sqrt[n]{w_n} < 1/q$, then the equation

$$Y(\lambda)(\lambda I - Q) = 0, \quad 0 \leq Y(\lambda) \in l \tag{6.3.5}$$

has only the trivial solution for some (and therefore for all) $\lambda > 0$. Suppose the contrary is true, then equation (6.3.5) has a non-trivial solution $\{y_n(\lambda) : n \geq 0\}$ which, after some rearrangements, satisfies, for all $n \geq 2$,

$$b_0 w_{n+2} y_{n+2} + (b_0 + b_1) w_{n+1} y_{n+1} = \sum_{k=0}^{n} y_k + \sum_{j=2}^{n} (\sum_{k=n+1}^{\infty} b_{k-j+2}) w_j y_j \tag{6.3.6}$$

here we have let $\lambda = 1$ and denoted $y_n = y_n(1) \quad (n \geq 0)$. Denote $h_n = w_n y_n \quad (n \geq 2)$ and $\sigma_n = \sum_{j=n}^{\infty} b_{j+2} \quad (n \geq 1)$. Note that $\{y_n\}$ is a non-negative summable solution of (6.3.5) and thus $\sum_{n=0}^{\infty} y_n s^n$ is finite for all $|s| \leq 1$ and therefore

$$\limsup_{n\to\infty} \sqrt[n]{y_n} \leq 1 \tag{6.3.7}$$

Since both $\{w_n\}$ and $\{y_n\}$ are non-negative sequence we have, by (6.3.7) and the assumed condition, that

$$\limsup_{n\to\infty} \sqrt[n]{h_n} \leq \limsup_{n\to\infty} \sqrt[n]{y_n} \cdot \limsup_{n\to\infty} \sqrt[n]{w_n} < 1/q$$

and hence the convergence radius of the power series $H(s) =: \sum_{n=2}^{\infty} h_n s^{n-2}$ is strictly greater than $q$. Hence there exists $\varepsilon > 0$ such that $H(s)$ is well-defined and finite on $[0, q+\varepsilon)$, here we may further assume $q + \varepsilon < 1$. Also let $Y(s) = \sum_{n=0}^{\infty} y_n s^n$ and $\sigma(s) = \sum_{n=1}^{\infty} \sigma_n s^{n-1}$ , then both $Y(s)$ and $\sigma(s)$ are well-defined and finite at least on $[0,1)$. Hence by (6.3.6) and some algebra we obtain

$$\begin{aligned} & b_0(H(s) - h_2 - h_3 s) + (b_0 + b_1) s (H(s) - h_2) \\ = & \frac{Y(s)}{1-s} - y_0 - (y_0 + y_1) s + s^2 \sigma(s) H(s) \end{aligned} \tag{6.3.8}$$

where all $H(s), Y(s)$ and $\sigma(s)$ are at least well-defined and finite on $[0, q+\varepsilon)$. But it is easily seen that

$$B(s) = (1-s)[b_0 + (b_0+b_1)s - s^2\sigma(s)]. \tag{6.3.9}$$

Substituting (6.3.9) into (6.3.8) and using the fact that $y_0 = b_0h_2$ and $y_1 = b_1h_2 + b_0h_3$, which come from the first two equation of (6.3.5), yields the next equation

$$H(s)B(s) = Y(s) \tag{6.3.10}$$

where all three functions are well-defined and finite at least on $[0, q+\varepsilon)$. In particular, letting $s = q$ in (6.3.10) yields

$$H(q)B(q) = Y(q).$$

Therefore $B(q) = 0$ and $H(q) < +\infty$ imply $Y(q) = 0$ which, in turn, implies $y_n \equiv 0 \quad (\forall n)$. A contradiction since $\{y_n : n \geq 0\}$ is a non-trivial solution of Equation (6.3.6) for $\lambda = 1$. □

We now summarise our main conclusions regarding uniqueness as follows.

**Theorem 6.3.5.** Suppose $Q$ is a WCB $q$-matrix as in (6.1.1)$-$(6.1.3). Then there exists exactly one WCBP if either

(i) $m_d \geq m_b$, or

(ii) $m_d < m_b \leq +\infty$ and $\limsup_{n\to\infty} \sqrt[n]{w_n} < 1/q$ where $0 < q < 1$ is the smallest root of $B(s) = 0$ on $[0,1]$, holds.

Note that for nearly all the models we are interested in, if $m_d < m_b \leq +\infty$, then they do satisfy the condition $\limsup_{n\to\infty} \sqrt[n]{w_n} < 1/q$. For example, the most interesting model discussed in Chapter 4, $w_n = \binom{n}{2}$ and thus $\limsup_{n\to\infty} \sqrt[n]{w_n} = 1 < 1/q$ if $m_d < m_b \leq +\infty$.

## 6.4. Extinction

We now turn to consider the extinction probability and extinction time. From now on, we shall only consider the Feller minimal WCBP and thus Lemma 6.2.2 is applicable.

Let $\{X(t);\ t \geq 0\}$ be the Feller minimal WCBP, and let $P(t) = (p_{ij}(t))$ denote its transition function. Define the extinction times $\tau_0$ and $\tau_1$ for states 0 and 1 by

$$\tau_0 = \begin{cases} \inf\{t > 0,\ X(t) = 0\} & \text{if } X(t) = 0 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 0 \text{ for all } t > 0 \end{cases}$$

$$\tau_1 = \begin{cases} \inf\{t > 0,\ X(t) = 1\} & \text{if } X(t) = 1 \text{ for some } t > 0 \\ +\infty & \text{if } X(t) \neq 1 \text{ for all } t > 0 \end{cases}$$

and denote the corresponding extinction probabilities by

$$a_{i0} = P(\tau_0 < +\infty | X(0) = i) \quad \textit{and} \quad a_{i1} = P(\tau_1 < +\infty | X(0) = i).$$

Note that $\tau =: \tau_0 \wedge \tau_1$ is the extinction time and $a_i =: P(\tau < +\infty | X(0) = i)$ is the corresponding extinction probability.

**Theorem 6.4.1.** For the Feller minimal WCBP starting at state $i \geq 2$.

(i) If $m_d \geq m_b$ then

$$\begin{cases} a_{i0} = (-q_* + q_*^i)/(1 - q_*) \\ a_{i1} = (1 - q_*^i)/(1 - q_*). \end{cases} \tag{6.4.1}$$

(ii) If $m_d < m_b \leq +\infty$ then

$$\begin{cases} a_{i0} = (-q_* q^i + q q_*^i)/(q - q_*) \\ a_{i1} = (q^i - q_*^i)/(q - q_*). \end{cases} \tag{6.4.2}$$

(iii) The overall extinction probability satisfies

$$a_i = a_{i0} + a_{i1}. \tag{6.4.3}$$

Therefore

$$a_{i0} + a_{i1} = 1, \quad if\ m_d \geq m_b, \tag{6.4.4}$$

$$a_{i0} + q a_{i1} = q^i < 1, \quad if\ m_d < m_b \leq +\infty. \tag{6.4.5}$$

**Proof.** The required results follow directly from Lemma 6.2.3 since $P(\tau_k < \infty | X(0) = i) = \lim_{t \to \infty} p_{ik}(t)$ $(i \geq 2, k = 0, 1)$ and $P(\tau < t | X(0) = i) = p_{i0}(t) + p_{i1}(t)$ $(i \geq 2)$. $\square$

Having pointed out the extinction probabilities, we now consider the mean extinction time $E_i[\tau]$ $(i \geq 2)$ and conditional mean extinction times $E_i[\tau_k|\tau_k < \infty]$ $(i \geq 2, k = 0, 1)$ and $E_i[\tau|\tau < \infty]$ $(i \geq 2)$, where $E_i$ denotes the expectation under the condition $X(0) = i$. It is clear that

$$E_i[\tau_k|\tau_k < \infty] = \frac{E_i[\tau_k I_{\{\tau_k<\infty\}}]}{P(\tau_k < \infty|X(0) = i)}, \quad k = 0, 1.$$

**Theorem 6.4.2.** For the Feller minimal WCBP starting at state $i \geq 2$, $E_i[\tau]$ is finite if and only if $m_d \geq m_b$ and $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$ hold and in which case we have

$$E_i[\tau] = \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \tag{6.4.6}$$

More specifically,

(i) If $m_d \geq m_b$, then $E_i[\tau]$ is finite if and only if $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$.

(ii) If $m_d < m_b \leq +\infty$, then $E_i[\tau] = +\infty$.

**Proof.** If $m_d < m_b \leq +\infty$, it is easily seen from Theorem 6.4.1 that $E_i[\tau] = +\infty$. So we can assume that $m_d \geq m_b$, then $(p_{ij}(t);\ i, j \in \mathbf{Z}_+)$ is honest and thus

$$\int_0^t P(\tau > s|X(0) = i)ds = \int_0^t (1 - p_{i0}(s) - p_{i1}(s))ds = \int_0^t (\sum_{k=2}^{\infty} p_{ik}(s))ds.$$

Letting $t \uparrow \infty$ and then using (6.2.26) yields

$$E_i[\tau] = \int_0^{\infty} P(\tau > s|X(0) = i)ds = \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!}.$$

Finally, it follows from (6.2.24) that $E_i[\tau] < \infty$ if and only if $\sum_{k=2}^{\infty}(g_{k-2}/w_k)$ $< \infty$. The proof is complete. $\square$

¿From Theorem 6.4.2 we see that $E_i[\tau_k] = +\infty$ $(i \geq 2,\ k = 0, 1)$ in some cases, so we consider the conditional mean extinction time $E_i[\tau_k|\tau_k < +\infty]$ $(i \geq 2,\ k = 0, 1)$.

**Theorem 6.4.3.** Suppose that the Feller minimal WCBP starts at state $i \geq 2$.

(i) If $m_d \geq m_b$ and $\sum_{k=2}^{\infty}(g_{k-2}/w_k) = \infty$, then $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ are all infinite while if $m_d \geq m_b$ and $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$, then $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ are all finite and given by

$$E_i[\tau_0|\tau_0 < \infty] = \frac{1}{(-q_* + q_*^i)} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (-q_* + q_*^k) \quad (6.4.7)$$

$$E_i[\tau_1|\tau_1 < \infty] = \frac{1}{(1 - q_*^i)} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (1 - q_*^k) \quad (6.4.8)$$

$$E_i[\tau] = \frac{-q_* + q_*^i}{1 - q_*} E_i[\tau_0|\tau_0 < \infty] + \frac{1 - q_*^i}{1 - q_*} E_i[\tau_1|\tau_1 < \infty]. \quad (6.4.9)$$

(ii) If $m_d < m_b \leq +\infty$ and $\Sigma_{k=2}^{\infty} \frac{g_{k-2}}{w_k} \cdot q^k = \infty$, then $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ and $E_i[\tau|\tau < \infty]$ are all infinite while if $m_d < m_b$ $\leq +\infty$ and $\Sigma_{k=2}^{\infty} \frac{g_{k-2}}{w_k} \cdot q^k < \infty$, then $E_i[\tau_k|\tau_k < \infty]$ $(k = 0, 1)$ and $E_i[\tau|\tau < \infty]$ are all finite and given by

$$\begin{aligned} & E_i[\tau_0|\tau_0 < \infty] \\ = \; & \frac{1}{-q_* q^i + q q_*^i} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (-q_* q^k + q q_*^k), \end{aligned} \quad (6.4.10)$$

$$E_i[\tau_1|\tau_1 < \infty] = \frac{1}{(q^i - q_*^i)} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (q^k - q_*^k) \quad (6.4.11)$$

$$\begin{aligned} & E_i[\tau|\tau < \infty] \\ = \; & \frac{(-q_* q^i + q q_*^i) E_i[\tau_0|\tau_0 < \infty] + (q^i - q_*^i) E_i[\tau_1|\tau_1 < \infty]}{q^i(1 - q_*) - q_*^i(1 - q)}. \end{aligned} \quad (6.4.12)$$

**Proof.** Note that

$$E_i[\tau_k|\tau_k < \infty] = \frac{E_i[\tau_k \cdot I_{\{\tau_k < \infty\}}]}{a_{ik}}, \quad k = 0, 1. \quad (6.4.13)$$

We only need to evaluate $\mu_{ik} =: E_i[\tau_k \cdot I_{\{\tau_k < \infty\}}]$, $k = 0, 1$.

We will prove (ii) first. Since $m_d < m_b \leq +\infty, 0 < q < 1$. It follows from (6.4.5) and $\Sigma_{j=0}^{\infty} p_{ij}(t) q^j = q^i$ that

$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) q = \sum_{k=2}^{\infty} p_{ik}(t) q^k. \quad (6.4.14)$$

Similarly, from (6.2.21) and the fact that $\Sigma_{j=0}^{\infty} p_{ij}(t) q_*^j = q_*^i$ we can get

$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) q_* = \sum_{k=2}^{\infty} p_{ik}(t) q_*^k. \quad (6.4.15)$$

Therefore, since $a_{ik} - p_{ik}(t) = P(t < \tau_k < \infty | X(0) = i)$, $k = 0, 1$, integrating (6.4.14) and (6.4.15) with respect to $t$ yields

$$\int_0^t P(s < \tau_0 < \infty | X(0) = i) ds + q \int_0^t P(s < \tau_1 < \infty | X(0) = i) ds$$
$$= \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot q^k,$$

$$\int_0^t P(s < \tau_0 < \infty | X(0) = i) ds - q_* \int_0^t P(s < \tau_1 < \infty | X(0) = i) ds$$
$$= \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot q_*^k$$

and hence

$$\int_0^t P(s < \tau_0 < \infty | X(0) = i) ds = \frac{1}{q - q_*} \cdot \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot (-q_* q^k + q q_*^k),$$

$$\int_0^t P(s < \tau_1 < \infty | X(0) = i) ds = \frac{1}{q - q_*} \cdot \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot (q^k - q_*^k).$$

Note that $0 < -q_* < q < 1$, letting $t \uparrow \infty$ and using Monotone Convergence Theorem yields that

$$\mu_{i0} = \frac{1}{q - q_*} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (-q_* q^k + q q_*^k) \tag{6.4.16}$$

$$\mu_{i1} = \frac{1}{q - q_*} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (q^k - q_*^k). \tag{6.4.17}$$

On the other hand, from the definition of $\tau$ we can see that

$$E_i[\tau \cdot I_{\{\tau < \infty\}}] = E_i[\tau_0 \cdot I_{\{\tau_0 < \infty\}}] + E_i[\tau_1 \cdot I_{\{\tau_1 < \infty\}}] \tag{6.4.18}$$

Thus the conclusions in (ii) follow from (6.4.16)–(6.4.18) and the fact that $0 < -q_* < q < 1$.

Now we turn to prove (i). Since $m_d \geq m_b$, $P(\tau < \infty | X(0) = i) = 1$. It follows from (6.4.4) and the honesty of $(p_{ij}(t))$ that

$$(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) = \sum_{k=2}^{\infty} p_{ik}(t). \tag{6.4.19}$$

Secondly, (6.4.15) still holds in this case. Thus, a similar argument yields

$$\int_0^t P(s < \tau_0 < \infty | X(0) = i) ds = \frac{1}{1 - q_*} \cdot \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot (-q_* + q_*^k),$$

$$\int_0^t P(s < \tau_1 < \infty | X(0) = i) ds = \frac{1}{1-q_*} \cdot \sum_{k=2}^{\infty} (\int_0^t p_{ik}(s) ds) \cdot (1 - q_*^k).$$

Note that $0 < -q_* < 1$, letting $t \uparrow \infty$ and using Monotone Convergence Theorem yields that

$$\mu_{i0} = \frac{1}{1-q_*} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (-q_* + q_*^k) \tag{6.4.20}$$

$$\mu_{i1} = \frac{1}{1-q_*} \cdot \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!} \cdot (1 - q_*^k). \tag{6.4.21}$$

On the other hand, it follows from (6.4.13) and the definition of $\tau$ that

$$E_i[\tau] = E_i[\tau \cdot I_{\{\tau < \infty\}}] = \frac{-q_* + q_*^i}{1-q_*} \cdot E_i[\tau_0 | \tau_0 < \infty] + \frac{1-q_*^i}{1-q_*} \cdot E_i[\tau_1 | \tau_1 < \infty]. \tag{6.4.22}$$

Thus the conclusions in (i) follow from (6.4.20)–(6.4.22) and the fact that $0 < -q_* < 1$. □

## 6.5. Explosion and Holding Time

Having obtained the extinction probability and mean extinction time, we are now in a position to consider the explosion probability and mean explosion time. By Theorem 6.3.1, we only need to consider the case that $m_d < m_b \leq +\infty$. Let $\tau_\infty$ denote the explosion time and $a_{i\infty} = P(\tau_\infty < \infty | X(0) = i)$ denote explosion probability.

Since we are dealing with the minimal process,

$$p_{i\infty}(t) := 1 - \sum_{j=0}^{\infty} p_{ij}(t) = P(\tau_\infty \leq t | X(0) = i)$$

is the probability of explosion by time $t$ starting at state $i$, and $p_{i\infty}(t) \to a_{i\infty}$ as $t \to \infty$.

**Theorem 6.5.1.** Suppose that the Feller minimal WCBP starts at state $i \geq 2$.

(i) If $m_d \geq m_b$, then $a_{i\infty} = 0$.

(ii) If $m_d < m_b \leq +\infty$ and $\Sigma_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$, then

$$a_{i\infty} = 1 - \frac{(1-q_*)q^i - (1-q)q_*^i}{q - q_*} > 0 \tag{6.5.1}$$

and

$$
\begin{aligned}
&a_{i0}E_i[\tau_0|\tau_0 < \infty] + a_{i1}E_i[\tau_1|\tau_1 < \infty] + a_{i\infty}E_i[\tau_\infty|\tau_\infty < \infty] \\
&= \sum_{k=2}^{\infty} \frac{1}{w_k} \cdot \frac{G_i^{(k-2)}(0)}{(k-2)!}.
\end{aligned} \tag{6.5.2}
$$

**Proof.** If $m_d \geq m_b$, then the Feller minimal WCBP is honest and hence $a_{i\infty} = 0$. If $m_d < m_b \leq +\infty$ and $\sum_{k=2}^{\infty}(g_{k-2}/w_k) < \infty$, it follows from Theorems 6.3.2 and 6.3.3 that the Feller minimal WCBP is dishonest, i.e.,

$$
p_{i\infty}(t) = 1 - \sum_{j=0}^{\infty} p_{ij}(t) > 0 \tag{6.5.3}
$$

which yields (6.5.1) by letting $t \uparrow \infty$ and using (6.2.27) together with (6.4.2). It follows from (6.5.1) and (6.5.3) that

$$
(a_{i0} - p_{i0}(t)) + (a_{i1} - p_{i1}(t)) + (a_{i\infty} - p_{i\infty}(t)) = \sum_{j=2}^{\infty} p_{ij}(t). \tag{6.5.4}
$$

Integrating (6.5.4) with respect to $t$ and noting that

$$
P_i(\tau_\infty \leq t|\tau_\infty < \infty) = \frac{p_{i\infty}(t)}{a_{i\infty}}
$$

yields (6.5.2). □

Our final result concerns the time spent in each state over the lifetime of the process. Let $T_k$ be the total time spent in state $k$ ($\geq 2$) and let $\mu_{ik} = E[T_k|X(0) = i]$, $i \geq 2$. Then,

$$
\mu_{ik} = E[\int_0^\infty I_{\{X(t)=k\}}dt|X(0) = i] = \int_0^\infty p_{ik}(t)dt.
$$

This expression was evaluated in (6.2.26). We have therefore proved the following result.

**Theorem 6.5.2.** All of $\mu_{ik}$, $i \geq 2$, $k \geq 2$, are finite and given by

$$
\mu_{ik} = \frac{1}{w_k}\frac{G_i^{(k-2)}(0)}{(k-2)!}.
$$

## 6.6. Notes

WCBP is a natural generalisation of CBP and GCBP. In fact, we recover GCBP by letting $w_n = n^\alpha(n-1)^\beta$ $(n \geq 2)$ and recover CBP by letting $w_n = n(n-1)/2$ $(n \geq 2)$.

The work discussed in this chapter is in preparation for submission in Chen, Pollett, Li and Zhang (2004). More specifically, the introduction of the function $G(z)$ (see (6.2.1)) is due to Li. Lemma 6.2.1, Lemma 6.2.2, Lemma 6.2.4, Theorem 6.3.2, Theorem 6.4.2 and Theorem 6.4.3 are due to Li. Lemma 6.2.3, Theorem 6.2.1, Theorem 6.3.3, Theorem 6.4.1 and Theorem 6.5.1 are due to Li and the other authors.

Similar as regarding WMBP, the regularity problem for WCBP remains unsolved in the case $m_b = +\infty$. By Theorem 6.3.2, if the WCBP is regular then $\Sigma_{k=2}^{\infty}(g_{k-2}/w_k) = +\infty$, i.e., the condition $\Sigma_{k=2}^{\infty}(g_{k-2}/w_k) = +\infty$ is necessary for the regularity. However, in most situations, such as the CBPs or GCBPs considered in Chapter 4 or 5, respectively, this condition is also sufficient. A natural problem is whether this condition is sufficient for regularity in the most general WCBP.

Theorem 6.3.5 gives some sufficient conditions for uniqueness. However, when $m_d < m_b \leq +\infty$, the uniqueness problem of $Q$-processes satisfying Kolmogorov forward equations is still not completely solved.

As mentioned in Section 3.8, by using a proper compound Poisson process as an underlying structure, the approach of random time changes could be applicable in studying WCBPs. We will study this interesting problem in future.

Because of the generality of the sequence $\{w_n; n \geq 2\}$, many special properties of CBP or GCBP are no longer true for the most general WCBP. For example, the partial differential equation (4.3.5) established for CBP and the important expression (5.2.16) established for GCBP are not now available for WCBP. However, this most general WCBP has a wide range of applications. We can use different forms of $\{w_n; n \geq 2\}$ to model different realistic cases, for instance, we can obtain an interesting

# 7. Conclusions and Future work

## 7.1. Conclusions

In the previous chapters, we have considered some generalised branching models and discussed some of the important characteristics of the corresponding $Q$-processes. This included the regularity criterion, extinction and explosion behaviour, recurrence and ergodic properties. We now summarise our main conclusions.

### 7.1.1. Markov Branching Processes with Immigration and Resurrection

Criteria for the regularity and uniqueness for Markov branching processes with immigration and resurrection are established. The effect of such immigration and resurrection is investigated in detail.

Explicit expressions for the extinction probability and mean extinction time of the process are presented. It is revealed that if the death rate is greater than the mean birth rate for the underlying branching structure, then a considerably large immigration is necessary to rescue a species from extinction, while if the death rate is equal to the mean birth rate, then a mild immigration will suffice.

Ergodicity and stability properties of the process incorporating resurrection structure are then investigated. The conditions for recurrence, ergodicity and exponential ergodicity are obtained. An explicit expression for the equilibrium distribution is also presented.

### 7.1.2. Weighted Markov Branching Processes

Regularity and uniqueness criteria for the weighted Markov branching processes, which are very easy to verify, are established.

Some important characteristics regarding the hitting times of such structure are obtained. In particular, closed expressions for the mean extinction time and conditional mean extinction time are presented.

The explosion behaviour of the process is investigated and the mean explosion time is derived. The mean global holding time and the mean total survival time are also obtained. The Harris regularity criterion for ordinary Markov branching process is extended to the more general case of non-linear Markov branching processes.

### 7.1.3. Collision Branching Processes and General Collision Branching Processes with 2 parameters

The collision branching process is explored further and a new class of branching models, the general collision branching process with 2 parameters, is also considered. The regularity and uniqueness criteria of the process are established.

Explicit expressions are then obtained for the extinction probability vector, the mean extinction times and the conditional mean extinction times of the process.

The explosion behaviour of such model is investigated and an explicit expression for mean explosion time is established. The mean global holding time is also obtained. It is discovered that these properties are substantially different between the super-explosive and sub-explosive cases.

### 7.1.4. Weighted Collision Branching Processes

For the most general WCBP, some conditions for the regularity and uniqueness are established.

The extinction behaviour of such processes is investigated and explicit expressions for mean extinction time and conditional mean extinction time are presented.

The explosion behaviour of the process is also investigated and the mean explosion time is derived. The mean global holding time and the mean total survival time are also obtained.

## 7.2. Future Work

The models discussed in the previous chapters play an important role in the study of complex branching systems and have a wide range of applications. The following are some significant related problems we would like to investigate in the future.

**Question 1.** Since $C = \{2, 3, \cdots\}$ is the transient and communicating class of a weighted collision branching process $(p_{ij}(t))$, the limit $\lim_{t\to\infty} p_{ij}(t)$ is equal to 0 for all $i, j \in C$ and therefore quite uninformative. The sample paths are simply thinning out with increasing $t$. In order to get more information about this process, it is natural and important to consider the so-called decay parameter of $C$, i.e., to find the limit

$$\lambda_C = -\lim_{t\to\infty} \frac{1}{t} \log p_{ij}(t), \qquad i, j \in C$$

and to consider the quasistationary distribution:

$$\tilde{p}_{ij}(\infty) = \lim_{t\to\infty} \frac{p_{ij}(t)}{\sum_{k\in C} p_{ik}(t)}, \quad i, j \in C.$$

For additional material regarding the decay parameter and quasistationary distribution, refer to Kingman (1963), Vere-Jones (1962, 1967, 1969), Pollett (1986, 1988), Pollett and Vere-Jones (1992) and Nair and Pollett (1993).

**Question 2.** In this thesis, the models we considered have at most two absorbing states. However, in many realistic situations, the branching events are effected by the interaction or collision of $k(\geq 3)$ particles. When the number of particles in the system is strictly less than $k$, such interaction or collision can not happen. Therefore, in these models, there are $k(\geq 3)$ absorbing states. A special case of such structures was considered in Kalinkin (1982, 2002). We are interested in studying the extinction and explosion behaviour for the general cases.

**Question 3.** Consider a system in which there are $d \geq 2$ different types of particle: the branching events are affected by the interaction of $k(\leq d)$ (maybe different types of) particles. For example (Asmussen and Hering (1983)), suppose there are two different types of particles, say, male and female, in the system. The system evolves as follows: only a couple, a male and a female, can give birth. Neither a single male nor a single female alone can give birth. Therefore, when only one gender is left, the system will stop. For the general multi-dimensional cases, it could be much more complicated. In order to describe such population systems, a multi-dimensional process is needed. It is worth considering the evolution of such systems.

Finally, we would like to point out that our discussion in this thesis is mainly theoretical. We understand that simulation is another important technique. Therefore, in future, we will use simulated realisations to illustrate the different types of behaviour of the models considered in this thesis and apply the theoretical results to tackle realistic problems in science, engineering and finance.

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